A nonsymmetric version of Okounkov's *BC*-type interpolation Macdonald polynomials

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(q-)Pochhammer symbols:

Let $q \in \mathbb{C}$, $0 < |q| < 1$, $n \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned} (a)_n &:= a(a+1)\ldots(a+n-1), \\ (a;q)_n &:= (1-a)(1-qa)\ldots(1-q^{n-1}a), \\ (a_1,\ldots,a_k;q)_n &:= (a_1;q)_n\ldots(a_k;q)_n. \end{aligned}$$

$$\begin{aligned} (a;q)_\infty &:= \prod_{j=0}^{\infty}(1-q^j a), \\ (a_1,\ldots,a_k;q)_\infty &:= (a_1;q)_\infty\ldots(a_k;q)_\infty. \end{aligned}$$

Interpolation:

1. $$x(x-1)\dots(x-n+1)=(-1)^n(-x)_n$$

   is the unique monic polynomial of degree n which vanishes at $x=0,1,\dots,n-1$.

2. $$(x-1)(x-q)\dots(x-q^{n-1})=x^n(x^{-1};q)_n$$

   is the unique monic polynomial of degree n which vanishes at $x=1,q,\dots,q^{n-1}$.

3. $$\prod_{j=0}^{n-1}(z+z^{-1}-aq^j-a^{-1}q^{-j})=\frac{(az,az^{-1};q)_n}{(-1)^n\,q^{\frac12 n(n-1)}\,a^n}$$

   is the unique monic symmetric Laurent polynomial of degree n which vanishes on $a,aq,\dots,aq^{n-1}$ (and their inverses) .

Askey–Wilson polynomials:

$$R_n(z;a,b,c,d\,|\,q)=\frac{p_n(\frac12(z+z^{-1});a,b,c,d\,|\,q)}{p_n(\frac12(a+a^{-1});a,b,c,d\,|\,q)}$$

$$:=\ {}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right)$$

$$=\sum_{k=0}^{n}\frac{(q^{-n},q^{n-1}abcd;q)_k\,(az,az^{-1};q)_k}{q^{-k}\,(ab,ac,ad,q;q)_k}.$$

Orthogonality. $0<q<1$; $|a|,|b|,|c|,|d|\le 1$ with products $ab,ac,\ldots,cd\ne 1$, and with non-real a,b,c,d occurring in complex conjugate pairs.

$$\Delta_+(z)=\Delta_+(z;a,b,c,d;q):=\frac{(z^2;q)_\infty}{(az,bz,cz,dz;q)_\infty},$$

$$\Delta(z):=\Delta_+(z)\Delta_+(z^{-1}).$$

$$\int_{|z|=1}R_n(z)\,R_m(z)\,\Delta(z)\,\frac{dz}{z}=0\qquad\text{if } n\ne m.$$

Partitions:
$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
Λ_n^+ is the set of all such partitions.
λ has *weight* $|\lambda| := \lambda_1 + \cdots + \lambda_n$.
$\delta := (n-1, n-2, \ldots, 0)$.

Dominance and inclusion partial ordering. For $\lambda, \mu \in \Lambda_n^+$:

$$\mu \leq \lambda \quad \text{iff} \quad \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \quad (i = 1, \ldots, n);$$
$$\mu \subseteq \lambda \quad \text{iff} \quad \mu_i \leq \lambda_i \quad (i = 1, \ldots, n).$$

Symmetrized monomials. For $\lambda \in \Lambda_n^+$:

$$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu, \qquad \widetilde{m}_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu \quad (W_n := S_n \ltimes (\mathbb{Z}_2)^n).$$

These are symmetric polynomials and symmetric Laurent polynomials, respectively.

