Additions to Chapters 9 and 14 in the book "Hypergeometric orthogonal polynomials and their q-analogues" by Koekoek, Lesky & Swarttouw

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April 30, 2012

Introduction

This informal manuscript contains some formulas about (q)-hypergeometric orthogonal polynomials which I missed but wanted to use while consulting Chapters 9 and 14 in the book

R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer-Verlag, 2010.

These chapters form together the (slightly extended) successor of the report

R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1998; `http://aw.twi.tudelft.nl/~koekoek/askey/`.

Usually, any type of formula I give for a special class of polynomials, will suggest a similar formula for many other classes, but I have not aimed at completeness by filling in a formula of such type at all places. The resulting choice of formulas is rather arbitrary, just depending on the formulas which I happened to need or which raised my interest. For each formula I give a suitable reference or I sketch a proof. It is my intention to gradually extend this collection of formulas.

Conventions The (x.y) and (x.y.z) type subsection numbers, the (x.y.z) type formula numbers, and the [x] type citation numbers refer to the book by Koekoek et al. The (x) type formula numbers refer to this manuscript and the [Kx] type citation numbers refer to citations which are not in the book. Some standard references like [DLMF] are given by special acronyms.

N is always a positive integer. Always assume n to be a nonnegative integer or, if N is present, to be in $\{0,1,\ldots,N\}$. Throughout assume $0<q<1$.

For each family the coefficient of the term of highest degree of the orthogonal polynomial of degree n can be found in the book as the coefficient of $p_n(x)$ in the formula after the main formula under the heading "Normalized Recurrence Relation". If that main formula is numbered as (x.y.z) then I will refer to the second formula as (x.y.zb).

Generalities

Even orthogonality measure If $\{p_n\}$ is a system of orthogonal polynomials with respect to an even orthogonality measure which satisfies the three-term recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + C_n p_{n-1}(x)$$

then

$$\frac{p_{2n}(0)}{p_{2n-2}(0)} = -\frac{C_{2n-1}}{A_{2n-1}}\,. \tag{1}$$

Appell's bivariate hypergeometric function F_4 This is defined by

$$F_4(a,b;c,c';x,y) := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} b_{m+n}}{(c)_m (c')_n\, m!\, n!}\, x^m y^n \qquad (|x|^{\frac12} + |y|^{\frac12} < 1), \tag{2}$$

see [HTF1, 5.7(9), 5.7(44)] or [DLMF, (16.3.4)]. There is the reduction formula

$$F_4\left(a,b;b,b;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right) = (1-x)^a(1-y)^a\ {}_2F_1\left(\begin{matrix}a,1+a-b\\ b\end{matrix};xy\right),$$

see [HTF1, 5.10(7)]. When combined with the quadratic transformation [HTF1, 2.11(34)] (here $a-b-1$ should be replaced by $a-b+1$), see also [DLMF, (15.8.15)], this yields

$$F_4\left(a,b;b,b;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right) = \left(\frac{(1-x)(1-y)}{1+xy}\right)^a\ {}_2F_1\left(\begin{matrix}\frac12 a,\frac12(a+1)\\ b\end{matrix};\frac{4xy}{(1+xy)^2}\right).$$

This can be rewritten as

$$F_4(a,b;b,b;x,y) = (1-x-y)^{-a}\ {}_2F_1\left(\begin{matrix}\frac12 a,\frac12(a+1)\\ b\end{matrix};\frac{4xy}{(1-x-y)^2}\right). \tag{3}$$

Note that, if $x,y \ge 0$ and $x^{\frac12} + y^{\frac12} < 1$, then $1-x-y > 0$ and $0 \le \frac{4xy}{(1-x-y)^2} < 1$.

9.5 Hahn

Special values From (9.5.3) and (1) it follows that

$$Q_{2n}(N;\alpha,\alpha,2N) = \frac{(\frac12)_n (N+\alpha+1)_n}{(-N+\frac12)_n(\alpha+1)_n}\,. \tag{4}$$

From (9.5.1) and [DLMF, (15.4.24)] it follows that

$$Q_N(x;\alpha,\beta,N) = \frac{(-N-\beta)_x}{(\alpha+1)_x} \qquad (x = 0,1,\ldots,N). \tag{5}$$

Duality The Remark on p.208 gives the duality between Hahn and dual Hahn polynomials:

$$Q_n(x;\alpha,\beta,N) = R_x(n(n+\alpha+\beta+1);\alpha,\beta,N) \quad (n,x \in \{0,1,\ldots N\}). \tag{6}$$

