A nonsymmetric version of Okounkov's *BC*-type interpolation Macdonald polynomials

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(q-)Pochhammer symbols:

Let $q \in \mathbb{C}$, $0 < |q| < 1$.

$$\begin{aligned} (a)_n &:= a(a+1)\ldots(a+n-1), \\ (a;q)_n &:= (1-a)(1-qa)\ldots(1-q^{n-1}a), \\ (a_1,\ldots,a_k;q)_n &:= (a_1;q)_n\ldots(a_k;q)_n. \end{aligned}$$

Interpolation:

1. $$x(x-1)\dots(x-n+1) = (-1)^n(-x)_n$$

   is the unique monic polynomial of degree n which vanishes at $x = 0, 1, \dots, n-1$.

2. $$x(x-q)\dots(x-q^{n-1}) = x^n(x^{-1};q)_n$$

   is the unique monic polynomial of degree n which vanishes at $x = 1, q, \dots, q^{n-1}$.

3. $$\prod_{j=0}^{n-1}(z+z^{-1}-aq^j-a^{-1}q^{-j}) = \frac{(az, az^{-1};q)_n}{(-1)^n\, q^{\frac{1}{2}n(n-1)}\, a^n}$$

   is the unique monic symmetric Laurent polynomial of degree n which vanishes on $a, aq, \dots, aq^{n-1}$ (and their inverses) .

Askey-Wilson polynomials:

$$R_n(z;a,b,c,d;q) = \frac{p_n(\frac{1}{2}(z+z^{-1});a,b,c,d\,|\,q)}{p_n(\frac{1}{2}(a+a^{-1});a,b,c,d\,|\,q)}$$

$$:= {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right)$$

$$= \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}abcd;q)_k\,(az, az^{-1};q)_k}{q^{-k}\,(ab, ac, ad, q;q)_k}.$$

Orthogonality. $0 < q < 1)$. $|a|, |b|, |c|, |d| \le 1$ with pairwise products of $a, b, c, d \neq 1$, and non-real a, b, c, d in complex conjugate pairs.

$$\Delta_+(z) = \Delta_+(z;a,b,c,d,;q) := \frac{(z^2;q)_\infty}{(az,bz,cz,dz;q)_\infty},$$

$$\Delta(z) := \Delta_+(z)\Delta_+(z^{-1}).$$

$$\int_{|z|=1} R_n(z)\,R_m(z)\,\Delta(z)\,\frac{dz}{z} = 0 \qquad \text{if } n \neq m.$$

Dual parameters $\widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}$:
$\widetilde{a} := (q^{-1}abcd)^{\frac{1}{2}}, \quad \widetilde{a}\,\widetilde{b} = ab, \quad \widetilde{a}\,\widetilde{c} = ac, \quad \widetilde{a}\,\widetilde{d} = ad.$
Then

$$(q^{-n}, q^{n-1}abcd; q)_k = \big(\widetilde{a}\,(\widetilde{a}q^n), \widetilde{a}\,(\widetilde{a}q^n)^{-1}; q\big)_k.$$

Hence

$$R_n(z; a, b, c, d; q) = \sum_{k=0}^{n} \frac{\big(\widetilde{a}\,(\widetilde{a}q^n), \widetilde{a}\,(\widetilde{a}q^n)^{-1}; q\big)_k\,(az, az^{-1}; q)_k}{q^{-k}\,(ab, ac, ad, q; q)_k}\,,$$

from which the **duality relation**

$$R_n(a^{-1}q^{-m}; a, b, c, d; q) = R_m\big(\widetilde{a}^{-1}q^{-n}; \widetilde{a}, \widetilde{b}, \widetilde{c}, \widetilde{d}; q\big) \quad (m, n \in \mathbb{Z}_{\geq 0}),$$

since both sides are equal to

$$\sum_{k=0}^{\min(m,n)} \frac{\big(q^{-n}, \widetilde{a}^2q^n; q\big)_k\,(q^{-m}, a^2q^m; q)_k}{q^{-k}\,(ab, ac, ad, q; q)_k}\,.$$

Partitions:
$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.
Λ_n^+ is the set of all such partitions.
λ has *weight* $|\lambda| := \lambda_1 + \cdots + \lambda_n$.
$\delta := (n-1, n-2, \ldots, 0)$.