Macdonald weight function $(0 < q, t < 1)$:

$$\Delta_+(x) = \Delta_+(x; q, t) := \prod_{1 \le i < j \le n} \frac{(x_i x_j^{-1}; q)_\infty}{(t\, x_i x_j^{-1}; q)_\infty},$$

$$\Delta(x) := \Delta_+(x)\Delta_+(x^{-1}).$$

Macdonald polynomials. Define for $\lambda \in \Lambda_n^+$

$$P_\lambda(x; q, t) = P_\lambda(x) = \sum_{\mu \le \lambda} u_{\lambda,\mu} m_\mu(x), \quad u_{\lambda,\lambda} = 1,$$

such that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu < \lambda.$$

(Note that $P_\mu(x^{-1}) = \overline{P_\mu(x)}$ for $x \in \mathbb{T}^n \subset \mathbb{C}^n$.)

Then $P_\lambda(x)$ is homogeneous of degree $|\lambda|$ in x and there is full orthogonality:

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu \ne \lambda.$$

Koornwinder weight function. $0 < q, t < 1$, $|a|, |b|, |c|, |d| \le 1$ with products $ab, ac, \ldots, cd \ne 1$, and with non-real a, b, c, d occurring in complex conjugate pairs.

$$\Delta_+(x) = \Delta_+(x; q, t; a, b, c, d)$$
$$:= \prod_{j=1}^{n} \frac{(x_j^2; q)_\infty}{(ax_j, bx_j, cx_j, dx_j; q)_\infty} \prod_{1 \le i < j \le n} \frac{(x_i x_j, x_i x_j^{-1}; q)_\infty}{(t x_i x_j, t x_i x_j^{-1}; q)_\infty},$$

$\Delta(x) := \Delta_+(x)\Delta_+(x^{-1})$. For $n = 1$ no t: Askey–Wilson case.

Koornwinder polynomials (1992). Define for $\lambda \in \Lambda_n^+$

$$P_\lambda(x; q, t; a, b, c, d) = P_\lambda(x) = \sum_{\mu \le \lambda} u_{\lambda,\mu} \widetilde{m}_\mu(x), \quad u_{\lambda,\lambda} = 1,$$

such that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x)\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu < \lambda.$$

Then full orthogonality (for $\mu \ne \lambda$).

5-parameter generalization of 3-parameter Macdonald BC_n polynomials.

A-type interpolation polynomials (Kostant & Sahi, Sahi, Knop, Knop & Sahi, 1991–1997)

Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1$.
Let $q, t \in \mathbb{C}$, $0 < |q|, |t| < 1$.
For $\lambda \in \Lambda_n^+$ define $P_\lambda^{\mathrm{ip,A}}(x; q, \tau)$ as the unique symmetric polynomial of degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$ such that its term involving x^λ has coefficient 1 and

$$P_\lambda^{\mathrm{ip,A}}(q^\mu \tau; q, \tau) = 0 \quad \text{if } \mu \in \Lambda_n^+,\ |\mu| \le |\lambda|,\ \mu \ne \lambda.$$

In the *principal specialization* $\tau := t^\delta = (t^{n-1}, t^{n-2}, \ldots, 1)$ moreover the following two properties hold:

Extra-vanishing property:
$P_\lambda^{\mathrm{ip,A}}(q^\mu t^\delta; q, t^\delta) = 0$ if $\mu \in \Lambda_n^+$ and $\lambda \not\subseteq \mu$.

Expansion in terms of Macdonald polynomials:

$$P_\lambda^{\mathrm{ip,A}}(x; q, t^\delta) = \sum_{\mu \subseteq \lambda} b_{\lambda,\mu} P_\mu(x; q, t), \quad b_{\lambda,\lambda} = 1.$$

Hence $\quad r^{-|\lambda|} P_\lambda^{\mathrm{ip,A}}(rx; q, t^\delta) \to P_\lambda(x; q, t) \quad$ as $r \to \infty$.

BC-type interpolation polynomials (Okounkov, 1998)

τ, q, t as before, and $s \in \mathbb{C}$, $0 < |s| < 1$.
For $\lambda \in \Lambda_n^+$ define $P_\lambda^{\mathrm{ip,B}}(x; q, \tau)$ as the unique W_n-invariant Laurent polynomial of degree $|\lambda|$ such that its term involving x^λ has coefficient 1 and

$$P_\lambda^{\mathrm{ip,B}}(q^\mu \tau; q, \tau) = 0 \quad \text{if } \mu \in \Lambda_n^+,\ |\mu| \le |\lambda|,\ \mu \ne \lambda.$$

In the *principal specialization* $\tau := st^\delta$ we have moreover:

Extra-vanishing property:
$P_\lambda^{\mathrm{ip,B}}(q^\mu st^\delta; q, st^\delta) = 0$ if $\mu \in \Lambda_n^+$ and $\lambda \not\subseteq \mu$.

