The hierarchy of hypergeometric functions and related algebras

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Hypergeometric series

Pochhammer symbol: $(a)_k := a(a+1)\ldots(a+k-1)$.

Hypergeometric series: ${}_rF_s(a_1,\ldots,a_r;b_1,\ldots,b_s;z)$

$$= {}_rF_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};z\right) := \sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_r)_k}{(b_1)_k\ldots(b_s)_k\,k!}\,z^k.$$

Terminating if $a_1=-n$ (n nonnegative integer).
If nonterminating and $s=r-1$ then converges for $|z|<1$.

Gauss hypergeometric series: ${}_2F_1(a,b;c;z)$.

Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(x) := \text{const.}\ {}_2F_1\big(-n,n+\alpha+\beta+1;\alpha+1;\tfrac12(1-x)\big).$$

Orthogonality $(\alpha,\beta>-1)$:

$$\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)\,(1-x)^\alpha\,(1+x)^\beta\,dx=0 \qquad (n\neq m).$$

Five different types of generalizations

The Gauss hypergeometric function / Jacobi polynomial case can be generalized in five different directions, which often can be combined, and ideally should always be combined.

1. Higher hypergeometric series; Askey scheme of hypergeometric orthogonal polynomials
2. q-hypergeometric series, elliptic and hyperbolic hypergeometric function
3. Non-symmetric functions (double affine Hecke algebras)
4. Four regular singularities (Heun equation)
5. Multivariable special functions associated with root systems (Heckman-Opdam, Macdonald, Macdonald-K, Cherednik, . . .)

I will not discuss items 4 and 5 here. However, item 3 was inspired by the multi-variable case.

Plan of the lecture

First part

Higher hypergeometric series and q- and elliptic analogues

Second part

Double affine Hecke algebra in the Askey-Wilson case and relationship with Zhedanov algebra

Criteria for the (q-)hypergeometric hierarchy

For hypergeometric and q-hypergeometric functions we will restrict to some cases which:

- have a rich set of transformations, which form a nice symmetry group;
- allow harmonic analysis: orthogonal polynomials or biorthogonal rational functions, or continuous analogues of these as kernels of integral transforms.

Then we mainly have:

- ${}_4F_3(1)$, ${}_7F_6(1)$, ${}_9F_8(1)$ hypergeometric functions, and q- and hyperbolic analogues, and only one elliptic analogue
- Moreover in these cases restrictions on parameters (balanced, very-well poised)
- Always distinction between terminating and non-terminating series
- In non-terminating cases alternative representations as hypergeometric (Mellin-Barnes type) integral; crucial role of gamma function (ordinary, q-, hyperbolic, elliptic)

Symmetries of ${}_3F_2(1)$

Thomae's transformation formula rediscovered by Ramanujan:

$$
{}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; 1\right) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(a)\,\Gamma(d+e-a-c)\,\Gamma(d+e-a-b)} \times {}_3F_2\left(\begin{matrix} d-a, e-a, d+e-a-b-c \\ d+e-a-c, d+e-a-b \end{matrix}; 1\right).
$$

Hardy (*Ramanujan, Twelve lectures on subject suggested by his life and work*, 1940):

$$
\frac{1}{\Gamma(d)\,\Gamma(e)\,\Gamma(d+e-a-b-c)}\ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; 1\right)
$$

is symmetric in d, e, $d+e-b-c$, $d+e-c-a$, $d+e-a-b$.
Symmetry group $S_5 = W(A_4)$ (Weyl group of root system A_4).

Balanced ${}_4F3(1)$

${}_rF_{r-1}(a_1,\ldots,a_r;b_1,\ldots,b_{r-1};z)$ is called *balanced* if $b_1+\ldots+b_{r-1}=a_1+\ldots+a_r+1$.

