ON SOME LIMIT CASES OF ASKEY-WILSON POLYNOMIALS

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ABSTRACT. We show that limit transitions from Askey-Wilson polynomials to q-Racah, little and big q-Jacobi polynomials can be made rigorous on the level of their orthogonality measures in a suitable weak sense. This allows us to derive the orthogonality relations and norm evaluations for the q-Racah polynomials, little and big q-Jacobi polynomials by taking limits in the orthogonality relations and norm evaluations for the Askey-Wilson polynomials.

1. INTRODUCTION

In this paper we consider three families of basic hypergeometric orthogonal polynomials as limit cases of the Askey-Wilson polynomials. The three limit cases we consider are the q-Racah polynomials, the little q-Jacobi polynomials and the big q-Jacobi polynomials. These limits are well known in the sense of pointwise convergence. We will prove these limit transitions in a suitable weak sense on the level of their orthogonality measures.

To be more precise, we show that the continuous part of the orthogonality measure of the Askey-Wilson polynomials disappears in each of the three limit transitions while the discrete part of the orthogonality measure tends to the discrete orthogonality measure of the q-Racah polynomials, the little q-Jacobi polynomials respectively the big q-Jacobi polynomials. We prove then the orthogonality relations and norm evaluations for the q-Racah polynomials, the little and the big q-Jacobi polynomials by taking limits in the orthogonality relations and norm evaluations for the Askey-Wilson polynomials.

The contents of this paper are as follows. In section 2 we introduce the Askey-Wilson polynomials and state their orthogonality relations and norm evaluations. Furthermore, we introduce the q-Racah polynomials, the little q-Jacobi polynomials and the big q-Jacobi polynomials as limits of the Askey-Wilson polynomials. In section 3, 4 respectively 5 we give new proofs of the orthogonality relations and norm evaluations for the q-Racah polynomials, little q-Jacobi polynomials respectively big q-Jacobi polynomials by proving these three limits in a suitable weak sense on the level of their orthogonality measures. In section 6 we give some concluding remarks on the methods presented in this paper.

2. PRELIMINARIES

Throughout the paper we assume that q is a real number between 0 and 1. We denote the q-shifted factorials by $\left(a;q\right)_k := \prod_{i=0}^{k-1}(1-aq^i)$ $(k \in \mathbb{N})$, $\left(a;q\right)_0 := 1$

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and $\big(a;q\big)_\infty := \lim_{k\to\infty}\big(a;q\big)_k$ and we use the notation

$$\big(a_1,\ldots,a_r;q\big)_k := \prod_{i=1}^{r}\big(a_i;q\big)_k$$

for products of q-shifted factorials. The basic hypergeometric series of type ${}_{s+1}\phi_s$ are then given by

$$ {}_{s+1}\phi_s\left(\begin{matrix} a_1,\ldots,a_{s+1} \\ b_1,\ldots,b_s \end{matrix}\;;q,z\right) = \sum_{m=0}^{\infty}\frac{(a_1,\ldots,a_{s+1};q)_m}{(b_1,\ldots,b_s,q;q)_m}z^m. \tag{2.1}$$

Askey and Wilson [AW2] introduced a very general family of basic hypergeometric orthogonal polynomials depending on four parameters a,b,c,d which is nowadays known as the family of Askey-Wilson polynomials. In terms of the basic hypergeometric series (2.1) they are given by

$$P_n^{AW}(z;a,b,c,d) := a^{-n}\big(ab,ac,ad;q\big)_n {}_4\phi_3\left(\begin{matrix} q^{-n},\ q^{n-1}abcd,\ az,\ az^{-1} \\ [0.5ex]ab,\ ac,\ ad \end{matrix}\;;q,q\right)$$

for $n\in\mathbb{Z}_+$. Then $P_n^{AW}(z)$ is a polynomial in $z+z^{-1}$ of degree n and the corresponding monic polynomial in $z+z^{-1}$ is given by

$$p_n^{AW}(z;a,b,c,d) := \big(abcdq^{n-1};q\big)_n^{-1}P_n^{AW}(z;a,b,c,d).$$

The orthogonality relations and norm evaluations for the monic Askey-Wilson polynomials can be stated as follows.