Sahi

Knop

Okounkov

Nonsymmetric Macdonald polynomials
$E_\lambda(x;q,t) \quad (\lambda \in \Lambda_n := (\mathbb{Z}_{\geq 0})^n)$ and
nonsymmetric Koornwinder polynomials
$E_\lambda(x;a,b,c,d,t;q) \quad (\lambda \in \mathbb{Z}^n)$ can be defined as eigenfunctions (polynomials or Laurent polynomials, respectively) of suitable q-difference-reflection operators (generalized Dunkl operators) coming from the polynomial representation of a suitable double affine Hecke algebra.

Expansion of symmetric polynomials in terms of non-symmetric polynomials $(\lambda \in \Lambda_n^+)$:

$$P_\lambda = \sum_{\mu \in S_n \lambda} a_{\lambda,\mu} E_\mu \quad \text{(Macdonald polynomials)},$$

$$P_\lambda = \sum_{\mu \in W_n \lambda} b_{\lambda,\mu} E_\mu \quad \text{(Koornwinder polynomials)}$$

for suitable coefficients $a_{\lambda,\mu}$ and $b_{\lambda,\mu}$.

Nonsymmetric interpolation, case A_{n-1} (Sahi, Knop)

$q \in \mathbb{C}$ with $0 < |q| < 1$.
$\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1$.
Choose simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n$.
Let $\alpha \in \Lambda_n := \mathbb{Z}^n_{\geq 0}$, $|\alpha| := \sum_{i=1}^d \alpha_i$.
Let w_α be shortest element in S_n such that $w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$.
Then, for $i < j$,

$$w_\alpha^{-1}(i) < w_\alpha^{-1}(j) \iff \alpha_i \geq \alpha_j.$$

Put $\overline{\alpha} := q^\alpha w_\alpha \tau$. Then $\overline{\alpha}_i = q^{\alpha_i} \tau_{w_\alpha^{-1} i}$.
Let $\Lambda_{n,d} := \{\alpha \in \Lambda_n \mid |\alpha| \leq d\}$.

Theorem

For any given $\{\overline{f}_\alpha\}_{\alpha \in \Lambda_{n,d}}$ *there is a unique polynomial* $f \in \mathrm{Span}\{x^\alpha\}_{\alpha \in \Lambda_{n,d}}$ *such that* $f(\overline{\alpha}) = \overline{f}_\alpha \ (\alpha \in \Lambda_{n,d})$.

Note that existence implies uniqueness.

Nonsymmetric interpolation, case BC_n (Disveld, Stokman, K)

Choose simple roots $e_1 - e_2, \ldots, e_{n-1} - e_n, e_n$.
Let $\alpha \in \mathbb{Z}^n$, $|\alpha| := \sum_{i=1}^d |\alpha_i|$.
Let w_α be shortest element in W_n such that $w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$.
Write $w_\alpha = \sigma_\alpha \pi_\alpha$ ($\sigma_\alpha \in \{\pm 1\}^n$, $\pi_\alpha \in S_n$). Then
$\sigma_\alpha = (\mathrm{sgn}(\alpha_1), \ldots, \mathrm{sgn}(\alpha_n))$, where $\mathrm{sgn}(0) = 1$,
and π_α is such that, for $i < j$,

$$\pi_\alpha^{-1}(i) < \pi_\alpha^{-1}(j) \iff |\alpha_i| > |\alpha_j| \text{ or } 0 \le \alpha_i = \pm\alpha_j.$$

Following Sahi this means the following rule for getting $\pi_\alpha^{-1}(i)$:

Reorder the α_i by decreasing $|\alpha_i|$, then, for fixed $|\alpha_i|$, first put the ones with $\alpha_i \ge 0$ from left to right, and next put the ones with $\alpha_i < 0$ from right to left.
Then α_i has moved from position i to position $\pi_\alpha^{-1}(i)$.

