The hierarchy of hypergeometric functions

Tom Koornwinder

University of Amsterdam, thk@science.uva.nl

lecture, March 17, 2008

16ème colloque de la SMT, Sousse, Tunisie

Definition of hypergeometric series

A series $\sum_{k=0}^{\infty} c_k$ is called

- *hypergeometric* if c_{k+1}/c_k is a rational function of k;
- *q-hypergeometric* if c_{k+1}/c_k is a rational function of q^k;
- *elliptic hypergeometric* if c_{k+1}/c_k is an elliptic function of k (meromorphic doubly periodic function).

Typical cases for c_{k+1}/c_k:

- ordinary: $\dfrac{(a_1+k)\ldots(a_r+k)\,z}{(b_1+k)\ldots(b_s+k)\,(1+k)}$;
- q-case: $\dfrac{(1-a_1q^k)\ldots(1-a_rq^k)\,(-q^k)^{s-r+1}\,z}{(1-b_1q^k)\ldots(1-b_sq^k)\,(1-q^{k+1})}$;
- elliptic: $\dfrac{\theta(a_1q^k;p)\ldots\theta(a_rq^k;p)\,z}{\theta(b_1q^k;p)\ldots\theta(b_{r-1}q^k;p)\,\theta(q^{k+1};p)}$,

  where $a_1\ldots a_r = b_1\ldots b_{r-1}\,q$

  and $\quad \theta(a;p) := \prod_{j=0}^{\infty}(1-ap^k)(1-a^{-1}p^{k+1}) \quad (|p|<1)$.

Definition of hypergeometric series (continued)

Pochhammer symbols and *hypergeometric series*:

- ordinary: $(a)_k := a(a+1)\ldots(a+k-1)$,

$$ {}_rF_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k\ldots(a_r)_k\,z^k}{(b_1)_k\ldots(b_s)_k\,k!}\,. $$

- q-case ($|q|<1$):
  $(a;q)_k := (1-a)(1-qa)\ldots(1-q^{k-1}a)$,
  $(a;q)_\infty := (1-a)(1-qa)(1-q^2a)\ldots$,

$$ {}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};q,z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k\ldots(a_r;q)_k\,\left((-1)^k q^{\frac12 k(k-1)}\right)^{s-r+1}\,z^k}{(b_1;q)_k\ldots,(b_s;q)_k\,(q;q)_k}\,. $$

Series are terminating if some $a_i=-n$ resp. q^{-n} $(n\in\mathbb{Z}_{\geq 0})$.

As $q\to 1$ we have $(1-q)^{-k}(q^a;q)_k \to (a)_k$ and

$$ {}_r\phi_{r-1}\left(\begin{matrix} q^{a_1},\ldots,q^{a_r} \\ q^{b_1},\ldots,q^{b_{r-1}} \end{matrix};q,z\right) \to {}_rF_{r-1}\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{r-1} \end{matrix};z\right). $$

Definition of hypergeometric series (continued)

Elliptic Pochhammer symbol and hypergeometric series:

$|q|, |p| < 1. \qquad \theta(a;p) := (a;p)_\infty \, (a^{-1}p;p)_\infty.$

$(a;q,p)_k := \theta(a;p)\theta(qa;p)\ldots\theta(q^{k-1}a;p).$

$${}_rE_{r-1}\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{r-1}\end{matrix};q,p;z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q,p)_k\ldots(a_r;q,p)_k\, z^k}{(b_1;q,p)_k\ldots(b_{r-1};q,p)_k\,(q;q,p)_k}$$

$$(a_1\ldots a_r = b_1\ldots b_{r-1}q).$$

Series is terminating if some $a_i = q^{-n} \quad (n \in \mathbb{Z}_{\geq 0})$.

$(a;q,p)_k \to (a;q)_k$ and ${}_rE_{r-1}(q,p,z) \to {}_r\phi_{r-1}(q,z)$ as $p \to 0$.

Nonterminating ${}_rF_s(z)$ and ${}_r\phi_s(z)$ series converge everywhere if $s > r-1$, converge for $|z| < 1$ if $s = r-1$, and converge nowhere outside 0 if $s < r-1$.

Nonterminating elliptic hypergeometric series do not converge.