Theorem 2.1. *(*[AW2, Theorem 2.3]*) Assume that pairwise products of a,b,c,d as a multiset (so both a^2 and ab are considered among the products) do not belong to the set $\{q^{-j}\}_{j\in\mathbb{Z}_+}$. Then the monic Askey-Wilson polynomials satisfy the orthogonality relations*

$$\frac{1}{2\pi\sqrt{-1}}\int_{z\in C}\big(p_m^{AW}p_n^{AW}\big)(z;a,b,c,d;q)\Delta_c^{AW}(z;a,b,c,d)\frac{dz}{z} = \delta_{m,n}\mathcal{N}^{AW}(n;a,b,c,d)$$

with weight function

$$\Delta_c^{AW}(z;a,b,c,d) := \frac{\big(z^2,z^{-2};q\big)_\infty}{\big(az,az^{-1},bz,bz^{-1},cz,cz^{-1},dz,dz^{-1};q\big)_\infty}.$$

Here C is a positively oriented, continuous differentiable Jordan curve containing 0 and the four sequences $\{eq^j\}_{j\in\mathbb{Z}_+}$ ($e=a,b,c,d$) and seperating them from $\{e^{-1}q^{-j}\}_{j\in\mathbb{Z}_+}$ ($e=a,b,c,d$). The quadratic norms $\mathcal{N}^{AW}(n)$ of the monic Askey-Wilson polynomials are explicitly given by

$$\mathcal{N}^{AW}(n;a,b,c,d) = \frac{2\big(q^{2n-1}abcd,q^{2n}abcd;q\big)_\infty}{\big(q^{n+1},q^{n-1}abcd,q^nab,q^nac,q^nad,q^nbc,q^nbd,q^ncd;q\big)_\infty}.$$

For the proof of the orthogonality relations and norm evaluations, Askey and Wilson [AW2] used the q-Pfaff-Saalschütz sum [AW2, (1.29)], [GR, (II.12), p. 237] and the explicit evaluation of the integral over the weight function,

$$\frac{1}{2\pi\sqrt{-1}}\int_{z\in C}\Delta_c^{AW}(z;a,b,c,d)\frac{dz}{z} = \frac{2\big(abcd;q\big)_\infty}{\big(q,ab,ac,ad,bc,bd,cd;q\big)_\infty} \tag{2.2}$$

(cf. [AW2, Theorem 2.1]). The integral (2.2) is a q-analogue of the classical beta integral and its evaluation is proved in [AW2] by summing up four sequences of residues by a summation formula of a very-well poised ${}_6\phi_5$ series [AW2, (2.2)], [GR, (II.20), p.238] and subsequently summing the four remaining terms with the help of an elliptic function identity. More elementary proofs of (2.2) were obtained, for instance, in [R], [IS] and [K3].

A partially discrete, partially continuous orthogonality measure can be obtained by deforming C over some of the poles of Δ_c^{AW} and picking up their residues. The poles of Δ_c^{AW} are simple for generic parameters $a, b, c, d \neq 0$ and are given by the eight sequences $\{eq^j\}_{j\in\mathbb{Z}_+}$, $\{e^{-1}q^{-j}\}_{j\in\mathbb{Z}_+}$ $(e = a, b, c, d)$. We write

$$\Delta_d^{AW}(eq^i; e; f, g, h) := \mathrm{res}_{z=eq^i}\Big(\frac{\Delta_c^{AW}(z; a, b, c, d)}{z}\Big) \tag{2.3}$$

for the residues, where f, g, h are such that $\{e, f, g, h\} = \{a, b, c, d\}$ (counted with multiplicity). When Δ_c^{AW} has a simple pole in eq^i, then

$$\mathrm{res}_{z=e^{-1}q^{-i}}\Big(\frac{\Delta_c^{AW}(z; a, b, c, d)}{z}\Big) = -\Delta_d^{AW}(eq^i; e; f, g, h) \tag{2.4}$$

by the invariance of $\Delta_c^{AW}(z)$ under the transformation $z \mapsto z^{-1}$, and we have the explicit formula

$$\begin{aligned}\Delta_d^{AW}(eq^i; e; f, g, h) := {} & \frac{\big(e^{-2}; q\big)_\infty}{\big(q, ef, f/e, eg, g/e, eh, h/e; q\big)_\infty} \\ & \times \frac{\big(e^2, ef, eg, eh; q\big)_i}{\big(q, qe/f, qe/g, qe/h; q\big)_i} \frac{(1 - e^2q^{2i})}{(1 - e^2)} \left(\frac{q}{efgh}\right)^i\end{aligned} \tag{2.5}$$

(cf. [AW2, Theorem 2.4] with a slight correction in [AW2, (2.10)]).