9.6 Dual Hahn

Re: (9.6.11). The generating function (9.6.11) can be written in a more conceptual way as

$$(1-t)^x \, {}_2F_1\left(\begin{matrix} x-N, x+\gamma+1 \\ -\delta-N \end{matrix}; t\right) = \frac{N!}{(\delta+1)_N} \sum_{n=0}^{N} \omega_n \, R_n(\lambda(x);\gamma,\delta,N)\, t^n, \tag{7}$$

where

$$\omega_n := \binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}, \tag{8}$$

i.e., the denominator on the right-hand side of (9.6.2). By the duality between Hahn polynomials and dual Hahn polynomials (see (6)) the above generating function can be rewritten in terms of Hahn polynomials:

$$(1-t)^n \, {}_2F_1\left(\begin{matrix} n-N, n+\alpha+1 \\ -\beta-N \end{matrix}; t\right) = \frac{N!}{(\beta+1)_N} \sum_{x=0}^{N} w_x \, Q_n(x;\alpha,\beta,N)\, t^x, \tag{9}$$

where

$$w_x := \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}, \tag{10}$$

i.e., the weight occurring in the orthogonality relation (9.5.2) for Hahn polynomials.

Special value By (5) and (6) we have

$$R_n(N(N+\gamma+\delta+1);\gamma,\delta,N) = \frac{(-N-\delta)_n}{(\gamma+1)_n}\,. \tag{11}$$

9.8 Jacobi

Orthogonality relation Write the right-hand side of (9.8.2) as $h_n\,\delta_{m,n}$. Then

$$\frac{h_n}{h_0} = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_n\, n!}\,. \tag{12}$$

Symmetry

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n \, P_n^{(\beta,\alpha)}(x). \tag{13}$$

Use (9.8.2) and (9.8.5b) or see [DLMF, Table 18.6.1].

Special values

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}\,, \quad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(\beta+1)_n}{n!}\,. \tag{14}$$

Use (9.8.1) and (13) or see [DLMF, Table 18.6.1].

Bilateral generating functions For $0 \le r < 1$ and $x,y \in [-1,1]$ we have in terms of F_4 (see (2)):

$$\sum_{n=0}^\infty \frac{(\alpha+\beta+1)_n\,n!}{(\alpha+1)_n(\beta+1)_n}\,r^n\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)=\frac{1}{(1+r)^{\alpha+\beta+1}}$$
$$\times F_4\Big(\tfrac12(\alpha+\beta+1),\tfrac12(\alpha+\beta+2);\alpha+1,\beta+1;\frac{r(1-x)(1-y)}{(1+r)^2},\frac{r(1+x)(1+y)}{(1+r)^2}\Big),\quad(15)$$

$$\sum_{n=0}^\infty \frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1}\,\frac{(\alpha+\beta+2)_n\,n!}{(\alpha+1)_n(\beta+1)_n}\,r^n\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)=\frac{1-r}{(1+r)^{\alpha+\beta+2}}$$
$$\times F_4\Big(\tfrac12(\alpha+\beta+2),\tfrac12(\alpha+\beta+3);\alpha+1,\beta+1;\frac{r(1-x)(1-y)}{(1+r)^2},\frac{r(1+x)(1+y)}{(1+r)^2}\Big).\quad(16)$$

Formulas (15) and (16) were first given by Bailey [91, (2.1), (2.3)]. See Stanton [485] for a shorter proof. (However, in the second line of [485, (1)] z and Z should be interchanged.) As observed in Bailey [91, p.10], (16) follows from (15) by applying the operator $r^{-\frac12(\alpha+\beta-1)}\,\frac{d}{dr}\circ r^{\frac12(\alpha+\beta+1)}$ to both sides of (15). In view of (12), formula (16) is the Poisson kernel for Jacobi polynomials. The right-hand side of (16) makes clear that this kernel is positive. See also the discussion in Askey [46, following (2.32)].

Quadratic transformations

$$\frac{C_{2n}^{(\alpha+\frac12)}(x)}{C_{2n}^{(\alpha+\frac12)}(1)}=\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}=\frac{P_n^{(\alpha,-\frac12)}(2x^2-1)}{P_n^{(\alpha,-\frac12)}(1)}\,,\quad(17)$$
$$\frac{C_{2n+1}^{(\alpha+\frac12)}(x)}{C_{2n+1}^{(\alpha+\frac12)}(1)}=\frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)}=\frac{x\,P_n^{(\alpha,\frac12)}(2x^2-1)}{P_n^{(\alpha,\frac12)}(1)}\,.\quad(18)$$

See p.221, Remarks, last two formulas together with (14) and (29). Or see [DLMF, (18.7.13), (18.7.14)].