Example of π_α

i:	1	2	3	4	5	6	7
α_i:	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:							

Example of π_α

i:	1	2	3	4	5	6	7
α_i:	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:		1					

Example of π_α

i:	1	2	3	4	5	6	7
α_i:	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:	2	1					

Example of π_α

i :	1	2	3	4	5	6	7
α_i :	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:	2	1	3				

Example of π_α

i:	1	2	3	4	5	6	7
α_i:	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:	2	1	3			4	

Example of π_α

i:	1	2	3	4	5	6	7
α_i:	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:	2	1	3			4	5

Example of π_α

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$\pi_\alpha^{-1}(i)$:	2	1	3	6		4	5

Example of π_α

i:	1	2	3	4	5	6	7
α_i:	-2	2	1	-1	0	1	-1
$\pi_\alpha^{-1}(i)$:	2	1	3	6	7	4	5

Two important cases of π_α

$$\begin{array}{rccccccc}
i: & 1 & \ldots & k-1 & k & k+1 & \ldots & n \\
\alpha_i: & * & \ldots & * & 0 & \neq 0 & \ldots & \neq 0 \\
\pi_\alpha^{-1}(i): & & \ldots & & n & & \ldots &
\end{array}$$

$$\begin{array}{rccccccc}
i: & 1 & \ldots & n-k & n-k+1 & n-k+2 & \ldots & n \\
\alpha_i: & \neq 0,-1 & \ldots & \neq 0,-1 & -1 & \neq 0 & \ldots & \neq 0 \\
\pi_\alpha^{-1}(i): & & \ldots & & n & & \ldots &
\end{array}$$

Main theorem for nonsymmetric BC_n interpolation

$q \in \mathbb{C}$ with $0 < |q| < 1$.
$\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1$.
$w_\alpha = \sigma_\alpha \pi_\alpha$.

Put $\overline{\alpha} := q^\alpha w_\alpha \tau$. Then

$$\overline{\alpha}_i = q^{\alpha_i} (\tau_{\pi_\alpha^{-1}(i)})^{\mathrm{sgn}(\alpha_i)}, \quad \text{where } \mathrm{sgn}(0) = 1.$$

Let $Z_{n,d} := \{\alpha \in \mathbb{Z}^n \mid |\alpha| \le d\}$.

Theorem

For any given $\{\overline{f}_\alpha\}_{\alpha \in Z_{n,d}}$ there is a unique Laurent polynomial $f \in \mathrm{Span}\{x^\alpha\}_{\alpha \in Z_{n,d}}$ such that $f(\overline{\alpha}) = \overline{f}_\alpha$ $(\alpha \in Z_{n,d})$.

Note that existence implies uniqueness.

Idea of the BC_n existence proof

$$T_{n,d,k} := \{\alpha \in Z_{n,d} \mid \alpha_{k+1}, \ldots, \alpha_n \neq 0\},$$
$$R_{n,d,k} := \{\alpha \in T_{n,d,0} \mid \alpha_1, \ldots, \alpha_{n-k} \neq -1\}.$$

Note that $T_{n,d,n} = Z_{n,d}$, $R_{n,d,n} = T_{n,d,0}$.

More generally prove existence of Laurent interpolation polynomials on $T_{n,d,k}$ and $R_{n,d,k}$:

Proposition (I($T_{n,d,k}$))

For any given $\{\bar f_\alpha\}_{\alpha \in T_{n,d,k}}$ *there is a Laurent polynomial* $f \in \mathrm{Span}\{x^\alpha \mid \alpha + e_J \in Z_{n,d} \ \ \forall J \subseteq [k+1, n]\}$ *such that* $f(\overline{\alpha}) = \bar f_\alpha \ \ (\alpha \in T_{n,d,k})$.