We end this section with introducing the q-Racah polynomials, big and little q-Jacobi polynomials as limit cases of the Askey-Wilson polynomials. The monic q-Racah polynomials $\{p_n^{qR}(.; a, b, c, N; q)\}_{n=0}^N$ for $N \in \mathbb{N}$ may be considered as limit case of the monic Askey-Wilson polynomials by sending d to $b^{-1}q^{-N}$,

$$p_n^{qR}(z; a, b, c; N) := p_n^{AW}(z; a, b, c, b^{-1}q^{-N}). \tag{2.6}$$

Note that for $d = b^{-1}q^{-N}$, the parameters do no longer satisfy the assumptions of Theorem 2.1.

The monic little q-Jacobi polynomials $\{p_n^L(.; a, b)\}_{n\in\mathbb{Z}_+}$ can be considered as limit cases of the monic Askey-Wilson polynomials by substituting

$$\underline{t}_L(\epsilon) := (\epsilon q^{\frac{1}{2}}b, \epsilon^{-1}q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}a) \tag{2.7}$$

for the four variables of the Askey-Wilson polynomials, rescaling of the z-variable, and taking the limit $\epsilon \downarrow 0$,

$$p_n^L(z; a, b) := \lim_{\epsilon\downarrow 0}\Big(\epsilon q^{-\frac{1}{2}}\Big)^n p_n^{AW}\Big(\epsilon^{-1}q^{\frac{1}{2}}z; \underline{t}_L(\epsilon)\Big) \tag{2.8}$$

$$= \frac{\big(qb; q\big)_n}{(qb)^n\big(q^{n+1}ab; q\big)_n}\,{}_3\phi_2\left(\begin{array}{c} q^{-n},\ q^{n+1}ab,\ qbz \\ {[0.5ex]}qb,\ 0 \end{array}; q, q\right) \tag{2.9}$$

$$= \frac{(-1)^n q^{\binom{n}{2}}\big(qa; q\big)_n}{\big(q^{n+1}ab; q\big)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},\ q^{n+1}ab, \\ {[0.5ex]}qa \end{array}; q, qz\right) \tag{2.10}$$

(cf. [K2, Proposition 6.3] and take into account that the Askey-Wilson polynomials used in [K2] are written as functions of $(z + z^{-1})/2$ and are normalized differently).

In fact, an easy calculation yields

(2.11)
$$\begin{aligned}\big(\epsilon q^{-\frac{1}{2}}\big)^n p_n^{AW}\big(\epsilon^{-1}q^{\frac{1}{2}}z;\underline{t}_L(\epsilon)\big) &= \frac{\big(qb;q\big)_n}{(qb)^n\big(q^{n+1}ab;q\big)_n}\sum_{m=0}^{n}\frac{\big(q^{-n},q^{n+1}ab;q\big)_m}{\big(q,qb;q\big)_m}q^m\\ &\quad\times\big(-\epsilon q^{m+1}b,-\epsilon q^{m+1}ab;q\big)_{n-m}\\ &\quad\times\prod_{i=0}^{m-1}\Big((1+\epsilon^2b^2q^{2i+1})-q^{i+1}b\epsilon q^{-\frac{1}{2}}h_1(\epsilon^{-1}q^{\frac{1}{2}}z)\Big)\end{aligned}$$

with $h_1(z) := z+z^{-1}$, so (2.9) follows directly from the the observation that $\lim_{\epsilon\downarrow 0}\big(u\epsilon;q\big)_m = 1$ and

$$\lim_{u\downarrow 0} uh_1(u^{-1}z) = z. \tag{2.12}$$

A transformation formula for terminating ${}_2\phi_1$ series [GR, (III.7), p. 241] yields (2.10) and shows that the little q-Jacobi polynomials are also defined for $b=0$. The little q-Jacobi polynomial $p_n^L(z;a,b)$ is a monic polynomial of degree n in the variable z. So in the limit (2.8) we go from a polynomial in $z+z^{-1}$ to a polynomial in z. This can be made more transparent as follows. Expand p_n^{AW} in powers of $z+z^{-1}$,

$$p_n^{AW}(z;a,b,c,d) = \sum_{r=0}^{n} c_{n,r}^{AW}(a,b,c,d)h_r(z) \qquad (c_{n,n}^{AW}=1)$$

with $h_r(z) := \big(h_1(z)\big)^r = (z+z^{-1})^r$. Then (2.12) extends to the limit

$$\lim_{u\downarrow 0} u^r h_r(u^{-1}z) = z^r \qquad (r\in\mathbb{N}) \tag{2.13}$$

so by (2.11) and (2.13) we conclude that

$$p_n^L(z;a,b) = \sum_{r=0}^{n} c_{n,r}^L(a,b)z^r$$

with

$$\lim_{\epsilon\downarrow 0}\big(\epsilon q^{-\frac{1}{2}}\big)^{n-r}c_{n,r}^{AW}(\underline{t}_L(\epsilon)) = c_{n,r}^L(a,b). \tag{2.14}$$