Dominance and inclusion partial ordering. For $\lambda, \mu \in \Lambda_n^+$:

$$\mu \leq \lambda \quad \text{iff} \quad \mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i \quad (i = 1, \ldots, n);$$
$$\mu \subseteq \lambda \quad \text{iff} \quad \mu_i \leq \lambda_i \quad (i = 1, \ldots, n).$$

Symmetrized monomials. For $\lambda \in \Lambda_n^+$:

$$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu, \qquad \widetilde{m}_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu \quad (W_n := S_n \ltimes (\mathbb{Z}_2)^n).$$

These are symmetric polynomials and symmetric Laurent polynomials, respectively.

Macdonald weight function $(0 < q, t < 1)$:

$$\Delta_+(x) = \Delta_+(x; q, t) := \prod_{1 \le i < j \le n} \frac{(x_i x_j^{-1}; q)_\infty}{(t\, x_i x_j^{-1}; q)_\infty},$$

$$\Delta(x) := \Delta_+(x) \Delta_+(x^{-1}).$$

Macdonald polynomials. Define for $\lambda \in \Lambda_n^+$

$$P_\lambda(x; q, t) = P_\lambda(x) = \sum_{\mu \le \lambda} u_{\lambda,\mu} m_\mu(x), \quad u_{\lambda,\lambda} = 1,$$

such that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu < \lambda.$$

Then $P_\lambda(x)$ is homogeneous of degree $|\lambda|$ in x and there is full orthogonality:

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu \ne \lambda.$$

Koornwinder weight function. $0 < q, t < 1$,
$|a|, |b|, |c|, |d| \leq 1$ with pairwise products of $a, b, c, d \neq 1$ and non-real a, b, c, d in complex conjugate pairs.

$$\Delta_+(x) = \Delta_+(x; a, b, c, d, t; q)$$
$$:= \prod_{j=1}^{n} \frac{(x_j^2; q)_\infty}{(ax_j, bx_j, cx_j, dx_j; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(x_i x_j, x_i x_j^{-1}; q)_\infty}{(tx_i x_j, tx_i x_j^{-1}; q)_\infty},$$

$\Delta(x) := \Delta_+(x)\Delta_+(x^{-1})$. For $n = 1$ no t: Askey-Wilson case.

Koornwinder polynomials (1992). Define for $\lambda \in \Lambda_n^+$

$$P_\lambda(x; a, b, c, d, t; q) = P_\lambda(x) = \sum_{\mu \leq \lambda} u_{\lambda,\mu} \widetilde{m}_\mu(x), \quad u_{\lambda,\lambda} = 1,$$

such that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x)\, \Delta(x)\, \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} = 0 \qquad \text{if } \mu < \lambda.$$

Then full orthogonality (for $\mu \neq \lambda$).

5-parameter generalization of 3-parameter Macdonald BC_n polynomials.

Interpolation Macdonald polynomials (Sahi, Knop, 1996-97).

Let $t \in \mathbb{C}$, $0 < |t| < 1$. Put $t^\delta := (t^{n-1}, t^{n-2}, \ldots, 1)$
For $\lambda \in \Lambda_n^+$ define $P_\lambda^{\mathrm{ip}}(x; q, t)$ as the unique symmetric polynomial of degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$ such that its term involving x^λ has coefficient 1 and

$$P_\lambda^{\mathrm{ip}}(q^\mu t^\delta; q, t) = 0 \quad \text{if } |\mu| \le |\lambda|,\ \mu \ne \lambda.$$

Then $P_\lambda^{\mathrm{ip}}(q^\lambda t^\delta; q, t \,|\, q) \ne 0$ (and may be put equal to 1 in a different standardization).

Extra-vanishing property:
$P_\lambda^{\mathrm{ip}}(q^\mu t^\delta a; a, t; q) = 0$ if μ is a partition not containing λ.