Differentiation formulas Each differentiation formula is given in two equivalent forms.

$$\frac{d}{dx}\Big((1-x)^\alpha P_n^{(\alpha,\beta)}(x)\Big)=-(n+\alpha)\,(1-x)^{\alpha-1}P_n^{(\alpha-1,\beta+1)}(x),$$
$$\Big((1-x)\frac{d}{dx}-\alpha\Big)\,P_n^{(\alpha,\beta)}(x)=-(n+\alpha)\,P_n^{(\alpha-1,\beta+1)}(x).\quad(19)$$

$$\frac{d}{dx}\Big((1+x)^\beta P_n^{(\alpha,\beta)}(x)\Big)=(n+\beta)\,(1+x)^{\beta-1}P_n^{(\alpha+1,\beta-1)}(x),$$
$$\Big((1+x)\frac{d}{dx}+\beta\Big)\,P_n^{(\alpha,\beta)}(x)=(n+\beta)\,P_n^{(\alpha+1,\beta-1)}(x).\quad(20)$$

Formulas (19) and (20) follow from [DLMF, (15.5.4), (15.5.6)] together with (9.8.1). They also follow from each other by (13).

Generalized Gegenbauer polynomials See [146, p.156]. These are defined by

$$S_{2m}^{(\alpha,\beta)}(x) := \text{const.}\, P_m^{(\alpha,\beta)}(2x^2-1), \qquad S_{2m+1}^{(\alpha,\beta)}(x) := \text{const.}\, x\, P_m^{(\alpha,\beta+1)}(2x^2-1). \tag{21}$$

Then for $\alpha, \beta > -1$ we have the orthogonality relation

$$\int_{-1}^{1} S_m^{(\alpha,\beta)}(x)\, S_n^{(\alpha,\beta)}(x)\, |x|^{2\beta+1}(1-x^2)^\alpha\, dx = 0 \qquad (m \neq n). \tag{22}$$

If we define the *Dunkl operator* T_μ by

$$(T_\mu f)(x) := f'(x) + \mu\, \frac{f(x)-f(-x)}{x} \tag{23}$$

and if we choose the constants in (21) as

$$S_{2m}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)_m}{(\beta+1)_m}\, P_m^{(\alpha,\beta)}(2x^2-1), \quad S_{2m+1}^{(\alpha,\beta)}(x) = \frac{(\alpha+\beta+1)_{m+1}}{(\beta+1)_{m+1}}\, x\, P_m^{(\alpha,\beta+1)}(2x^2-1) \tag{24}$$

then (see [K2, (1.6)])

$$T_{\beta+\frac{1}{2}} S_n^{(\alpha,\beta)} = 2(\alpha+\beta+1)\, S_{n-1}^{(\alpha+1,\beta)}. \tag{25}$$

Formula (25) with (24) substituted gives rise to two differentiation formulas involving Jacobi polynomials which are equivalent to (9.8.7) and (20).

Composition of (25) with itself gives

$$T_{\beta+\frac{1}{2}}^2 S_n^{(\alpha,\beta)} = 4(\alpha+\beta+1)(\alpha+\beta+2)\, S_{n-2}^{(\alpha+2,\beta)},$$

which is equivalent to the composition of (9.8.7) and (20):

$$\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right) P_n^{(\alpha,\beta)}(2x^2-1) = 4(n+\alpha+\beta+1)(n+\beta)\, P_{n-1}^{(\alpha+2,\beta)}(2x^2-1). \tag{26}$$

Formula (26) was also given in [322, (2.4)].

9.8.1 Gegenbauer / Ultraspherical

Notation Here the Gegenbauer polynomial is denoted by C_n^λ instead of $C_n^{(\lambda)}$.

Orthogonality relation Write the right-hand side of (9.8.20) as $h_n\, \delta_{m,n}$. Then

$$\frac{h_n}{h_0} = \frac{\lambda}{\lambda+n}\, \frac{(2\lambda)_n}{n!}\,. \tag{27}$$

Hypergeometric representation Beside (9.8.19) we have also

$$C_n^\lambda(x) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} \frac{(-1)^\ell (\lambda)_{n-\ell}}{\ell!\,(n-2\ell)!}\,(2x)^{n-2\ell} = (2x)^n\, \frac{(\lambda)_n}{n!}\; {}_2F_1\!\left(\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n+\frac{1}{2} \\ 1-\lambda-n \end{matrix}; \frac{1}{x^2}\right). \tag{28}$$

See [DLMF, (18.5.10)].

Special value

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!}\,. \tag{29}$$

Use (9.8.19) or see [DLMF, Table 18.6.1].