Beyer-Louck-Stein rediscovered Hardy's S_5-symmetry for ${}_3F_2(1)$, and found symmetry group $S_6=W(A_5)$ for terminating balanced ${}_4F_3(1)$:

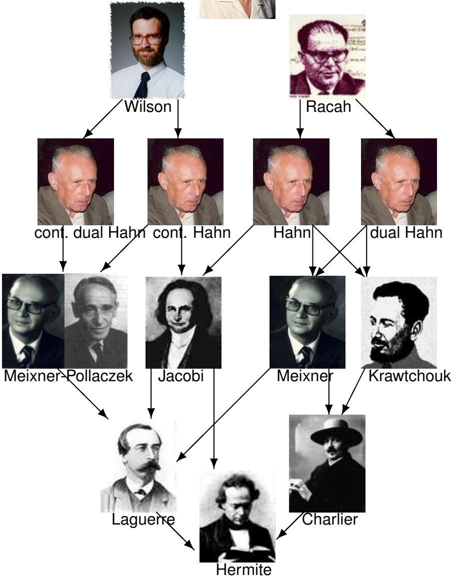
$$ {}_4F_3\left(\begin{matrix}-n,a,b,c\\ d,e,f\end{matrix};1\right)\quad (d+e+f=-n+a+b+c+1). $$

Related orthogonal polynomials: *Wilson polynomials* $W_n(x^2):=$

$$ \text{const.}\ {}_4F_3\left(\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix};1\right), $$

and *Racah polynomials*. These form the top level of the Askey scheme of hypergeometric orthogonal polynomials.

Askey  scheme



Wilson functions

For *Wilson functions* (non-polynomial analogues of Wilson polynomials) one has to go to the ${}_7F_6$ level.

Well-poised hypergeometric series:

$$ {}_rF_{r-1}\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ 1+a_1-a_2, \ldots, 1+a_1-a_r \end{matrix}; z\right). $$

This is *very well-poised* (VWP) if $a_2 = 1 + \frac{1}{2}a_1$.

Terminating VWP ${}_7F_6(1) = \text{const.} \times$ terminating balanced ${}_4F_3(1)$.

Non-terminating VWP ${}_7F_6(1) =$ linear combination of two balanced ${}_4F_3(1)$'s.

Wilson function transform (Groenevelt).

The ${}_9F_8$ top level

Terminating 2-balanced VWP ${}_9F_8(1)$:

Transformation formula (Bailey, Whipple).

Biorthogonal rational functions (J. Wilson).

Non-terminating 2-balanced VWP ${}_9F_8(1)$:

Four-term transformation formula (Bailey).

Let $0 < q < 1$.

q-Pochhammer symbol:

$$\begin{aligned}(a;q)_k &:= (1-a)(1-qa)\ldots(1-q^{k-1}a),\\ (a;q)_\infty &:= (1-a)(1-qa)(1-q^2a)\ldots\ ,\\ (a_1,\ldots,a_r;q)_k &:= (a_1;q)_k\ldots(a_r;q)_k.\end{aligned}$$

q-hypergeometric ${}_r\phi_{r-1}$ series:

$${}_r\phi_{r-1}\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_{r-1}\end{matrix};q,z\right) := \sum_{k=0}^\infty \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_{r-1};q)_k\,(q;q)_k}\,z^k.$$

Terminating if $a_1 = q^{-n}$ (n nonnegative integer).
If nonterminating then converges for $|z| < 1$.

Balanced if $b_1\ldots b_{r-1} = qa_1\ldots a_r$.

Askey-Wilson polynomials and functions

Terminating balanced ${}_4\phi_3$ of argument q:

- Symmetry group $S_6 = W(A_5)$ (Van der Jeugt & S. Rao).
- Askey-Wilson polynomials:

$$p_n\left(\tfrac{1}{2}(z+z^{-1}\right) := \text{const.}\ {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

- Askey-Wilson polynomials together with q-Racah polynomials form the top level of the q-Askey scheme.