Proposition (I($R_{n,d,k}$))

For any given $\{\bar f_\alpha\}_{\alpha \in R_{n,d,k}}$ *there is a Laurent polynomial* $f \in \mathrm{Span}\{x^\alpha \mid \alpha + e_J \in Z_{n,d-n+k} \ \ \forall J \subseteq [n-k+1, n]\}$ *such that* $f(\overline{\alpha}) = \bar f_\alpha \ \ (\alpha \in R_{n,d,k})$.

Recursion scheme for $\mathrm{I}(T_{n,d,k})$ and $\mathrm{I}(R_{n,d,k})$

$\mathrm{I}(T_{n-1,d,k-1})$ & $\mathrm{I}(T_{n,d,k-1}) \Rightarrow \mathrm{I}(T_{n,d,k})$

$\mathrm{I}(T_{n,d,0}) \Leftrightarrow \mathrm{I}(R_{n,d,n})$

$\mathrm{I}(R_{n-1,d-1,k-1})$ & $\mathrm{I}(R_{n,d,k-1}) \Rightarrow \mathrm{I}(R_{n,d,k})$

$\mathrm{I}(T_{n,d-n,n})$ for $q\tau \quad \Leftrightarrow \quad \mathrm{I}(R_{n,d,0})$ for τ

Suppose that $\mathrm{I}(T_{m,c,\ell})$ and $\mathrm{I}(R_{m,c,\ell})$ are proved for all ℓ, for all $m \le n$, $c \le d$ with $(m,c) \ne (n,d)$ and for all τ.
Then successively prove
$\mathrm{I}(R_{n,d,0}), \ldots, \mathrm{I}(R_{n,d,n}), \mathrm{I}(T_{n,d,0}), \ldots, \mathrm{I}(T_{n,d,n})$.

Note that $T_{n,d,k} = \emptyset$ if $d + k < n$, and $R_{n,d,k} = \emptyset$ if $d < n$.
Statements $\mathrm{I}(\emptyset)$ are trivially true. Thus, in the above recursion,
$\mathrm{I}(T_{d,d,0}) \Rightarrow \mathrm{I}(T_{d+1,d,1}) \Rightarrow \cdots \Rightarrow \mathrm{I}(T_{n-1,d,n-d-1}) \Rightarrow \mathrm{I}(T_{n,d,n-d})$,
where $d < n$.

Recursion ends with $d = 0$ (then $f(x) = \overline{f}_0$) or $n = 1$ (then more simple recursion on next slide).

The case $n=1$: $\overline{\alpha}=q^{\alpha}\tau^{\mathrm{sgn}(\alpha)}$ $(\alpha\in\mathbb{Z},\ 0<|q|,|\tau|<1)$.

For given $\{\overline{f}_\alpha\}_{\alpha\in[-d,d]}$ find Laurent polynomial $f(x)$ of degree $\le d$ such that $f(\overline{\alpha})=\overline{f}_\alpha$ $(\alpha\in[-d,d])$. Proved by recursion.

$\mathrm{I}(T_{1,d,0})\Rightarrow\mathrm{I}(T_{1,d,1})$:
$f(x)=\overline{f}(0)+(x-\tau)g(x)$. Then $f(\tau)=\overline{f}(0)$ and
$f(\overline{\alpha})=\overline{f}(0)+(\overline{\alpha}-\tau)g(\overline{\alpha})$ $(\alpha\in[-d,d],\ \alpha\ne0)$.
So solve $\mathrm{I}(T_{1,d,0})$, i.e., $\mathrm{I}(R_{1,d,1})$, for $g(x)$.

$\mathrm{I}(R_{1,d,0})\Rightarrow\mathrm{I}(R_{1,d,1})$:
$g(x)=\overline{g}(-1)+(x^{-1}-q\tau)h(x)$. Then $g(q^{-1}\tau^{-1})=\overline{g}(-1)$ and
$g(\overline{\alpha})=\overline{g}(-1)+(\overline{\alpha}^{-1}-q\tau)h(\overline{\alpha})$ $(\alpha\in[-d,d],\ \alpha\ne0,-1)$.
So solve $\mathrm{I}(R_{1,d,0})$ for $h(x)$.