Expression in terms of Jacobi

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} = \frac{P_n^{(\lambda-\frac12,\lambda-\frac12)}(x)}{P_n^{(\lambda-\frac12,\lambda-\frac12)}(1)}\,, \qquad C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda+\frac12)_n}\,P_n^{(\lambda-\frac12,\lambda-\frac12)}(x). \tag{30}$$

Re: (9.8.21) By iteration of recurrence relation (9.8.21):

$$x^2 C_n^\lambda(x) = \frac{(n+1)(n+2)}{4(n+\lambda)(n+\lambda+1)}\,C_{n+2}^\lambda(x) + \frac{n^2+2n\lambda+\lambda-1}{2(n+\lambda-1)(n+\lambda+1)}\,C_n^\lambda(x) + \frac{(n+2\lambda-1)(n+2\lambda-2)}{4(n+\lambda)(n+\lambda-1)}\,C_{n-2}^\lambda(x). \tag{31}$$

Bilateral generating functions

$$\sum_{n=0}^\infty \frac{n!}{(2\lambda)_n}\,r^n\,C_n^\lambda(x)\,C_n^\lambda(y) = \frac{1}{(1-2rxy+r^2)^\lambda}\,{}_2F_1\!\left(\begin{matrix}\frac12\lambda,\frac12(\lambda+1)\\ \lambda+\frac12\end{matrix};\frac{4r^2(1-x^2)(1-y^2)}{(1-2rxy+r^2)^2}\right) \qquad (r\in(-1,1),\ x,y\in[-1,1]). \tag{32}$$

For the proof put $\beta:=\alpha$ in (15), then use (3) and (30). The Poisson kernel for Gegenbauer polynomials can be derived in a similar way from (16), or alternatively by applying the operator $r^{-\lambda+1}\frac{d}{dr}\circ r^\lambda$ to both sides of (32):

$$\sum_{n=0}^\infty \frac{\lambda+n}{\lambda}\,\frac{n!}{(2\lambda)_n}\,r^n\,C_n^\lambda(x)\,C_n^\lambda(y) = \frac{1-r^2}{(1-2rxy+r^2)^{\lambda+1}} \times {}_2F_1\!\left(\begin{matrix}\frac12(\lambda+1),\frac12(\lambda+2)\\ \lambda+\frac12\end{matrix};\frac{4r^2(1-x^2)(1-y^2)}{(1-2rxy+r^2)^2}\right) \qquad (r\in(-1,1),\ x,y\in[-1,1]). \tag{33}$$

Formula (33) was obtained by Gasper & Rahman [234, (4.4)] as a limit case of their formula for the Poisson kernel for continuous q-ultraspherical polynomials.

A trigonometric expansion By [DLMF, (14.13.1), (14.3.21), (5.5.5)]:

$$C_n^\lambda(\cos\theta) = \frac{\Gamma(2\lambda+1)}{2^{2\lambda}\Gamma(\lambda+1)^2}\,\frac{(2\lambda)_n}{(\lambda+1)_n}\,(\sin\theta)^{1-2\lambda}\sum_{k=0}^\infty \frac{(1-\lambda)_k(n+1)_k}{(n+\lambda+1)_k k!}\,\sin\big((2k+n+1)\theta\big) \qquad (\lambda>0,\ 0<\theta<\pi). \tag{34}$$

For $\lambda\in\mathbb{Z}_{>0}$ the above series terminates after the term with $k=\lambda-1$.

Fourier transform

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_{-1}^{1}\frac{C_n^\lambda(y)}{C_n^\lambda(1)}\,(1-y^2)^{\lambda-\frac{1}{2}}\,e^{ixy}\,dy = i^n\,2^\lambda\,\Gamma(\lambda+1)\,x^{-\lambda}\,J_{\lambda+n}(x). \tag{35}$$

See [DLMF, (18.17.17) and (18.17.18)].

Laplace transforms

$$\frac{2}{n!\,\Gamma(\lambda)}\int_0^\infty H_n(tx)\,t^{n+2\lambda-1}\,e^{-t^2}\,dt = C_n^\lambda(x). \tag{36}$$

See Nielsen [K5, p.48, (4) with p.47, (1) and p.28, (10)] (1918) or Feldheim [K3, (28)] (1942).

$$\frac{2}{\Gamma(\lambda+\frac{1}{2})}\int_0^1\frac{C_n^\lambda(t)}{C_n^\lambda(1)}\,(1-t^2)^{\lambda-\frac{1}{2}}\,t^{-1}\,(x/t)^{n+2\lambda+1}\,e^{-x^2/t^2}\,dt = 2^{-n}\,H_n(x)\,e^{-x^2}\quad(\lambda>-\tfrac{1}{2}). \tag{37}$$

Use Askey & Fitch [K1, (3.29)] for $\alpha=\pm\frac{1}{2}$ together with (13), (17), (18), (48) and (49).