Very well-poised (VWP) q-hypergeometric series:

$${}_rV_{r-1}(a_1; a_4, \ldots, a_r; q, z) := {}_r\phi_{r-1}\left(\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \ldots, a_r \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \ldots, qa_1/a_r \end{matrix}; q, z\right)$$

Non-terminating very well-poised ${}_8\phi_7$ of argument $\dfrac{q^2\, a_1^2}{a_4\, a_5\, a_6\, a_7\, a_8}$:

- Sum of two non-terminating balanced ${}_4\phi_3$'s of argument q.
- Symmetry group $W(D_5)$ (Van der Jeugt & S. Rao).
- Askey-Wilson functions (Stokman).

Bailey's two-term ${}_{10}\phi_9$ function

$$\Phi(a;b;c,d,e,f,g,h;q) :=$$
$$(aq/c, aq/d, aq/e, aq/f, aq/g, aq/h;q)_\infty$$
$$\times (bc/a, bd/a, be/a, bf/a, bg/a, bh/a;q)_\infty/(b/a, aq;q)_\infty$$
$$\times {}_{10}V_9(a;b,c,d,e,f,g,h;q,q)$$
$$+ \frac{(bq/c, bq/d, bq/e, bq/f, bq/g, bq/h, c, d, e, f, g, h;q)_\infty}{(a/b, b^2q/a;q)\infty}$$
$$\times {}_{10}V_9(b^2/a;b, bc/a, bd/a, be/a, bf/a, bg/a, bh/a;q,q),$$

where $a^3q^2 = bcdefgh$.

Bailey's four-term transformation formula:

$$\Phi(a;b;c,d,e,f,g,h;q) = \Phi\Big(\frac{a^2q}{cde};b;\frac{aq}{de},\frac{aq}{ce},\frac{aq}{cd},f,g,h;q\Big).$$

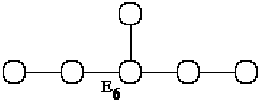
Symmetry group $W(E_6)$ (Lievens & Van der Jeugt).

${}_{10}\phi_9$: the terminating case

Terminating balanced very well-poised ${}_{10}\phi_9$'s of argument q:

- Bailey's two-term transformation formula.
- Same symmetry group $W(E_6)$.
- Biorthogonal rational functions (Rahman, J. Wilson)

Dynkin diagram of E_6:



The elliptic hypergeometric integral

Let $p, q \in \mathbb{C}$ $(|p|, |q| < 1)$.

Elliptic gamma function (Ruijsenaars):

$$\Gamma_e(z; p, q) := \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}.$$

Elliptic hypergeometric integral (Spiridonov):

$$S_e(t; p, q) := \int_{\mathcal{C}} \frac{\prod_{j=1}^{8} \Gamma_e(t_j z^{\pm 1}; p, q)}{\Gamma_e(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z} \quad \left(\prod_{j=1}^{8} t_j = p^2 q^2\right),$$

where $\mathcal{C}$ is a deformation of the unit circle which separates the poles $t_j p^m q^n$ $(m, n = 0, 1, \ldots)$ from the poles $t_j^{-1} p^{-m} q^{-n}$ $(m, n = 0, 1, \ldots)$.

The transformations of $S_e(t; p, q)$ form a symmetry group which is isomorphic to $W(E_7)$ (Rains).

Elliptic hypergeometric differential equation and series

Put $t_6 = az$, $t_7 = a/z$, $f(z) = S_e(t;p,q)$. Then $f(z)$ satisfies the *elliptic hypergeometric differential equation* (Spiridonov):

$$A(z)\big(f(qz) - f(z)\big) + A(z^{-1})\big(f(q^{-1}z) - f(z)\big) + \nu f(z) = 0,$$

where $A(z)$ and ν are suitable products of *theta functions*

$$\theta(b;p) := (b, pb^{-1};p)_\infty.$$

Elliptic Pochhammer symbol:

$$(a;q,p)_k := \theta(a;p)\theta(qa;p)\ldots\theta(q^{k-1}a;p).$$

Elliptic hypergeometric series:

$${}_rE_{r-1}\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_{r-1} \end{matrix};q,p;z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q,p)_k\ldots(a_r;q,p)_k}{(b_1;q,p)_k\ldots(b_{r-1};q,p)_k(q;q,p)_k}\, z^k,$$

where $a_1\ldots a_r = b_1\ldots b_{r-1}q$.