The monic big q-Jacobi polynomials $\{p_n^B(.;a,b,c,d)\}_{n\in\mathbb{Z}_+}$ may be considered as limit cases of the monic Askey-Wilson polynomials by substituting

$$\underline{t}_B(\epsilon) := (\epsilon a(qd/c)^{\frac{1}{2}},\epsilon^{-1}(qc/d)^{\frac{1}{2}},-\epsilon^{-1}(qd/c)^{\frac{1}{2}},-\epsilon b(qc/d)^{\frac{1}{2}}) \tag{2.15}$$

for the four variables of the Askey-Wilson polynomials, rescaling of the z-variable, and taking the limit $\epsilon\downarrow 0$:

(2.16)
$$\begin{aligned}p_n^B(z;a,b,c,d) &:= \lim_{\epsilon\downarrow 0}\Big(\epsilon(cd/q)^{\frac{1}{2}}\Big)^n p_n^{AW}\Big(\epsilon^{-1}(q/cd)^{\frac{1}{2}}z;\underline{t}_B(\epsilon)\Big)\\ &= \frac{\big(qa,-qad/c;q\big)_n}{\big(q^{n+1}ab;q\big)_n(qa/c)^n}\,{}_3\phi_2\left(\begin{matrix}q^{-n},\ q^{n+1}ab,\ qza/c\\ \text{[0.5ex]}qa,\ -qad/c\end{matrix};q,q\right)\end{aligned}$$

(cf. [K2, Proposition 6.1]). Note that $p_n^B(z;a,b,c,d)$ is a monic polynomial of degree n in the variable z. Similarly as in the little q-Jacobi case, we have

$$p_n^B(z;a,b,c,d) = \sum_{r=0}^{n} c_{n,r}^B(a,b,c,d)z^r$$

with

$$c^B_{n,r}(a,b,c,d)=\lim_{\epsilon\downarrow 0}\bigl(\epsilon(cd/q)^{\frac{1}{2}}\bigr)^{n-r}c^B_{n,r}(\underline{t}_B(\epsilon)). \tag{2.17}$$

3. Limit to q-Racah polynomials.

The orthogonality relations and norm evaluations for the monic q-Racah polynomials can be stated as follows.

Theorem 3.1. *(*[AW1, section 2]*) Let $N\in\mathbb{N}$. For generic parameters a,b,c we have the orthogonality relations*

$$\sum_{i=0}^{N}\bigl(p^{qR}_m p^{qR}_n\bigr)(bq^i;a,b,c;N)\Delta^{qR}(bq^i;b;a,c,b^{-1}q^{-N})=\delta_{m,n}\mathcal{N}^{qR}(n;b;a,c,b^{-1}q^{-N})$$

for $m,n\in\{0,\ldots,N\}$, with

$$\Delta^{qR}(bq^i;b;a,c,d):=\frac{(1-b^2q^{2i})\bigl(ab,b^2,bc,bd;q\bigr)_i}{\bigl(abcdq^{-1}\bigr)^i(1-b^2)\bigl(q,qa^{-1}b,qc^{-1}b,qd^{-1}b;q\bigr)_i}.$$

The quadratic norms of the monic q-Racah polynomials are explicitly given by

$$\mathcal{N}^{qR}(n;b;a,c,d):=\frac{\bigl(q,ab,ac,ad,bc,bd,cd;q\bigr)_n}{\bigl(q^{n-1}abcd;q\bigr)_n\bigl(abcd;q\bigr)_{2n}}\frac{\bigl(a/b,c/b,d/b,abcd;q\bigr)_\infty}{\bigl(ac,ad,cd,b^{-2};q\bigr)_\infty}.$$

Proof. In view of continuity we may take as generic conditions on the parameters a,b,c that $a,b,c\in\mathbb{C}\setminus\{0\}$ and that the 6 arguments $\arg(e),\arg(e^{-1})\in[0,2\pi)$ $(e=a,b,c)$ are mutually different. Let $d\in\mathbb{C}\setminus\{0\}$ be such that the 8 arguments $\arg(e),\arg(e^{-1})$ $(e=a,b,c,d)$ are mutually different. Then the poles of $\Delta^{AW}_c(z;a,b,c,d)$ are simple and the conditions of Theorem 2.1 are satisfied. The residue Δ^{AW}_d (2.3) at $z=bq^i$ can then be written as