9.10 Meixner

History In 1934 Meixner [406] (see (1.1) and case IV on pp. 10, 11 and 12) gave the orthogonality measure for the polynomials P_n given by the generating function

$$e^{xu(t)}\,f(t) = \sum_{n=0}^\infty P_n(x)\,\frac{t^n}{n!}\,,$$

where

$$e^{u(t)} = \left(\frac{1-\beta t}{1-\alpha t}\right)^{\frac{1}{\alpha-\beta}},\qquad f(t) = \frac{(1-\beta t)^{\frac{k_2}{\beta(\alpha-\beta)}}}{(1-\alpha t)^{\frac{k_2}{\alpha(\alpha-\beta)}}}\qquad (k_2<0;\ \alpha>\beta>0\ \text{ or }\ \alpha<\beta<0).$$

Then P_n can be expressed as a Meixner polynomial:

$$P_n(x) = (-k_2(\alpha\beta)^{-1})_n\,\beta^n\,M_n\left(-\frac{x+k_2\alpha^{-1}}{\alpha-\beta},-k_2(\alpha\beta)^{-1},\beta\alpha^{-1}\right).$$

In 1938 Gottlieb [K4, §2] introduces polynonials l_n “of Laguerre type” which turn out to be special Meixner polynomials: $l_n(x) = e^{-n\lambda}M_n(x;1,e^{-\lambda})$.

9.11 Krawtchouk

Special values By (9.11.1) and the binomial formula:

$$K_n(0;p,N)=1,\qquad K_n(N;p,N) = (-1)^n p^{-n}(1-p)^n. \tag{38}$$

Symmetry By the orthogonality relations:

$$\frac{K_n(N-x;p,N)}{K_n(N;p,N)} = K_n(x;1-p,N), \tag{39}$$

in particular:

$$K_n(N-x;\tfrac12,N) = (-1)^n K_n(x;\tfrac12,N). \tag{40}$$

Hence

$$K_{2m+1}(N;\tfrac12,2N) = 0. \tag{41}$$

From (9.11.11):

$$K_{2m}(N;\tfrac12,2N) = \frac{(\frac12)_m}{(-N+\frac12)_m}\,. \tag{42}$$

Quadratic transformations

$$K_{2m}(x+N;\tfrac12,2N) = \frac{(\frac12)_m}{(-N+\frac12)_m}\,R_m(x^2;-\tfrac12,-\tfrac12,N), \tag{43}$$

$$K_{2m+1}(x+N;\tfrac12,2N) = -\frac{(\frac32)_m}{N\,(-N+\frac12)_m}\,x\,R_m(x^2-1;\tfrac12,\tfrac12,N-1), \tag{44}$$

$$K_{2m}(x+N+1;\tfrac12,2N+1) = \frac{(\frac12)_m}{(-N-\frac12)_m}\,R_m(x(x+1);-\tfrac12,\tfrac12,N), \tag{45}$$

$$K_{2m+1}(x+N+1;\tfrac12,2N+1) = \frac{(\frac32)_m}{(-N-\frac12)_{m+1}}\,(x+\tfrac12)\,R_m(x(x+1);\tfrac12,-\tfrac12,N), \tag{46}$$

where R_m is a dual Hahn polynomial (9.6.1). For the proofs use (9.6.2), (9.11.2), (9.6.4) and (9.11.4).

9.12 Laguerre

Notation Here the Laguerre polynomial is denoted by L_n^α instead of $L_n^{(\alpha)}$.

Special value

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}\,. \tag{47}$$

Use (9.12.1) or see [DLMF, (18.6.1)].

Quadratic transformations

$$H_{2n}(x) = (-1)^n\,2^{2n}\,n!\,L_n^{-1/2}(x^2), \tag{48}$$

$$H_{2n+1}(x) = (-1)^n\,2^{2n+1}\,n!\,x\,L_n^{1/2}(x^2). \tag{49}$$

See p.244, Remarks, last two formulas. Or see [DLMF, (18.7.19), (18.7.20)].

Fourier transform

$$\frac{1}{\Gamma(\alpha+1)}\int_0^\infty \frac{L_n^\alpha(y)}{L_n^\alpha(0)}\,e^{-y}\,y^\alpha\,e^{ixy}\,dy = i^n\,\frac{y^n}{(iy+1)^{n+\alpha+1}}\,, \tag{50}$$

see [DLMF, (18.17.34)].