Equivalently, solve $\mathrm{I}(T_{1,d-1,1})$ with τ replaced by $q\tau$ for $h(x)$.
Indeed, $\overline{\alpha}=q^{\alpha-\mathrm{sgn}(\alpha)}(q\tau)^{\mathrm{sgn}(\alpha)}$.

Proof details for $\mathrm{I}(T_{n-1,d,k-1})$ & $\mathrm{I}(T_{n,d,k-1}) \Rightarrow \mathrm{I}(T_{n,d,k})$

$T_{n,d,k} = \{\alpha \in Z_{n,d} \mid \alpha_{k+1}, \ldots, \alpha_n \neq 0\}$

Proposition ($\mathrm{I}(T_{n,d,k})$)

For any given $\{\bar{f}_\alpha\}_{\alpha \in T_{n,d,k}}$ there is a Laurent polynomial $f(x) = f(x, \tau, \{\bar{f}_\alpha\}_{\alpha \in T_{n,d,k}}) \in \mathrm{Span}\{x^\alpha \mid \alpha + e_J \in Z_{n,d} \ \ \forall J \subseteq [k+1, n]\}$ such that $f(\overline{\alpha}) = \bar{f}_\alpha$ $(\alpha \in T_{n,d,k})$.

$$f\big(x, \tau, \{\bar{f}_\alpha\}_{\alpha \in T_{n,d,k}}\big) = g\big(x^{(k)}, \tau', \{\bar{f}_\alpha\}_{\alpha_k = 0,\ \alpha^{(k)} \in T_{n-1,d,k-1}}\big) + (x_k - \tau_n)\, h\big(x, \tau, \{\bar{h}_\alpha\}_{\alpha \in T_{n,d,k-1}}\big).$$

Here $x^{(k)}$ is x with k-th coordinate omitted, and τ' is τ with last coordinate omitted.

$g(x^{(k)})$ from $\mathrm{I}(T_{n-1,d,k-1})$; $h(x)$ from $\mathrm{I}(T_{n,d,k-1})$.

If $\alpha \in T_{n,d,k}$ and $\alpha_k = 0$ then: $\overline{\alpha}_k = \tau_n$ and $\overline{\alpha}(\tau)^{(k)} = \overline{\alpha^{(k)}}(\tau')$.

If $\alpha \in T_{n,d,k}$ and $\alpha_k \neq 0$ then get $\bar{h}_\alpha$ from:

$$\bar{f}_\alpha = g(\overline{\alpha}^{(k)}) + (\overline{\alpha}_k - \tau_n)\, \bar{h}_\alpha.$$

Proof details for $\mathrm{I}(R_{n-1,d-1,k-1})$ & $\mathrm{I}(R_{n,d,k-1}) \Rightarrow \mathrm{I}(R_{n,d,k})$

$$R_{n,d,k} = \{\alpha \in Z_{n,d} \mid \alpha_1, \ldots, \alpha_n \neq 0,\ \alpha_1, \ldots, \alpha_{n-k} \neq -1\}$$

Proposition ($\mathrm{I}(R_{n,d,k})$)

For any given $\{\overline{f}_\alpha\}_{\alpha \in R_{n,d,k}}$ there is a Laurent polynomial $f(x) = f\big(x, \tau, \{\overline{f}_\alpha\}_{\alpha \in R_{n,d,k}}\big) \in \mathrm{Span}\{x^\alpha\}_{\alpha + e_J \in Z_{n,d-n+k},\, J \subseteq [n-k+1,n]}$ such that $f(\overline{\alpha}) = \overline{f}_\alpha$ $(\alpha \in R_{n,d,k})$.

$$f\big(x, \tau, \{\overline{f}_\alpha\}_{\alpha \in R_{n,d,k}}\big) = g\big(x^{(n-k+1)}, \tau', \{\overline{f}_\alpha\}_{\alpha_{n-k+1}=-1,\ \alpha^{(n-k+1)} \in R_{n-1,d-1,k-1}}\big) + (x_{n-k+1}^{-1} - q\tau_n)\, h\big(x, \tau, \{\overline{h}_\alpha\}_{\alpha \in R_{n,d,k-1}}\big).$$

$g(x^{(n-k+1)})$ from $\mathrm{I}(R_{n-1,d-1,k-1})$; $h(x)$ from $\mathrm{I}(R_{n,d,k-1})$.