(q-)Hypergeometric series: elementary cases

Exponential series: $\quad {}_0F_0(-;-;z)=\sum_{k=0}^\infty \frac{z^k}{k!}=e^z.$

Binomial series: $\quad {}_1F_0(a;-;z)=\sum_{k=0}^\infty \frac{(a)_k}{k!}\,z^k=(1-z)^{-a}.$

q-exponential series:

$$e_q(z):={}_1\phi_0(0;-;q,z)=\sum_{k=0}^\infty \frac{z^k}{(q;q)_k}=\frac{1}{(z;q)_\infty}\quad(|z|<1),$$

$$E_q(z):={}_0\phi_0(-;-;q,-z)=\sum_{k=0}^\infty \frac{q^{\frac12 k(k-1)}z^k}{(q;q)_k}=(-z;q)_\infty\quad(|z|<\infty).$$

q-binomial series:

$${}_1\phi_0(a;-;q,z)=\sum_{k=0}^\infty \frac{(a;q)_k}{(q;q)_k}\,z^k=\frac{(az;q)_\infty}{(z;q)_\infty}\quad(|z|<1).$$

Intermezzo: Chaundy-Bullard identity

$$(x+y)^{m+n+1} = y^{n+1}\sum_{i=0}^{m}\binom{m+n+1}{i}x^i y^{m-i} + x^{m+1}\sum_{j=0}^{n}\binom{m+n+1}{j}x^{n-j}y^j,$$

$$1 = (1-x)^{n+1}\sum_{i=0}^{m}\binom{m+n+1}{i}x^i(1-x)^{m-i} + x^{m+1}\sum_{j=0}^{n}\binom{m+n+1}{j}x^{n-j}(1-x)^j,$$

$$1 = (1-x)^{n+1}P_{m,n}(x) + x^{m+1}P_{n,m}(1-x) \quad (\deg(P_{m,n}) \le m),$$

$$(1-x)^{-(n+1)} = P_{m,n}(x) + \mathcal{O}(x^{m+1}) \qquad (x \to 0),$$

$$= \sum_{k=0}^{m}\frac{(n+1)_k}{k!}x^k + \mathcal{O}(x^{m+1}) \quad (x \to 0).$$

Chaundy-Bullard identity, continued

Hence (**Chaundy & Bullard, 1960**):

$$1 = (1-x)^{n+1} P_{m,n}(x) + x^{m+1} P_{n,m}(1-x),$$

where

$$P_{m,n}(x) = \sum_{k=0}^{m} \frac{(n+1)_k}{k!}\, x^k.$$

Two-variable analogue (**TK & Schlosser, 2008**):

$$1 = (1-x-y)^{l+1} P_{m,n,l}(x,y) + x^{m+1} P_{n,l,m}(y, 1-x-y) \\ + y^{n+1} P_{l,m,n}(1-x-y, x),$$

where

$$P_{m,n,l}(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{(l+1)_{i+j}}{i!\, j!}\, x^i y^j.$$

Similarly, an *n*-variable analogue.

Jacobi polynomials

Gauss hypergeometric series: $\quad {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}\, z^k.$

Terminating case gives *Jacobi polynomials*:

$$P_n^{(\alpha,\beta)}(x) := \text{const.}\ {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \tfrac{1}{2}(1-x)\right)$$
$$= \text{const.} \sum_{k=0}^{n} \frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!} \left(\frac{1-x}{2}\right)^k.$$

Orthogonality $\quad (\alpha,\beta > -1)$:

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta\,dx = 0 \qquad (n \neq m).$$

Differential equation:

$$\left((1-x^2)\frac{d^2}{dx^2} + \left(\beta-\alpha-(\alpha+\beta+2)\right)\frac{d}{dx}\right) P_n^{(\alpha,\beta)}(x)$$
$$= -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x).$$

Jacobi functions

Nonterminating ${}_2F_1$ gives *Jacobi functions*:

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1\left(\tfrac12(\alpha+\beta+1+i\lambda), \tfrac12(\alpha+\beta+1-i\lambda); \alpha+1; -\sinh^2 t\right).$$

Transform pair $(\alpha \pm \beta > -1)$:

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_0^\infty f(t)\,\phi_\lambda^{(\alpha,\beta)}(t)\,\Delta_{\alpha,\beta}(t)\,dt, \\ f(t) = \displaystyle\int_0^\infty \widehat{f}(\lambda)\,\phi_\lambda^{(\alpha,\beta)}(t)\,\frac{d\lambda}{|c_{\alpha,\beta}(\lambda)|^2}, \end{cases}$$

where $\Delta_{\alpha,\beta}(t) := (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}$ and $c_{\alpha,\beta}(\lambda)$ is a certain quotient of products of Gamma functions.