Differentiation formulas Each differentiation formula is given in two equivalent forms.

$$\frac{d}{dx}\left(x^\alpha L_n^\alpha(x)\right) = (n+\alpha)\,x^{\alpha-1}L_n^{\alpha-1}(x), \qquad \left(x\frac{d}{dx}+\alpha\right)L_n^\alpha(x) = (n+\alpha)\,L_n^{\alpha-1}(x). \tag{51}$$

$$\frac{d}{dx}\left(e^{-x}L_n^\alpha(x)\right) = -e^{-x}L_n^{\alpha+1}(x), \qquad \left(\frac{d}{dx}-1\right)L_n^\alpha(x) = -L_n^{\alpha+1}(x). \tag{52}$$

Formulas (51) and (52) follow from [DLMF, (13.3.18), (13.3.20)] together with (9.12.1).

Generalized Hermite polynomials See [146, p.156]. These are defined by

$$H_{2m}^\mu(x) := \text{const.}\,L_m^{\mu-\frac12}(x^2), \qquad H_{2m+1}^\mu(x) := \text{const.}\,x\,L_m^{\mu+\frac12}(x^2). \tag{53}$$

Then for $\mu > -\frac12$ we have orthogonality relation

$$\int_{-\infty}^\infty H_m^\mu(x)\,H_n^\mu(x)\,|x|^{2\mu}e^{-x^2}\,dx = 0 \qquad (m\neq n). \tag{54}$$

Let the Dunkl operator T_μ be defined by (23). If we choose the constants in (53) as

$$H_{2m}^\mu(x) = \frac{(-1)^m(2m)!}{(\mu+\frac12)_m}\,L_m^{\mu-\frac12}(x^2), \qquad H_{2m+1}^\mu(x) = \frac{(-1)^m(2m+1)!}{(\mu+\frac12)_{m+1}}\,x\,L_m^{\mu+\frac12}(x^2) \tag{55}$$

then (see [K2, (1.6)])

$$T_\mu H_n^\mu = 2n\,H_{n-1}^\mu. \tag{56}$$

Formula (56) with (55) substituted gives rise to two differentiation formulas involving Laguerre polynomials which are equivalent to (9.12.6) and (51).

Composition of (56) with itself gives

$$T_\mu^2 H_n^\mu = 4n(n-1)\,H_{n-2}^\mu,$$

which is equivalent to the composition of (9.12.6) and (51):

$$\left(\frac{d^2}{dx^2}+\frac{2\alpha+1}{x}\,\frac{d}{dx}\right)L_n^\alpha(x^2) = -4(n+\alpha)\,L_{n-1}^\alpha(x^2). \tag{57}$$

9.15 Hermite

Fourier transforms

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H_n(y)\,e^{-\frac{1}{2}y^2}\,e^{ixy}\,dy = i^n\,H_n(x)\,e^{-\frac{1}{2}x^2}, \tag{58}$$

see [AAR, (6.1.15) and Exercise 6.11].

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} H_n(y)\,e^{-y^2}\,e^{ixy}\,dy = i^n\,x^n\,e^{-\frac{1}{4}x^2}, \tag{59}$$

see [DLMF, (18.17.35)].

$$\frac{i^n}{2\sqrt{\pi}}\int_{-\infty}^{\infty} y^n\,e^{-\frac{1}{4}y^2}\,e^{-ixy}\,dy = H_n(x)\,e^{-x^2}, \tag{60}$$

see [AAR, (6.1.4)].

14.1 Askey-Wilson

Symmetry The Askey-Wilson polynomials $p_n(x;a,b,c,d\mid q)$ are symmetric in a,b,c,d. This follows from the orthogonality relation (14.1.2) together with the value of its coefficient of x^n given in §14.1 under Normalized recurrence relation. Alternatively, combine (14.1.1) with [GR, (III.15)].

Special value

$$p_n\big(\tfrac{1}{2}(a+a^{-1});a,b,c,d\mid q\big) = a^{-n}\,(ab,ac,ad;q)_n\,, \tag{61}$$

and similarly for arguments $\frac{1}{2}(b+b^{-1})$, $\frac{1}{2}(c+c^{-1})$ and $\frac{1}{2}(d+d^{-1})$ by symmetry of p_n in a,b,c,d.

Trivial symmetry

$$p_n(-x;a,b,c,d\mid q) = (-1)^n p_n(x;-a,-b,-c,-d\mid q). \tag{62}$$

Both (61) and (62) are obtained from (14.1.1).