$$\Delta^{AW}_d(bq^i;b;a,c,d)=K(b;a,c,d)\Delta^{qR}(bq^i;b;a,c,d)$$

with $K(b;a,c,d)$ given by

$$K(b;a,c,d)=\frac{\bigl(b^{-2};q\bigr)_\infty}{\bigl(q,ab,a/b,bc,c/b,bd,d/b;q\bigr)_\infty} \tag{3.1}$$

in view of (2.5). The factor $K(b;a,c,d)$ is non zero and independent of i. By Cauchy's Theorem and (2.4) we obtain

$$\begin{aligned}\sum_{i=0}^{N}\bigl(p^{AW}_m p^{AW}_n\bigr)(bq^i;a,b,c,d)\Delta^{qR}(bq^i;b;a,c,d)&=\frac{\mathcal{N}^{AW}(n;a,b,c,d)}{2K(b;a,c,d)}\delta_{n,m}\\ -\bigl(K(b;a,c,d)\bigr)^{-1}\frac{1}{4\pi\sqrt{-1}}\int_{z\in C}\bigl(p^{AW}_m p^{AW}_n\bigr)&(z;a,b,c,d)\Delta^{AW}_c(z;a,b,c,d)\frac{dz}{z}\end{aligned} \tag{3.2}$$

where C is a positively oriented, continuous differentiable Jordan curve containing 0 together with the sequences $\{bq^{N+1+j}\}_{j\in\mathbb{Z}_+}$, $\{xq^j\}_{j\in\mathbb{Z}_+}$ $(x=a,c,d)$, $\{b^{-1}q^{-j}\}_{j=0}^N$ and seperating them from the sequences $\{b^{-1}q^{-N-1-j}\}_{j\in\mathbb{Z}_+}$, $\{x^{-1}q^{-j}\}_{j\in\mathbb{Z}_+}$ $(x=a,c,d)$ and $\{bq^j\}_{j=0}^N$. Consider a sequence $\{d_k\}_{k\in\mathbb{Z}_+}$ converging to $b^{-1}q^{-N}$ such that the 8 arguments $\arg(e),\arg(e^{-1})$ $(e=a,b,c,d_k)$ are mutually different for all k. Then the limit

$$\lim_{k\to\infty}\int_{z\in C}\bigl(p^{AW}_m p^{AW}_n\bigr)(z;a,b,c,d_k)\Delta^{AW}_c(z;a,b,c,d_k)\frac{dz}{z}$$

exists, since it equals

$$\int_{z\in C}\bigl(p_m^{qR}p_n^{qR}\bigr)(z;a,b,c;N)\Delta_c^{AW}(z;a,b,c,b^{-1}q^{-N})\frac{dz}{z}$$

by the Bounded Convergence Theorem, compactness of C and by (2.6). For the constant $K(b;a,c,d)$, we have

$$\lim_{k\to\infty}\bigl(K(b;a,c,d_k)\bigr)^{-1}=0$$

because of the factor $\bigl(bd;q\bigr)_\infty$ in the denominator of $K(b;a,c,d)$. Since

$$\frac{\mathcal{N}^{AW}(n;a,b,c,d)}{2K(b;a,c,d)}=\mathcal{N}^{qR}(n;b;a,c,d),$$

the theorem follows by taking the limit $d\to b^{-1}q^{-N}$ at both sides of (3.2) along the sequence $\{d_k\}_{k\in\mathbb{Z}_+}$. □

In other words, the continuous part of the orthogonality measure vanishes in the limit from Askey-Wilson polynomials to q-Racah polynomials because the residues Δ_d^{AW} at $z=bq^i$ $(i=0,\ldots,N)$ contain a common factor which blows up in the limit $d\to b^{-1}q^{-N}$.

Askey and Wilson [AW1] obtained the orthogonality relations and norm evaluations for the q-Racah polynomials from a summation formula for very well poised terminating ${}_6\phi_5$ series [AW1, (2.3)], [GR, (II.21),p.238] and the q-Pfaff-Saalschütz sum [AW1, (2.5)], [GR, (II.12), p.237]. In particular, they obtained the summation formula

$$\sum_{i=0}^{N}\Delta^{qR}(bq^i;b;a,c,b^{-1}q^{-N})=\frac{\bigl(qb^2,q/ac;q\bigr)_N}{\bigl(qb/a,qb/c;q\bigr)_N}\tag{3.3}$$

using a summation formula for very well poised terminating ${}_6\phi_5$ series [AW1, (2.3)], [GR, (II.21),p.238].