If $\alpha \in R_{n,d,k}$ and $\alpha_{n-k+1} = -1$ then:

$$\overline{\alpha}_{n-k+1} = q^{-1}\tau_n^{-1} \text{ and } \overline{\alpha}(\tau)^{(n-k+1)} = \overline{\alpha^{(n-k+1)}}(\tau').$$

If $\alpha \in R_{n,d,k}$ and $\alpha_{n-k+1} \neq -1$ then get $\overline{h}_\alpha$ from:

$$\overline{f}_\alpha = g(\overline{\alpha}^{(n-k+1)}) + (\overline{\alpha}_{n-k+1}^{-1} - q\tau_n)\, \overline{h}_\alpha.$$

Proof details for $\mathrm{I}(T_{n,d-n,n})(q\tau) \Leftrightarrow \mathrm{I}(R_{n,d,0})(\tau)$

$$T_{n,d-n,n} = Z_{n,d-n}, \quad R_{n,d,0} = \{\alpha \in Z_{n,d} \mid \alpha_1, \ldots, \alpha_n \neq 0. - 1\}.$$

Proposition ($\mathrm{I}(T_{n,d-n,n})(q\tau)$)

For any given $\{\bar{f}_\alpha\}_{\alpha \in Z_{n,d-n}}$ there is a Laurent polynomial $f(x) = f\big(x, q\tau, \{\bar{f}_\alpha\}_{\alpha \in Z_{n,d-n}}\big) \in \mathrm{Span}\{x^\alpha\}_{\alpha \in Z_{n,d-n}}$ such that $f(\overline{\alpha}(q\tau)) = \bar{f}_\alpha \ \ (\alpha \in Z_{n,d-n})$.

Proposition ($\mathrm{I}(R_{n,d,0})(\tau)$)

For any given $\{\bar{g}_\alpha\}_{\alpha \in R_{n,d,0}}$ there is a Laurent polynomial $g(x) = g\big(x, \tau, \{\bar{f}_\alpha\}_{\alpha \in R_{n,d,0}}\big) \in \mathrm{Span}\{x^\alpha\}_{\alpha \in Z_{n,d-n}}$ such that $g(\overline{\alpha}(\tau)) = \bar{g}_\alpha \ \ (\alpha \in R_{n,d,0})$.

$$f\big(x, q\tau, \{\bar{f}_\alpha\}_{\alpha \in Z_{n,d-n}}\big) = g\big(x, \tau, \{\bar{f}_{\beta - \mathrm{sgn}(\beta)}\}_{\beta \in R_{n,d,0}}\big).$$

$$\overline{\alpha}(\tau)_i = q^{\alpha_i}(\tau_{\pi_\alpha^{-1}(i)})^{\mathrm{sgn}(\alpha_i)}, \text{ hence } \quad \overline{\beta - \mathrm{sgn}(\beta)}(q\tau)_i = \overline{\beta}(\tau)_i.$$

Symmetric BC-type interpolation polynomials again
(now in different normalization):
For $\lambda \in \Lambda_n^+$ there is a unique W_n-invariant Laurent polynomial $R_\lambda(x;q,\tau)$ of degree $|\lambda|$ such that

$$R_\lambda(\overline{\mu};q,\tau) = \delta_{\lambda,\mu} \quad \text{if } \mu \in \Lambda_n^+,\ |\mu| \le |\lambda|, \overline{\mu} = q^\mu \tau.$$

Nonsymmetric BC-type interpolation polynomials
(now with interpolation values $\delta_{\alpha,\beta}$):
For $\alpha \in \mathbb{Z}^n$ there is a unique Laurent polynomial $G_\alpha(x;q,\tau)$ of degree $|\alpha|$ such that

$$G_\alpha(\overline{\beta};q,\tau) = \delta_{\alpha,\beta} \quad \text{if } \beta \in \mathbb{Z}^n,\ |\beta| \le |\alpha|, \overline{\beta} = q^\beta w_\beta \tau.$$