Expansion in terms of Macdonald polynomials:

$$P_\lambda^{\mathrm{ip}}(x; q, t) = \sum_{\mu \subseteq \lambda} b_{\lambda,\mu}\, P_\mu(x; q, t), \quad b_{\lambda,\lambda} = 1.$$

BC_n-type interpolation Macdonald polynomials

(Okounkov, 1998)

Let $a, t \in \mathbb{C}$, $0 < |a|, |t| < 1$. For $\lambda \in \Lambda_n^+$ define $P_\lambda^{\rm ip}(x; a, t; q)$ as the unique W_n-invariant Laurent polynomial of degree $|\lambda|$ such that its term involving x^λ has coefficient 1 and

$$P_\lambda^{\rm ip}(q^\mu t^\delta a; a, t; q) = 0 \quad \text{if } |\mu| \le |\lambda|,\ \mu \ne \lambda.$$

Then $P_\lambda^{\rm ip}(q^\lambda t^\delta a; a, t; q) \ne 0$ (and may be put equal to 1 in a different standardization).

Extra-vanishing property:
$P_\lambda^{\rm ip}(q^\mu t^\delta a; a, t; q) = 0$ if μ is a partition not containing λ.

The case $n = 2$ of Okounkov's interpolation polynomials

$$P^{\mathrm{ip}}_{m_1,m_2}(x_1,x_2;a,t;q) = \frac{q^{-\frac12 m_1(m_1-1)-\frac12 m_2(m_2-1)}}{(-t)^{m_1-m_2}a^{m_1+m_2}}\,(t,q^{2m_2}ta^2;q)_{m_1-m_2}$$

$$\times\,(ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}\sum_{\substack{j,k\ge0\\ j+k\le m_1-m_2}}\frac{(q^{-m_1+m_2};q)_{j+k}q^{j+k}}{(q^{2m_2}ta^2;q)_{j+k}}$$

$$\times\,\frac{(q^{m_2}ax_1,q^{m_2}ax_1^{-1};q)_j}{(q^{1-m_1+m_2}t^{-1},q;q)_j}\,\frac{(q^{m_2}ax_2,q^{m_2}ax_2^{-1};q)_k}{(q^{1-m_1+m_2}t^{-1},q;q)_k}$$

$$= \frac{q^{-\frac12 m_1(m_1-1)-\frac12 m_2(m_2-1)}}{(-t)^{m_1-m_2}a^{m_1+m_2}}\,(ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}$$

$$\times\,(q^{m_2}tax_1,q^{m_2}tax_1^{-1};q)_{m_1-m_2}$$

$$\times\;{}_4\phi_3\left(\begin{matrix}q^{-m_1+m_2},t,q^{m_2}ax_2,q^{m_2}ax_2^{-1}\\ q^{1-m_1+m_2}t^{-1},q^{m_2}tax_1,q^{m_2}tax_1^{-1}\end{matrix};q,q\right).$$

Binomial formula:

$$R_\lambda(x; a, b, c, d, t; q) := \frac{P_\lambda(x; a, b, c, d, t; q)}{P_\lambda(t^\delta a; a, b, c, d, t; q)} = \sum_{\nu \subseteq \lambda} \frac{P_\nu^{\mathrm{ip}}(q^\lambda t^\delta \tilde{a}; \tilde{a}, t; q)}{P_\nu^{\mathrm{ip}}(q^\nu t^\delta \tilde{a}; \tilde{a}, t; q])} \frac{P_\nu^{\mathrm{ip}}(x; a, t; q)}{P_\nu(t^\delta a; a, b, c, d, t; q)}.$$

For $n = 2$ an explicit double sum of products of two single sums.

For general n and $q \to 1$ tends the left-hand side of the binomial formula to a BC_n type Jacobi polynomial (Heckman-Opdam), the first factor in the sum on the right remains an interpolation polynomial and the second factor becomes a Jack polynomial.

For $n = 2$ and $q \to 1$ the first factor in the sum on the right becomes a balanced terminating ${}_4F_3(1)$ and the second factor is expressed in terms of ultraspherical polynomials. One recovers an old formula in K & Sprinkhuizen (1978).