4. Limit to little q-Jacobi polynomials.

Let V_{AW} be the set of parameters (a,b,c,d) which are real or appear in conjugate pairs, and which satisfy the additional conditions that the pairwise products $ab,ac,ad,bc,bd,cd\notin\mathbb{R}_{\geq 1}:=\{x\in\mathbb{R}\,|\,x\geq 1\}$. If $(a,b,c,d)\in V_{AW}$, then there are at most two parameters with modulus >1. Parameters with moduli >1 are then necessarily real, and if two parameters have moduli >1 then they have opposite sign. For parameters $(a,b,c,d)\in V_{AW}$, the polynomials p_n^{AW} are orthogonal with respect to a (partly continuous, partly discrete) positive measure,

$$\langle p_n^{AW}(.;a,b,c,d),p_m^{AW}(.;a,b,c,d)\rangle_{AW}^{a,b,c,d}=\delta_{m,n}\mathcal{N}^{AW}(n;a,b,c,d),\tag{4.1}$$

where

$$\begin{aligned}\langle f,g\rangle_{AW}^{a,b,c,d}:=&\frac{1}{2\pi\sqrt{-1}}\int_{z\in T}f(z)g(z)\Delta_c^{AW}(z;a,b,c,d)\frac{dz}{z}\\&+2\sum_{\substack{i=0,\ldots,N_e\\e=a,b,c,d}}f(eq^i)g(eq^i)\Delta_d^{AW}(eq^i;e;f,g,h).\end{aligned}\tag{4.2}$$

Here T is the unit circle in the complex plane traversed in the counterclockwise direction, $\{e,f,g,h\}=\{a,b,c,d\}$ (counted with multiplicity) and $N_e=-1$ if $|e|\leq 1$, respectively N_e is the largest positive integer such that $|eq^{N_e}|>1$ if $|e|>1$.

We use here the convention that sums over empty sets are zero, so the sum in the right hand side of (4.2) is over parameters e with modulus > 1 only. The orthogonality relations and norm evaluations (4.1) follow from Theorem 2.1, (2.3), (2.4), Cauchy's Theorem and by a continuity argument in the parameters (see [AW2, Theorem 2.4]). In fact, the orthogonality relations and norm evaluations (4.1) hold for generic parameter values a, b, c, d, with $\langle .,. \rangle_{AW}^{a,b,c,d}$ given by (4.2).

We will obtain the orthogonality relations and norm evaluations for the little q-Jacobi polynomials by taking suitable limits in the orthogonality relations and norm evaluations (4.1). We will need some elementary limits and estimates involving q-shifted factorials, which we collect in the following lemma.

Lemma 4.1. *For given* $\epsilon_0 \in \mathbb{R}$*, we set* $\epsilon_k := \epsilon_0 q^k$*.*
(a) *Let* $c \in \mathbb{C}$*. For* $\epsilon_0 > 0$ *with* $|c|\epsilon_0 \notin \{q^{-l}\}_{l\in\mathbb{Z}_+}$ *there exist positive constants* $M^\pm > 0$ *which only depend on* ϵ_0 *and* $|c|$*, such that* $M^- \leq |(c\epsilon_k;q)_\infty| \leq M^+$ *for all* $k \in \mathbb{Z}_+$*. Furthermore, we have* $\lim_{k\to\infty}(c\epsilon_k;q)_\infty = 1$*.*
(b) *Let* $a, b \in \mathbb{C}\setminus\{0\}$*, and set*

$$f_{\{l,m\}}(\epsilon;a,b) := \frac{(\epsilon^{-1}aq^{1-m};q)_m}{(\epsilon^{-1}bq^{1-l-m};q)_m}, \qquad l,m \in \mathbb{Z}_+.$$

Let $\epsilon_0 > 0$ *such that* $\epsilon_0^{-1}|b| \notin \{q^k\}_{k\in\mathbb{Z}_+}$*. Then there exists a positive constant* $M > 0$ *which depends only on* ϵ_0*,* $|a|$ *and* $|b|$*, such that* $|f_{\{l,m\}}(\epsilon_k;a,b)| \leq M|q^l a/b|^m$ *for all* $k, l, m \in \mathbb{Z}_+$*. Furthermore, we have* $\lim_{k\to\infty} f_{\{l,m\}}(\epsilon_k;a,b) = (q^l a/b)^m$*.*
(c) *Let* $u_i, v_j \in \mathbb{C}\setminus\{0\}$ *for* $i \in \{1,\ldots,r\}$*,* $j \in \{1,\ldots,s\}$ *and assume that* $r < s$*, or that* $r = s$ *and* $|u_1\ldots u_r| < |v_1\ldots v_r|$*. Set*

$$g(\epsilon) := \frac{(\epsilon^{-1}u_1,\ldots,\epsilon^{-1}u_r;q)_\infty}{(\epsilon^{-1}v_1,\ldots,\epsilon^{-1}v_s;q)_\infty}.$$