This is the *elliptic balancing condition* in order that $(k+1)$-th term / k-th term is doubly periodic in k.

Two-index biorthogonal rational elliptic hypergeometric functions

Very well-poised elliptic hypergeometric series:

$$
{}_rV_{r-1}(a_1;a_6,\ldots,a_r;q,p) := \\
{}_rE_{r-1}\left(\begin{matrix} a_1, qa_1^{\frac12}, -qa_1^{\frac12}, q(a_1/p)^{\frac12}, -q(a_1p)^{\frac12}, a_6,\ldots,a_r \\ a_1^{\frac12}, -a_1^{\frac12}, (pa_1)^{\frac12}, -(a_1/p)^{\frac12}, qa_1/a_6,\ldots,qa_1/a_r \end{matrix}; q,p;-1\right),
$$

where $a_6\ldots a_r = q^{\frac12 r-4}a_1^{\frac12 r-3}$.

A certain terminating ${}_{12}V_{11}$ satisfies the elliptic hypergeometric equation. It was first introduced by Frenkel & Turaev (elliptic 6j-symbol). They gave a transformation formula, and a ${}_{10}V_9$ summation formula as a degenerate case.

Products $R_n(z;q,p)\,R_m(z;p,q)$ of such rational functions satisfy a *two-index biorthogonality* (Spiridonov).

Hyperbolic hypergeometric series

In elliptic hypergeometric theory there are no transformation formulas below the ${}_{12}V_{11}$ level.

However, there is a limit case of the elliptic hypergeometric function, called *hyperbolic hypergeometric function*, started by Ruijsenaars, which is still above the q-case and with the following features:

- On top level again $W(E_7)$ symmetry.
- There is also a hyperbolic Askey-Wilson function.
- Has analytic continuation to q on unit circle.
- Explicit expressions as products of two q-hypergeometric functions or a sum of two such products.

For details see the Thesis by Fokko van de Bult, *Hyperbolic Hypergeometric Functions*, 2007 (partly based on papers jointly with Rains and Stokman).

Second part

Double affine Hecke algebra in the Askey-Wilson case and relationship with Zhedanov algebra

Askey-Wilson polynomials

Askey-Wilson operator acting on symmetric Laurent polynomials $f[z] = f[z^{-1}]$:

$$(D_{\rm sym} f)[z] := A[z]\big(f[qz] - f[z]\big) + A[z^{-1}]\big(f[q^{-1}z] - f[z]\big) + (1 + q^{-1}abcd)\, f[z],$$

where

$$A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,.$$

Askey-Wilson polynomials (monic symmetric Laurent polynomials $P_n[z] = P_n[z^{-1}] = z^n + \cdots + z^{-n}$):

$$P_n[z] := \mathrm{const.}\ {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right),$$

Eigenvalue equation:

$$D_{\rm sym} P_n = \lambda_n P_n, \quad \text{where} \quad \lambda_n := q^{-n} + q^{n-1}abcd.$$

Double affine Hecke algebra of type $(C_1^\vee, C_1)$

Let $0 < q < 1$, $a, b, c, d \in \mathbb{C}\backslash\{0\}$, $abcd \neq q^{-m}$ $(m = 0, 1, 2, \ldots)$.