Duality:

$$R_\lambda(q^\mu t^\delta a; a, b, c, d, t; q) = R_\mu(q^\lambda t^\delta \tilde{a}; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, t; q).$$

Non-symmetric Askey-Wilson polynomials

(Sahi, 1999; Noumi & Stokman, 2004)

$$E_n(z;a,b,c,d;q) :=$$
$$R_n(z;a,b,c,d;q) - \frac{q^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-qab)(1-ab)(1-ac)(1-ad)}$$
$$\times\, az^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d;q), \quad n\ge 0,$$
$$E_{-n}(z;a,b,c,d;q) :=$$
$$R_n(z;a,b,c,d;q) - \frac{q^{1-n}(1-q^n ab)(1-q^{n-1}abcd)}{(1-qab)(1-ab)(1-ac)(1-ad)}$$
$$\times\, b^{-1}z^{-1}(1-az)(1-bz)R_{n-1}(z;qa,qb,c,d;q), \quad n\ge 1.$$

Eigenfunctions of a q-difference-reflection operator (generalized Dunkl operator) coming from a generalization of Cherednik's double affine Hecke algebras.
The symmetric AW polynomials are symmetrizations of the non-symmetric AW-polynomials.

$$p_k^+(z;a;q) := (az, az^{-1};q)_k \quad (k \ge 0),$$
$$p_k^-(z;a,b;q) := az^{-1}(z-a)(z-b)(qaz, qaz^{-1};q)_{k-1} \quad (k \ge 1),$$
$$z_{a,q}(n) := aq^n \quad (n \ge 0), \qquad z_{a,q}(-n) := a^{-1}q^{-n} \quad (n > 0).$$

Binomial formula ($n \in \mathbb{Z}$):

$$E_n(z^{-1};a,b,c,d;q) = \sum_{k=0}^{|n|} \frac{p_k^+(z_{\tilde a,q}(n);\tilde a;q)\, p_k^+(z;a;q)}{q^{-k}(ab,ac,ad,q;q)_k}$$
$$- \sum_{k=1}^{|n|} \frac{p_k^-(z_{\tilde a,q}(n);\tilde a,\tilde b\, q)\, p_k^-(z;a,b;q)}{q^{-k}ab(ab;q)_{k+1}(ac,ad;q)_k(q;q)_{k-1}}\,.$$

Duality ($m,n \in \mathbb{Z}$):

$$E_n(z_{a,q}(m)^{-1};a,b,c,d;q) = E_m(z_{\tilde a,q}(n)^{-1};\tilde a,\tilde b,\tilde c,\tilde d;q)$$
$$= \sum_{k=0}^{\min(|m|,|n|)} \frac{p_k^+(z_{\tilde a,q}(n);\tilde a;q)\, p_k^+(z_{a,q}(m);a;q)}{q^{-k}(ab,ac,ad,q;q)_k}$$
$$- \sum_{k=1}^{\min(|m|,|n|)} \frac{p_k^-(z_{\tilde a,q}(n);\tilde a,\tilde b;q)\, p_k^-(z_{a,q}(m);a,b;q)}{q^{-k}ab(ab;q)_{k+1}(ac,ad;q)_k(q;q)_{k-1}}\,.$$

Nonsymmetric Macdonald polynomials
$E_\lambda(x;q,t)$ $(\lambda \in \Lambda_n := (\mathbb{Z}_{\ge 0})^n)$ and
nonsymmetric Koornwinder polynomials
$E_\lambda(x;a,b,c,d,t;q)$ $(\lambda \in \mathbb{Z}^n)$ can be defined as eigenfunctions (polynomials or Laurent polynomials, respectively) of suitable q-difference-reflection operators (generalized Dunkl operators) coming from the polynomial representation of a suitable double affine Hecke algebra.

Expansion of symmetric polynomials in terms of non-symmetric polynomials $(\lambda \in \Lambda_n^+)$:

$$P_\lambda = \sum_{\mu \in S_n \lambda} a_{\lambda,\mu} E_\mu \quad \text{(Macdonald polynomials)},$$

$$P_\lambda = \sum_{\mu \in W_n \lambda} b_{\lambda,\mu} E_\mu \quad \text{(Koornwinder polynomials)}.$$

Four cases of unique interpolation

Let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1$.
Let $|\alpha| := |\alpha_1| + \cdots + |\alpha_n| \quad (\alpha \in \mathbb{Z}^n)$.