Re: (14.1.5) Let

$$p_n(x) := \frac{p_n(x;a,b,c,d\mid q)}{2^n(abcdq^{n-1};q)_n} = x^n + \widetilde{k}_n x^{n-1} + \cdots\,. \tag{63}$$

Then

$$\widetilde{k}_n = -\frac{(1-q^n)(a+b+c+d-(abc+abd+acd+bcd)q^{n-1})}{2(1-q)(1-abcdq^{2n-2})}\,. \tag{64}$$

This follows because $\tilde{k}_n - \tilde{k}_{n+1}$ equals the coefficient $\frac{1}{2}\big(a+a^{-1}-(A_n+C_n)\big)$ of $p_n(x)$ in (14.1.5).

14.2 q-Racah

Trivial symmetry Clearly from (14.2.1):

$$R_n(y;\alpha,\beta,\gamma,\delta\mid q) = R_n(y;\beta\delta,\alpha\delta^{-1},\gamma,\delta\mid q).$$

14.8 Al-Salam-Chihara

q^{-1}-Al-Salam-Chihara

Re: (14.8.1) For $x \in \mathbb{Z}_{\geq 0}$:

$$
\begin{aligned}
Q_n(\tfrac{1}{2}(aq^{-x}+a^{-1}q^x);a,b \mid q^{-1}) &= (-1)^n b^n q^{-\frac{1}{2}n(n-1)} \big((ab)^{-1};q\big)_n \\
&\qquad \times {}_3\phi_1\left(\begin{matrix} q^{-n}, q^{-x}, a^{-2}q^x \\ (ab)^{-1} \end{matrix}; q, q^n ab^{-1}\right) \qquad (65)\\
&= (-ab^{-1})^x\, q^{-\frac{1}{2}x(x+1)}\, \frac{(qba^{-1};q)_x}{(a^{-1}b^{-1};q)_x}\, {}_2\phi_1\left(\begin{matrix} q^{-x}, a^{-2}q^x \\ qba^{-1} \end{matrix}; q, q^{n+1}\right) \qquad (66)\\
&= (-ab^{-1})^x\, q^{-\frac{1}{2}x(x+1)}\, \frac{(qba^{-1};q)_x}{(a^{-1}b^{-1};q)_x}\, p_x(q^n; ba^{-1}, (qab)^{-1}; q). \qquad (67)
\end{aligned}
$$

Formula (65) follows from the first identity in (14.8.1). Next (66) follows from [GR, (III.8)]. Finally (67) gives the little q-Jacobi polynomials (14.12.1). See also [79, §3].

Orthogonality

$$
\begin{aligned}
\sum_{x=0}^{\infty} \frac{(1-q^{2x}a^{-2})(a^{-2},(ab)^{-1};q)_x}{(1-a^{-2})(q,bqa^{-1};q)_x}\,(ba^{-1})^x q^{x^2}(Q_nQ_m)(\tfrac{1}{2}(aq^{-x}+a^{-1}q^x);a,b;q)\\
= \frac{(qa^{-2};q)_\infty}{(ba^{-1}q;q)_\infty}\,(q,(ab)^{-1};q)_n\,(ab)^n q^{-n^2}\,\delta_{n,m} \quad (ab>1,\ qb<a). \qquad (68)
\end{aligned}
$$

This follows from (29) together with (14.12.2) and the completeness of the orthogonal systerm of the little q-Jacobi polynomials, See also [79, §3]. An alternative proof is given in [64]. There combine (3.82) with (3.81), (3.67), (3.40).

Normalized recurrence relation

$$
xp_n(x) = p_{n+1}(x) + \tfrac{1}{2}(a+b)q^{-n}p_n(x) + \tfrac{1}{4}(q^{-n}-1)(abq^{-n+1}-1)p_{n-1}(x), \qquad (69)
$$

where

$$
Q_n(x;a,b \mid q^{-1}) = 2^n p_n(x).
$$

14.10.1 Continuous q-ultraspherical / Rogers

Re: (14.10.17)

$$
C_n(\cos\theta;\beta \mid q) = \frac{(\beta^2;q)_n}{(q;q)_n}\,\beta^{-\frac{1}{2}n}\,{}_4\phi_3\left(\begin{matrix} q^{-\frac{1}{2}n}, \beta q^{\frac{1}{2}n}, \beta^{\frac{1}{2}}e^{i\theta}, \beta^{\frac{1}{2}}e^{-i\theta} \\ -\beta, \beta^{\frac{1}{2}}q^{\frac{1}{4}}, -\beta^{\frac{1}{2}}q^{\frac{1}{4}} \end{matrix}; q^{\frac{1}{2}}, q^{\frac{1}{2}}\right), \qquad (70)
$$

see [GR, (7.4.13), (7.4.14)].

