

A fractional generalisation of an operational formula and the Gauss hypergeometric function

*Informal note, 11 February 2002, by Tom Koornwinder, thk@science.uva.nl;
major extension of an informal note of 10 August 2001*

Formula HTF 2.8 (21)

$$D^n(z^{a+n-1}F(a, b; c; z)) = (a)_n z^{a-1} F(a+n, b; c; z) \quad (1)$$

can be understood as an iteration of its case $n = 1$

$$D(z^a F(a, b; c; z)) = a z^{a-1} F(a+1, b; c; z) \quad (2)$$

by using the identity

$$z^n D^n z^n = (zDz)^n. \quad (3)$$

This last identity can be shown by induction. First observe

$$z^n D^n z^n = \sum_{k=0}^n \frac{n!^2}{k!^2 (n-k)!} z^{n+k} D^k. \quad (4)$$

Denote the RHS of the last identity by S_n . Then show that

$$(zDz)S_n = S_{n+1}.$$

We can rewrite (3) as

$$z^n \circ D^n \circ z^n = (z \circ D \circ z)^n = z^{-1} \circ (z^2 D)^n \circ z = \iota \circ z \circ (-D)^n \circ z^{-1} \circ \iota, \quad (5)$$

where $(\iota f)(z) := f(z^{-1})$. Thus (1) can be rewritten as

$$(zDz)^n (z^{a-1} F(a, b; c; z)) = (a)_n z^{a+n-1} F(a+n, b; c; z), \quad (6)$$

and as

$$(-D)^n (z^{-a} F(a, b; c; z^{-1})) = (a)_n z^{-a-n} F(a+n, b; c; z^{-1}). \quad (7)$$

Recall the Riemann-Liouville type fractional integral

$$(I_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_0^x f(y) (x-y)^{\mu-1} dy, \quad (8)$$

and the Weyl type fractional integral

$$(W_\mu f)(x) := \frac{1}{\Gamma(\mu)} \int_x^\infty f(y) (y-x)^{\mu-1} dy. \quad (9)$$

Bateman's fractional integral formula HTF 2.4 (2)

$$\frac{1}{\Gamma(\mu)} \int_0^x \Gamma(a + \mu) y^{a-1} F(a + \mu, b; c; y) (x - y)^{\mu-1} dy = \Gamma(a) x^{a+\mu-1} F(a, b; c; x) \quad (10)$$

(Re $a > 0$, Re $\mu > 0$)

can be seen as a fractional iteration of (2) by observing that

$$x^{-\mu} \circ I_{\mu} \circ x^{-\mu} = \iota \circ x \circ W_{\mu} \circ x^{-1} \circ \iota. \quad (11)$$

Formula (11) follows because

$$g(x) = x^{-\mu} \frac{1}{\Gamma(\mu)} \int_0^x y^{-\mu} f(y) (x - y)^{\mu-1} dy$$

implies that

$$g(x^{-1}) = x \frac{1}{\Gamma(\mu)} \int_x^{\infty} y^{-1} f(y^{-1}) (y - x)^{\mu-1} dy.$$

Formula (11) also implies the equality of the first and last part of (5). We can rewrite (10) by use of (11) as

$$\frac{1}{\Gamma(\mu)} \int_x^{\infty} \Gamma(a + \mu) y^{-a-\mu} F(a + \mu, b; c; y^{-1}) (y - x)^{\mu-1} dy = \Gamma(a) x^{-a} F(a, b; c; x^{-1}). \quad (12)$$