Differential equation:

$$\frac{1}{\Delta_{\alpha,\beta}(t)}\,\frac{d}{dt}\circ\Delta_{\alpha,\beta}(t)\,\frac{d}{dt}\,\phi_\lambda^{(\alpha,\beta)}(t) = -\big(\lambda^2+(\alpha+\beta+1)^2\big)\,\phi_\lambda^{(\alpha,\beta)}(t).$$

Five different types of generalizations

The Gauss hypergeometric function / Jacobi polynomial / Jacobi function case can be generalized in five different directions, which often can be combined, and ideally should always be combined.

1. Higher hypergeometric series; Askey scheme of hypergeometric orthogonal polynomials
2. q-hypergeometric series, elliptic and hyperbolic hypergeometric function
3. Non-symmetric functions (double affine Hecke algebras)
4. Four regular singularities (Heun equation)
5. Multivariable special functions associated with root systems (Heckman-Opdam, Macdonald, Macdonald-K, Cherednik, . . .)

I will not discuss items 3, 4 and 5 here. Item 3 was inspired by the multi-variable case.

Criteria for the (q-)hypergeometric hierarchy

For hypergeometric and q-hypergeometric functions we will restrict to some cases which:

- have a rich set of transformations, which form a nice symmetry group;
- allow harmonic analysis: orthogonal polynomials or biorthogonal rational functions, or continuous analogues of these as kernels of integral transforms.

Then we mainly have:

- ${}_4F_3(1)$, ${}_7F_6(1)$, ${}_9F_8(1)$ hypergeometric functions, and q- and hyperbolic analogues, and only one elliptic analogue.
- Moreover in these cases restrictions on parameters (balanced, very-well poised).
- Always distinction between terminating and non-terminating series.
- In non-terminating cases alternative representations as hypergeometric (Mellin-Barnes type) integral; crucial role of gamma function (ordinary, q-, hyperbolic, elliptic).

Symmetries of ${}_3F_2(1)$

Thomae's transformation formula rediscovered by Ramanujan:

$$ {}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};1\right) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(a)\,\Gamma(d+e-a-c)\,\Gamma(d+e-a-b)} \times {}_3F_2\left(\begin{matrix} d-a,e-a,d+e-a-b-c \\ d+e-a-c,d+e-a-b \end{matrix};1\right). $$

Hardy (*Ramanujan, Twelve lectures on subjects suggested by his life and work*, 1940):

$$ \frac{1}{\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-a-b-c)}\ {}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};1\right) $$

is symmetric in d, e, $d+e-b-c$, $d+e-c-a$, $d+e-a-b$.
Symmetry group $S_5 = W(A_4)$ (Weyl group of root system A_4).

Dynkin diagram for A_4: 

Balanced ${}_4F3(1)$

${}_rF_{r-1}(a_1,\ldots,a_r;b_1,\ldots,b_{r-1};z)$ is called *balanced*
if $b_1+\ldots+b_{r-1}=a_1+\ldots+a_r+1$.

Beyer-Louck-Stein rediscovered Hardy's S_5-symmetry for ${}_3F_2(1)$, and found symmetry group $S_6=W(A_5)$ for terminating balanced ${}_4F_3(1)$:

$${}_4F_3(-n,a,b,c;d,e,f;1) \qquad (d+e+f=-n+a+b+c+1).$$

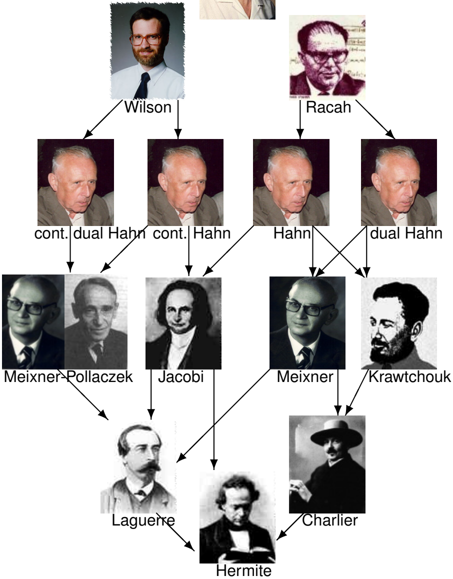
Dynkin diagram for A_5: 

Related orthogonal polynomials: *Wilson polynomials* $W_n(x^2):=$

$$\text{const.}\ {}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix};1\right),$$

and *Racah polynomials*. These form the top level of the Askey scheme of hypergeometric orthogonal polynomials.

Askey  scheme



Wilson functions

For *Wilson functions* (non-polynomial analogues of Wilson polynomials) one has to go to the ${}_7F_6$ level.