Expansion of R_λ in terms of the G_α:

$$R_\lambda(x) = \sum_{\alpha \in W_n \lambda} G_\alpha(x) \quad (\lambda \in \Lambda_n^+).$$

The case $n = 1$:

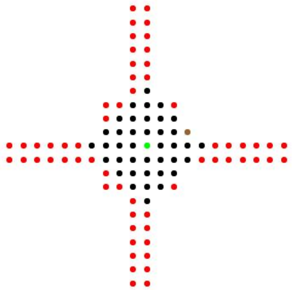
$$G_m(x;q,s) = \frac{\left(qsx, sx^{-1};q\right)_m}{\left(q^{1+m}s^2, q^{-m};q\right)_m}, \qquad m \in \mathbb{Z}_{\geq 0},$$

$$G_{-m}(x;q,s) = \frac{q^m sx \left(qsx;q\right)_{m-1} \left(sx^{-1};q\right)_{m+1}}{\left(q^m s^2;q\right)_{m+1} \left(q^{1-m};q\right)_{m-1}}, \quad m \in \mathbb{Z}_{>0}.$$

Then, with $R_m(x;q,s) = \frac{(sx,sx^{-1};q)_m}{(q^m s^2, q^{-m};q)_m} \quad (m \in \mathbb{Z}_{\geq 0})$,

$$R_0(x) = G_0(x), \qquad R_m(x) = G_m(x) + G_{-m}(x) \quad (m \in \mathbb{Z}_{>0}).$$

Extra-vanishing (present in nonsymmetric A_{n-1} interpolation):
By computer algebra experiments there is indication of extra-vanishing for $G_\alpha(x;q,\tau)$ in the principal specialization $\tau := st^\delta \quad (|s|,|t|<1)$, i.e.,
$G_\alpha(q^\beta st^\delta; q, st^\delta) = 0$ not only if $\beta \in \mathbb{Z}^n$, $|\beta| \le |\alpha|$, $\beta \ne \alpha$, but also for certain other $\beta \in \mathbb{Z}^n$, depending on α, but not on q, s, t.



$\alpha = (3,1)$

$\alpha = (2,2)$

green dot $= (0,0)$, brown dot $= \alpha$, black dots = other β with $|\beta| \le |\alpha|$,
red dots = points γ with $\overline{\gamma}$ extravanishing

Further motivation for our choice of interpolation points $\overline{\alpha}$

In principal specialization $\tau_i = st^{n-i}$:

$$\overline{\alpha}_i = q^{\alpha_i} \big(st^{n-\pi_\alpha^{-1}(i)}\big)^{\mathrm{sgn}(\alpha_i)}.$$

Put $s = \sqrt{q^{-1}abcd}$. Then (Sahi, 1999):
Nonsymmetric Koornwinder polynomials $E_\alpha(x; a, b, c, d, t; q)$ are eigenfunctions of operators Y_i for eigenvalue $\overline{\alpha}_i$.

Compare with A_{n-1} case (Knop, 1997).
In principal specialization $\tau_i = t^{n-i}$ we have $\overline{\alpha}_i = q^{\alpha_i} t^{n-\pi_\alpha^{-1}(i)}$.
Nonsymmetric Macdonald polynomials $E_\alpha(x; q, t)$ are eigenfunctions of operators ξ_i for eigenvalue $\overline{\alpha}_i$.
Moreover, the nonsymmetric Macdonald interpolation polynomials $G_\alpha(x; q, t)$ ($G_\alpha(\overline{\beta}; q, t) = \delta_{\alpha,\beta}$, $|\beta| \le |\alpha|$) are eigenfunctions of operators Ξ_i for eigenvalue $\overline{\alpha}_i^{-1}$.
Analogues of operators Ξ_i are missing in the BC_n case. There symmetric interpolation polynomilas satisfy an eigenvalue equation with a $q^{\frac{1}{2}}$-shift in the s-parameter (Rains, 2005).