Definition

The *double affine Hecke algebra* $\tilde{\mathfrak{H}}$ of type $(C_1^\vee, C_1)$ is the algebra with generators Z, Z^{-1}, T_1, T_0 and with relations

$$\begin{aligned}(T_1 + ab)(T_1 + 1) &= 0,\\ (T_0 + q^{-1}cd)(T_0 + 1) &= 0,\\ (T_1 Z + a)(T_1 Z + b) &= 0,\\ (qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) &= 0.\end{aligned}$$

(Sahi; Noumi & Stokman; Macdonald's 2003 book)

T_1 and T_0 are invertible. Let

$$Y := T_1 T_0, \qquad D := Y + q^{-1} abcd Y^{-1}.$$

Polynomial representation of $\tilde{\mathfrak{H}}$

Let $\mathcal{A}$ be the space of Laurent polynomials $f[z]$.

The *polynomial representation* of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$ is given by

$$(Zf)[z] := z\,f[z],$$

$$(T_1 f)[z] := -ab\,f[z] + \frac{(1-az)(1-bz)}{1-z^2}\,\big(f[z^{-1}] - f[z]\big),$$

$$(T_0 f)[z] := -q^{-1}cd\,f[z] + \frac{(c-z)(d-z)}{q-z^2}\,\big(f[z] - f[qz^{-1}]\big)$$

(q-difference-reflection operators; q-analogues of the Dunkl operator). Then

$$(T_1 f)[z] = -ab\,f[z] \quad \text{iff} \quad f[z] = f[z^{-1}],$$

and

$$(Df)[z] = (D_{\rm sym} f)[z] \quad \text{if} \quad f[z] = f[z^{-1}].$$

Eigenspaces of D

Let

$$Q_n[z] := a^{-1}b^{-1}z^{-1}(1-az)(1-bz)P_{n-1}[z; qa, qb, c, d \mid q].$$

Then

$$\begin{aligned} DQ_n &= \lambda_n Q_n, \qquad & T_1 Q_n &= -Q_n. \\ DP_n &= \lambda_n P_n, \qquad & T_1 P_n &= -abQ_n. \end{aligned}$$

D has eigenvalues λ_n $(n = 0, 1, 2, \ldots)$.

T_1 has eigenvalues $-1, -ab$.

D and T_1 commute.

The eigenspace of D for λ_n has basis P_n, Q_n $(n = 1, 2, \ldots)$ or P_0 $(n = 0)$.

Non-symmetric Askey-Wilson polynomials

Let

$$E_{-n} := \frac{ab}{ab-1}\,(P_n - Q_n) \qquad (n = 1, 2, \ldots),$$

$$E_n := \frac{(1-q^n ab)(1-q^{n-1}abcd)}{(1-ab)(1-q^{2n-1}abcd)}\,P_n - \frac{ab(1-q^n)(1-q^{n-1}cd)}{(1-ab)(1-q^{2n-1}abcd)}\,Q_n \qquad (n = 1, 2, \ldots).$$

Then

$$YE_{-n} = q^{-n}\,E_{-n} \qquad (n = 1, 2, \ldots),$$

$$YE_n = q^{n-1}abcd\,E_n \qquad (n = 0, 1, 2, \ldots).$$

The $E_n[z]$ ($n \in \mathbb{Z}$) are the nonsymmetric Askey-Wilson polynomials. They form a biorthogonal system with respect to a suitable inner product given by a contour integral.

Zhedanov's algebra $AW(3)$

Definition

Zhedanov's algebra $AW(3)$ is the algebra generated by K_0, K_1 with relations

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0,$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1.$$

The *Casimir operator*

$$\begin{aligned} Q := {} & K_1K_0K_1K_0 - (q^2+1+q^{-2})K_0K_1K_0K_1 \\ & + (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\ & + B\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\ & + (q+1+q^{-1})\big(D_0K_0 + D_1K_1\big). \end{aligned}$$

commutes in $AW(3)$ with the generators K_0, K_1.

The polynomial representation of $AW(3)$

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d.