Let $\epsilon_0 > 0$ *such that* $\epsilon_0^{-1}|v_j| \notin \{q^l\}_{l\in\mathbb{Z}}$ *for* $j \in \{1,\ldots,s\}$*. Then there exists a positive constant* $M > 0$ *which depends only on* ϵ_0*,* $|u_i|$ *and* $|v_j|$*, such that* $\sup_{k\in\mathbb{Z}_+}|g(\epsilon_k)| \leq M$*. Furthermore, we have* $\lim_{k\to\infty} g(\epsilon_k) = 0$*.*

Proof. The proof of **(a)** is straightforward. For **(b)** and **(c)** use the formula

$$(xq^{1-m};q)_m = q^{-\binom{m}{2}}(-x)^m(x^{-1};q)_m \tag{4.3}$$

for q-shifted factorials to rewrite $f_{\{l,m\}}$ as

$$f_{\{l,m\}}(\epsilon;a,b) = (q^l a/b)^m \frac{(a^{-1}\epsilon;q)_m}{(b^{-1}q^l\epsilon;q)_m},$$

and to rewrite $g(\epsilon_k)$ as

$$g(\epsilon_k) = \left(\frac{u_1\ldots u_r}{v_1\ldots v_s}\left(-q^{(k+1)/2}\epsilon_0\right)^{s-r}\right)^k \frac{(q\epsilon_0u_1^{-1},\ldots,q\epsilon_0u_r^{-1};q)_k}{(q\epsilon_0v_1^{-1},\ldots,q\epsilon_0v_s^{-1};q)_k} g(\epsilon_0). \tag{4.4}$$

The limits for $f_{\{l,m\}}$ and g given in **(b)** respectively **(c)** are now immediately clear. Furthermore we have the estimate $|f_{\{l,m\}}(\epsilon_k;a,b)| \leq M|q^l a/b|^m$ with

$$M = \frac{(-|a|^{-1}\epsilon_0;q)_\infty}{(|b|^{-1}\epsilon_0q^{k_0};q)_\infty} \prod_{\{i\in\mathbb{Z}_+ \,|\, 1<|b|^{-1}\epsilon_0q^i<2\}} \left(|b|^{-1}\epsilon_0q^i - 1\right)^{-1} > 0,$$

where k_0 is the smallest positive integer such that $|b|^{-1}\epsilon_0 q^{k_0} < 1$. Here we use the convention that an empty product is equal to 1. The estimate for $|g(\epsilon_k)|$ in **(c)** is easily derived from (4.4), the assumptions on r, s and on the parameters u_i, v_j, and from estimates similar to the estimate for M in the proof of **(b)**. □

For the formulation of the orthogonality relations and norm evaluations of the little q-Jacobi polynomials we use the definition of the Jackson q-integral and the q-gamma function. The Jackson q-integral of a (continuous) function f over an interval $[u,v]$ is defined by

$$\begin{aligned}\int_u^v f(x)d_qx &:= \int_0^v f(x)d_qx - \int_0^u f(x)d_qx,\\ \int_0^v f(x)d_qx &:= (1-q)\sum_{i=0}^{\infty} f(vq^i)vq^i.\end{aligned}$$

When $q \uparrow 1$, the q-integral of f becomes the usual Lebesgue integral of f over the interval $[u,v]$. The q-gamma function $\Gamma_q(z)$ is defined by

$$\Gamma_q(z) := \frac{\big(q;q\big)_\infty}{\big(q^z;q\big)_\infty}(1-q)^{1-z}, \quad z \notin -\mathbb{Z}_+. \tag{4.5}$$

The q-gamma function $\Gamma_q(z)$ tends to the gamma function $\Gamma(z)$ when $q \uparrow 1$.

The orthogonality relations and norm evaluations for the monic little q-Jacobi polynomials can now be stated as follows.

Theorem 4.2. *(*[AA1, Theorem 9]*) Let* $0 < a < 1/q$ *and* $b < 1/q$. *Then*

$$\int_0^1 \big(p_m^L p_n^L\big)(z;a,b)\Delta^L(z;a,b)d_qz = \delta_{m,n}\mathcal{N}^L(n;a,b),$$

with

$$\Delta^L(z;a,b) := \frac{\big(qz;q\big)_\infty}{\big(qbz;q\big)_\infty} z^\alpha \qquad (a = q^\alpha).$$

The quadratic norms $\mathcal{N}^L(n)$ *of the monic little q-Jacobi polynomials are explicitly given by*

$$\mathcal{N}^L(n;a,b) = \frac{\Gamma_q(n+1)\Gamma_q(n+1+\alpha)\Gamma_q(n+1+\beta)\Gamma_q(n+1+\alpha+\beta)}{\Gamma_q(2n+1+\alpha+\beta)\Gamma_q(2n+2+\alpha+\beta)} q^{(n+\alpha)n},$$

where $b = q^\beta$.