Well-poised hypergeometric series:

$$ {}_rF_{r-1}\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ 1+a_1-a_2, \ldots, 1+a_1-a_r \end{matrix}; z\right). $$

This is *very well-poised* (VWP) if $a_2 = 1 + \frac{1}{2}a_1$.

Terminating VWP ${}_7F_6(1) = \mathrm{const.} \times$ terminating balanced ${}_4F_3(1)$.

Non-terminating VWP ${}_7F_6(1) =$ linear combination of two balanced ${}_4F_3(1)$'s.

Wilson function transform (Groenevelt).

The ${}_9F_8$ top level

Terminating 2-balanced VWP ${}_9F_8(1)$:

Transformation formula (Bailey, Whipple).

Biorthogonal rational functions (J. Wilson).

Non-terminating 2-balanced VWP ${}_9F_8(1)$:

Four-term transformation formula (Bailey).

Askey-Wilson polynomials

${}_r\phi_{r-1}(a_1,\ldots,a_r;b_1,\ldots,b_{r-1};q,z)$ is called *balanced* if $b_1\ldots b_{r-1}=qa_1\ldots a_r$.

Terminating balanced ${}_4\phi_3$ of argument q:

- Symmetry group $S_6=W(A_5)$ (Van der Jeugt & S. Rao).
- Askey-Wilson polynomials:

$$p_n\big(\tfrac12(z+z^{-1}\big):=\text{const.}\ {}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ab,ac,ad\end{matrix};q,q\right).$$

- $P_n(z):=p_n\big(\tfrac12(z+z^{-1}\big)$ satisfies

$$A(z)P_n(qz)+A(z^{-1})P_n(q^{-1}z)-(A(z)+A(z^{-1}))P_n(z)\\=(q^{-n}-1)(1-abcdq^{n-1})P_n(z),$$

$$\text{where}\quad A(z):=\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

- These are on the top level of the q-Askey scheme.

Askey-Wilson functions

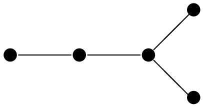
Very well-poised (VWP) q-hypergeometric series:

$$ {}_rV_{r-1}(a_1; a_4, \ldots, a_r; q, z) := {}_r\phi_{r-1}\left(\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \ldots, a_r \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \ldots, qa_1/a_r \end{matrix}; q, z\right) $$

Non-terminating very well-poised ${}_8\phi_7$ of argument $\frac{q^2\, a_1^2}{a_4\, a_5\, a_6\, a_7\, a_8}$:

- Sum of two non-terminating balanced ${}_4\phi_3$'s of argument q.
- Symmetry group $W(D_5)$ (Van der Jeugt & S. Rao).
- Askey-Wilson functions (Stokman).

Dynkin diagram for D_5:



Bailey's two-term ${}_{10}\phi_9$ function

$$\Phi(a;b;c,d,e,f,g,h;q) :=$$

$$(aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)_\infty$$

$$\times\, (bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q)_\infty / (b/a, aq; q)_\infty$$

$$\times\, {}_{10}V_9(a; b, c, d, e, f, g, h; q, q)$$

$$+ \frac{(bq/c, bq/d, bq/e, bq/f, bq/g, bq/h, c, d, e, f, g, h; q)_\infty}{(a/b, b^2q/a; q)\infty}$$

$$\times\, {}_{10}V_9(b^2/a; b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a; q, q),$$

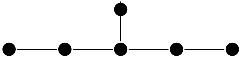
where $\quad a^3q^2 = bcdefgh$

Bailey's four-term transformation formula:

$$\Phi(a;b;c,d,e,f,g,h;q) = \Phi\Big(\frac{a^2q}{cde}; b; \frac{aq}{de}, \frac{aq}{ce}, \frac{aq}{cd}, f, g, h; q\Big).$$

Symmetry group $W(E_6)$

(Lievens & Van der Jeugt):



${}_{10}\phi_9$: the terminating case

Terminating balanced very well-poised ${}_{10}\phi_9$'s of argument q:

- Bailey's two-term transformation formula.
- Same symmetry group $W(E_6)$.
- Biorthogonal rational functions (Rahman, J. Wilson)

Summary of important (q-)hypergeometric cases

- ${}_9F_8$ and ${}_{10}\phi_9$ (very well-poised):
  Four- and two-term transformation formulas, E_6 symmetry, biorthogonal rational functions in terminating case.
- ${}_7F_6$ and ${}_8\phi_7$ (very well-poised):
  non-terminating, D_5 symmetry, Wilson and Askey-Wilson functions as kernels in integral transform pairs.
- ${}_4F_3$ and ${}_4\phi_3$ (balanced):
  terminating, A_5 symmetry, Wilson, Racah and Askey-Wilson, q-Racah polynomials.