1. Let $\overline{\lambda} := q^\lambda \tau$ ($\lambda \in \Lambda_n^+$). Symmetric polynomials of degree $\le d$ are uniquely determined by their values on $\overline{\lambda}$ ($\lambda \in \Lambda_n^+$, $|\lambda| \le d$).
2. Idem for W_n-invariant Laurent polynomials.
3. Let $\overline{\alpha} := q^\alpha w_\alpha \tau$ ($\alpha \in \Lambda_n$), w_α shortest element in S_n such that $w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$. Polynomials of degree $\le d$ are uniquely determined by their values on $\overline{\alpha}$ ($\alpha \in \Lambda_n$, $|\alpha| \le d$).
4. For $\alpha \in \mathbb{Z}^n$ let w_α be the shortest element in W_n such that $w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$. Write $w_\alpha = \sigma_\alpha \pi_\alpha$ ($\sigma_\alpha \in \{\pm 1\}^n$, $\pi_\alpha \in S_n$). Write $(w_\alpha \tau)_i := (\tau_{\pi_\alpha^{-1}(i)})^{\mathrm{sgn}(\alpha_i)}$. Let $\overline{\alpha} := q^\alpha w_\alpha \tau$ ($\alpha \in \mathbb{Z}^n$). Laurent polynomials of degree $\le d$ are uniquely determined by their values on $\overline{\alpha}$ ($\alpha \in \mathbb{Z}^n$, $|\alpha| \le d$).

Items 1 and 3 are results of Sahi (for specialized τ also of Knop), item 2 is due to Okounkov (also in our paper), and item 4 is a new result in our paper.

ad item 3:

$\overline{\alpha} := q^\alpha w_\alpha \tau$ $(\alpha \in \Lambda_n)$, $\quad w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$ (shortest element).
Then, for $i < j$, $w_\alpha^{-1}(i) < w_\alpha^{-1}(j)$ iff $\alpha_i \ge \alpha_j$.
Principal specialization: $\tau = t^\delta$ $(0 < |t| < 1)$.

ad item 4:

$\overline{\alpha}_i := q^{\alpha_i}\big(\tau_{\pi_\alpha^{-1}(i)}\big)^{\mathrm{sgn}(\alpha_i)}$ $(\alpha \in \mathbb{Z}^n)$,
$w_\alpha^{-1}\alpha = \alpha^+ \in \Lambda_n^+$ (shortest element),
$w_\alpha = \sigma_\alpha \pi_\alpha$ $(\sigma_\alpha \in \{\pm 1\}^n,\ \pi_\alpha \in S_n)$. Then, for $i < j$,

$$\pi_\alpha^{-1}(i) < \pi_\alpha^{-1}(j) \;\Leftrightarrow\; |\alpha_i| > |\alpha_j| \text{ or } 0 \le \alpha_i = \pm\alpha_j.$$

Principal specialization: $\tau = a t^\delta$ $(0 < |a|, |t| < 1)$..

Symmetric Okounkov interpolation polynomials:

$0 < |\tau_1| < |\tau_2| < \cdots < |\tau_n| < 1.\quad \overline{\lambda} := q^\lambda \tau \quad (\lambda \in \Lambda_n^+).$

For $\lambda \in \Lambda_n^+$ there is a unique W_n-invariant Laurent polynomial $R_\lambda(x; q, \tau)$ of degree $|\lambda|$ such that

$$R_\lambda(\overline{\mu}; q, \tau) = \delta_{\lambda,\mu} \quad \text{if } \mu \in \Lambda_n^+,\ |\mu| \le |\lambda|.$$

Nonsymmetric Okounkov interpolation polynomials:

τ as above, $\overline{\alpha}_i := q^{\alpha_i} \big(\tau_{\pi_\alpha^{-1}(i)}\big)^{\mathrm{sgn}(\alpha_i)}$ $(\alpha \in \mathbb{Z}^n)$.