Re: (14.10.21) (another q-difference equation). Let $C_n[e^{i\theta};\beta \mid q] := C_n(\cos\theta;\beta \mid q)$.

$$
\frac{1-\beta z^2}{1-z^2}\,C_n[q^{\frac{1}{2}}z;\beta \mid q] + \frac{1-\beta z^{-2}}{1-z^{-2}}\,C_n[q^{-\frac{1}{2}}z;\beta \mid q] = (q^{-\frac{1}{2}n}+q^{\frac{1}{2}n}\beta)\,C_n[z;\beta \mid q], \qquad (71)
$$

see [351, (6.10)].

Re: (14.10.23) This can also be written as

$$C_n[q^{\frac{1}{2}}z;\beta \mid q] - C_n[q^{-\frac{1}{2}}z;\beta \mid q] = q^{-\frac{1}{2}n}(\beta-1)(z-z^{-1})C_{n-1}[z;q\beta \mid q]. \tag{72}$$

Two other shift relations follow from the previous two equations:

$$(\beta+1)C_n[q^{\frac{1}{2}}z;\beta \mid q] = (q^{-\frac{1}{2}n}+q^{\frac{1}{2}n}\beta)C_n[z;\beta \mid q] + q^{-\frac{1}{2}n}(\beta-1)(z-\beta z^{-1})C_{n-1}[z;q\beta \mid q], \tag{73}$$

$$(\beta+1)C_n[q^{-\frac{1}{2}}z;\beta \mid q] = (q^{-\frac{1}{2}n}+q^{\frac{1}{2}n}\beta)C_n[z;\beta \mid q] + q^{-\frac{1}{2}n}(\beta-1)(z^{-1}-\beta z)C_{n-1}[z;q\beta \mid q]. \tag{74}$$

14.20 Little q-Laguerre / Wall

Re: (14.20.11) The right-hand side of this generating function converges for $|xt| < 1$. We can rewrite the left-hand side by use of the transformation

$${}_2\phi_1\left(\begin{matrix}0,0\\ c\end{matrix};q,z\right) = \frac{1}{(z;q)_\infty}\,{}_0\phi_1\left(\begin{matrix}-\\ c\end{matrix};q,cz\right).$$

Then we obtain:

$$(t;q)_\infty\,{}_2\phi_1\left(\begin{matrix}0,0\\ aq\end{matrix};q,xt\right) = \sum_{n=0}^{\infty}\frac{(-1)^n\,q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\,p_n(x;a;q)\,t^n \qquad (|xt|<1). \tag{75}$$

Expansion of x^n

Divide both sides of (75) by $(t;q)_\infty$. Then coefficients of the same power of t on both sides must be equal. We obtain:

$$x^n = (a;q)_n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\,q^{nk}\,p_k(x;a;q). \tag{76}$$

Quadratic transformations

Little q-Laguerre polynomials $p_n(x;a;q)$ with $a=q^{\pm\frac{1}{2}}$ are related to discrete q-Hermite I polynomials $h_n(x;q)$:

$$p_n(x^2;q^{-1};q^2) = \frac{(-1)^n q^{-n(n-1)}}{(q;q^2)_n}\,h_{2n}(x;q), \tag{77}$$

$$xp_n(x^2;q;q^2) = \frac{(-1)^n q^{-n(n-1)}}{(q^3;q^2)_n}\,h_{2n+1}(x;q). \tag{78}$$

14.21 q-Laguerre

Expansion of x^n

$$x^n = q^{-\frac{1}{2}n(n+2\alpha+1)}\,(q^{\alpha+1};q)_n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}\,q^k\,L_k^{\alpha}(x;q). \tag{79}$$

This follows from (76) by the equality given in the Remark at the end of §14.20. Alternatively, it can be derived in the same way as (76) from the generating function (14.21.14).

Quadratic transformations

q-Laguerre polynomials $L_n^{\alpha}(x;q)$ with $\alpha = \pm\frac{1}{2}$ are related to discrete q-Hermite II polynomials $\widetilde{h}_n(x;q)$:

$$L_n^{-1/2}(x^2;q^2) = \frac{(-1)^n q^{2n^2-n}}{(q^2;q^2)_n}\,\widetilde{h}_{2n}(x;q), \tag{80}$$

$$xL_n^{1/2}(x^2;q^2) = \frac{(-1)^n q^{2n^2+n}}{(q^2;q^2)_n}\,\widetilde{h}_{2n+1}(x;q). \tag{81}$$

These follows from (77) and (78), respectively, by applying the equalities given in the Remarks at the end of §14.20 and §14.28.

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