Elliptic case: Frenkel-Turaev formulas

Very well-poised elliptic hypergeometric series:

$$
{}_rV_{r-1}(a_1; a_6, \ldots, a_r; q, p) :=
$$
$$
{}_rE_{r-1}\left(\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, q(a_1/p)^{\frac{1}{2}}, -q(a_1p)^{\frac{1}{2}}, a_6, \ldots, a_r \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, (pa_1)^{\frac{1}{2}}, -(a_1/p)^{\frac{1}{2}}, qa_1/a_6, \ldots, qa_1/a_r \end{matrix}; q, p; -1\right),
$$

where $a_6 \ldots a_r = q^{\frac{1}{2}r-4} a_1^{\frac{1}{2}r-3}$.

Frenkel & Turaev gave a transformation formula for terminating ${}_{12}V_{11}$ and a summation formula for terminating ${}_{10}V_9$ as a degenerate case.

This followed from their study of the elliptic 6j-symbol, which is a solution of the Yang-Baxter equation for the fused eight-vertex model.

Spriridonov gave two-index biorthogonality relations for products of two terminating ${}_{12}V_{11}$ functions.

The elliptic hypergeometric integral

Elliptic gamma function (Ruijsenaars):

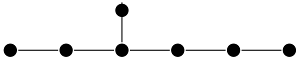
$$\Gamma_e(z;p,q) := \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^jq^k}\,.$$

Elliptic hypergeometric integral (Spiridonov):

$$S_e(t;p,q) := \int_{\mathcal{C}} \frac{\prod_{j=1}^{8}\Gamma_e(t_j z^{\pm 1};p,q)}{\Gamma_e(z^{\pm 2};p,q)}\,\frac{dz}{2\pi i z} \quad \left(\prod_{j=1}^{8} t_j = p^2q^2\right),$$

where $\mathcal{C}$ is a deformation of the unit circle which separates the poles $t_jp^mq^n$ $(m,n=0,1,\ldots)$ from the poles $t_j^{-1}p^{-m}q^{-n}$ $(m,n=0,1,\ldots)$.

The transformations of $S_e(t;p,q)$ form a symmetry group which is isomorphic to $W(E_7)$

(Rains): 

Hyperbolic hypergeometric series

In elliptic hypergeometric theory there are no transformation formulas below the ${}_{12}V_{11}$ level.

However, there is a limit case of the elliptic hypergeometric function, called *hyperbolic hypergeometric function*, started by Ruijsenaars, which is still above the q-case and with the following features:

- On top level again $W(E_7)$ symmetry.
- There is also a hyperbolic Askey-Wilson function.
- Has analytic continuation to q on unit circle.
- Explicit expressions as products of two q-hypergeometric functions or a sum of two such products.

For details see the Thesis by Fokko van de Bult, *Hyperbolic Hypergeometric Functions*, 2007 (partly based on papers jointly with Rains and Stokman).

References

On hypergeometric series:

W. N. Bailey, *Generalized hypergeometric series*, Cambridge University Press, 1935.

On *q*-hypergeometric series:

G. Gasper and M. Rahman, *Basic hypergeometric series*, 2nd edn., Cambridge University Press, 2004.

On the Askey and the *q*-Askey scheme:

R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, `http://aw.twi.tudelft.nl/~koekoek/askey/`.

On the Chaundy-Bullard identity:

T. H. Koornwinder and M. J. Schlosser, *On an identity by Chaundy and Bullard. I*, `arXiv:0712.2125v2 [math.CA]`, March 2008.

References, continued

On elliptic hypergeometric functions:

V. P. Spiridonov, *Classical elliptic hypergeometric functions and their applications*, `arXiv:math/0511579v2`.

V. P. Spiridonov, *Elliptic hypergeometric functions*, `arXiv:0704.3099v1`.

On groups of transformations of hypergeometric functions:

J. Van der Jeugt & K. S. Rao, *Invariance groups of transformations of basic hypergeometric series*, J. Math. Phys. 40 (1999), 6692–6700.

S. Lievens & J. Van der Jeugt, *Symmetry groups of Bailey's transformations for* ${}_{10}\phi_9$*-series*, J. Comput. Appl. Math. 206 (2007), 498–519.

F. J. van de Bult, E. M. Rains & J. V. Stokman, *Properties of generalized univariate hypergeometric functions*, `arXiv:math/0607250v1`.