For $\alpha \in \mathbb{Z}^n$ there is a unique Laurent polynomial $G_\alpha(x; q, \tau)$ of degree $|\alpha|$ such that

$$G_\alpha(\overline{\beta}; q, \tau) = \delta_{\alpha,\beta} \quad \text{if } \beta \in \mathbb{Z}^n,\ |\beta| \le |\alpha|.$$

Expansion of R_λ in terms of the G_α:

$$R_\lambda(x) = \sum_{\alpha \in W_n \lambda} G_\alpha(x) \quad (\lambda \in \Lambda_n^+).$$

The case $n = 1$:

$$R_m(x;q,s) = \frac{(sx, sx^{-1};q)_m}{(q^m s^2, q^{-m};q)_m}, \qquad m \in \mathbb{Z}_{\geq 0},$$

$$G_m(x;q,s) = \frac{(qsx, sx^{-1};q)_m}{(q^{1+m} s^2, q^{-m};q)_m}, \qquad m \in \mathbb{Z}_{\geq 0},$$

$$G_{-m}(x;q,s) = \frac{q^m sx(qsx;q)_{m-1}(sx^{-1};q)_{m+1}}{(q^m s^2;q)_{m+1}(q^{1-m};q)_{m-1}}, \qquad m \in \mathbb{Z}_{>0}.$$

Then

$$R_0(x) = G_0(x), \qquad R_m(x) = G_m(x) + G_{-m}(x) \quad (m \in \mathbb{Z}_{>0}).$$

Idea of the existence proof (constructive by recurrence)

$I \subseteq [1,n] := \{1,2,\ldots,n\}$.

$$\Lambda_{n,d} := \{\mu \in \mathbb{Z}^n \mid |\mu| \le d\},$$
$$R(n,d,I) := \{\alpha \in \Lambda_{n,d} \mid \alpha_j \ne 0 \text{ for all } j \text{ and } \alpha_i \ne -1 \text{ if } i \in I^c\},$$
$$T(n,d,I) := \{\alpha \in \Lambda_{n,d} \mid \alpha_i \ne 0 \text{ if } i \in I^c\}.$$

Note that

$$R\big(n,d,[1,n]\big) = T(n,d,\emptyset), \qquad T\big(n,d,[1,n]\big) = \Lambda_{n,d}.$$

By complete induction with respect to $n+d$:

1. For every map $\overline{f}\colon R(n,d,I) \to \mathbb{C}$ there is a Laurent polynomial $f \in \mathcal{P}_n$ such that $f(\overline{\alpha}(q,\tau)) = \overline{f}(\alpha)$ for all $\alpha \in R(n,d,I)$ and $\deg\big(x_J f(x)\big) \le d - n + k \quad \forall J \subseteq I$.
2. For every map $\overline{f}\colon T(n,d,I) \to \mathbb{C}$ there is a Laurent polynomial $f \in \mathcal{P}_n$ such that $f(\overline{\alpha}(q,\tau)) = \overline{f}(\alpha)$ for all $\alpha \in T(n,d,I)$ and $\deg\big(x_J f(x)\big) \le d \qquad \forall J \subseteq I^c$.

Both items are successively proved by complete induction with respect to $|I|$.

Some yet unsolved problems

- *Affine Hecke algebras*.
  In the case of nonsymmetric polynomial interpolation Knop and Sahi use suitable representations of affine Hecke algebras for proving deeper properties. In our case only representations of Hecke algebras until now. However, our main obtained results could be proved without using Hecke algebras.
- *Extra-vanishing*
  There is extra-vanishing for non-symmetric polynomial interpolation in the principal specialization (Knop, 1997).

  By computer algebra experiments there is indication of extra-vanishing for non-symmetric Laurent polynomial interpolation in the principal specialization.

- *Binomial formula*
  For non-symmetric Macdonald polynomials there is in the principal specialization a binomial formula involving sums of products of two different non-symmetric interpolation polynomials (Sahi, 1998).

  The case $n = 1$ indicates that there is little chance of such a binomial formula for nonsymmetric Koornwinder polynomials involving the nonsymmetric Laurent interpolation polynomials.

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