Put for the structure constants:

$$\begin{aligned} B &:= (1-q^{-1})^2(e_3+qe_1),\\ C_0 &:= (q-q^{-1})^2,\\ C_1 &:= q^{-1}(q-q^{-1})^2 e_4,\\ D_0 &:= -q^{-3}(1-q)^2(1+q)(e_4+qe_2+q^2),\\ D_1 &:= -q^{-3}(1-q)^2(1+q)(e_1e_4+qe_3). \end{aligned}$$

Then the *polynomial representation* of $AW(3)$ on the space $\mathcal{A}_{\rm sym}$ of symmetric Laurent polynomials in z is given by

$$\begin{aligned} (K_0f)[z] &:= (D_{\rm sym}f)[z],\\ (K_1f)[z] &:= (z+z^{-1})f[z]. \end{aligned}$$

The quotient algebra $AW(3, Q_0)$

In the polynomial representation (which is irreducible for generic values of a, b, c, d), Q becomes a constant scalar:

$$(Qf)[z] = Q_0\, f[z], \quad \text{where}$$

$$Q_0 := q^{-4}(1-q)^2\Big(q^4(e_4 - e_2) + q^3(e_1^2 - e_1 e_3 - 2e_2)$$

$$- q^2(e_2 e_4 + 2e_4 + e_2) + q(e_3^2 - 2e_2 e_4 - e_1 e_3) + e_4(1 - e_2)\Big).$$

Definition

$AW(3, Q_0)$ is the algebra $AW(3)$ with further relation $Q = Q_0$.

Theorem (K, 2007)

A basis of $AW(3, Q_0)$ is given by

$$K_0^n (K_1 K_0)^l K_1^m \quad (m, n = 0, 1, 2, \ldots, \quad l = 0, 1).$$

The polynomial representation of $AW(3, Q_0)$ on $\mathcal{A}_{\rm sym}$ is faithful.

Central extension of $AW(3)$

Let the algebra $\widetilde{AW}(3, Q_0)$ be generated by K_0, K_1, T_1 such that T_1 commutes with K_0, K_1 and with further relations

$$
\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0, \\
(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 &= B\,K_1 + C_0\,K_0 + D_0 \\
&\quad + E\,K_1(T_1+ab) + F_0(T_1+ab), \\
(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 &= B\,K_0 + C_1\,K_1 + D_1 \\
&\quad + E\,K_0(T_1+ab) + F_1(T_1+ab), \\
\widetilde{Q} :=&(K_1K_0)^2 - (q^2+1+q^{-2})K_0(K_1K_0)K_1 \\
&+ (q+q^{-1})K_0^2K_1^2 + (q+q^{-1})(C_0K_0^2 + C_1K_1^2) \\
&+ \big(B + E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1 + K_1K_0\big) \\
&+ (q+1+q^{-1})\big(D_0 + F_0(T_1+ab)\big)K_0 \\
&+ (q+1+q^{-1})\big(D_1 + F_1(T_1+ab)\big)K_1 + G(T_1+ab) = Q_0,
\end{aligned}
$$

where E, F_0, F_1, G can be explicitly specified.
Then $\widetilde{Q}$ commutes with all elements of $\widetilde{AW}(3)$.

Connecting $\widetilde{AW}(3, Q_0)$ with $\tilde{\mathfrak{H}}$

Theorem (K, 2007)

$\widetilde{AW}(3, Q_0)$ acts on $\mathcal{A}$ such that K_0, K_1, T_1 act as D, $Z + Z^{-1}$, T_1, respectively, in the polynomial representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$.

This representation is faithful.

$\widetilde{AW}(3, Q_0)$ has an injective embedding in $\tilde{\mathfrak{H}}$.

Theorem (K, 2007)

Let $ab \neq 1$.

$AW(3, Q_0)$ is naturally isomorphic to the spherical subalgebra $(T_1 + 1)\tilde{\mathfrak{H}}(T_1 + 1)$.

$\widetilde{AW}(3, Q_0)$ is the centralizer of T_1 in $\tilde{\mathfrak{H}}$.

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See the following two papers and references given there.

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