Proof. We assume throughout the proof that $b \neq 0$. At the end of the proof we can remove this assumption by continuity. For given $\epsilon_0 \in \mathbb{R}$, we set $\epsilon_k := \epsilon_0 q^k$. We claim that there exists an $\epsilon_0 > 0$ such that

$$\begin{aligned}&\lim_{k\to\infty}\big(-\epsilon_k^{-1}q, -\epsilon_k^{-1}qa;q\big)_\infty \big(\epsilon_k q^{-\frac{1}{2}}\big)^{m+n}\langle h_m, h_n\rangle_{AW}^{t_L(\epsilon_k)}\\ &\qquad = 2\big(q;q\big)_\infty^{-2}(1-q)^{-1}\int_0^1 z^{m+n}\Delta^L(z;a,b)d_qz\end{aligned} \tag{4.6}$$

for all $m,n\in\mathbb{Z}_+$, where $h_r(z):=(z+z^{-1})^r$ and $\underline{t}_L(\epsilon)$ is given by (2.7). Then we obtain from (2.13), (2.14) and (4.6),

$$\begin{aligned}
\lim_{k\to\infty}\big(-\epsilon_k^{-1}q,-\epsilon_k^{-1}qa;q\big)_\infty\big(\epsilon_kq^{-\frac{1}{2}}\big)^{m+n}\langle p_m^{AW},p_n^{AW}\rangle_{AW}^{\underline{t}_L(\epsilon_k)}&\\
=\sum_{r,s}\lim_{k\to\infty}\Big\{\big(\epsilon_kq^{-\frac{1}{2}}\big)^{m-r+n-s}\big(c_{m,r}^{AW}c_{n,s}^{AW}\big)\big(\underline{t}_L(\epsilon_k)\big)&\\
\times\big(-\epsilon_k^{-1}q,-\epsilon_k^{-1}qa;q\big)_\infty\big(\epsilon_kq^{-\frac{1}{2}}\big)^{r+s}\langle h_r,h_s\rangle_{AW}^{\underline{t}_L(\epsilon_k)}\Big\}&\\
=2\big(q;q\big)_\infty^{-2}(1-q)^{-1}\sum_{r,s}\big(c_{m,r}^{L}c_{n,s}^{L}\big)(a,b)\int_0^1 z^{r+s}\Delta^L(z;a,b)d_qz&\\
=2\big(q;q\big)_\infty^{-2}(1-q)^{-1}\int_0^1\big(p_m^Lp_n^L\big)(z;a,b)\Delta^L(z;a,b)d_qz&
\end{aligned}$$

where the sum is over $r\in\{0,\ldots,m\}$ and $s\in\{0,\ldots,n\}$. On the other hand, a straightforward calculation gives

$$\begin{aligned}
\lim_{k\to\infty}\big(-\epsilon_k^{-1}q,-\epsilon_k^{-1}qa;q\big)_\infty\big(\epsilon_kq^{-\frac{1}{2}}\big)^{2n}\mathcal{N}^{AW}\big(n;\underline{t}_L(\epsilon_k)\big)&\\
=2\big(q;q\big)_\infty^{-2}(1-q)^{-1}\mathcal{N}^L(n;a,b),&
\end{aligned}$$

hence the theorem follows from (4.6) and from the orthogonality relations and norm evaluations (4.1) for the Askey-Wilson polynomials. So it remains to prove that there exists an $\epsilon_0>0$ such that (4.6) is valid for all $m,n\in\mathbb{Z}_+$. Note that the modulus of the parameter $\epsilon^{-1}q^{\frac{1}{2}}$ in $\underline{t}_L(\epsilon)$ blows up for $\epsilon\downarrow 0$, so it contributes to the discrete part of the symmetric form $\langle .,.\rangle_{AW}^{\underline{t}_L(\epsilon)}$. The parameter $-aq^{\frac{1}{2}}$ in $\underline{t}_L(\epsilon)$ gives rise to a discrete term in $\langle .,.\rangle_{AW}^{\underline{t}_L(\epsilon)}$ if $q^{-\frac{1}{2}}<a<q^{-1}$. So for $\epsilon>0$ sufficiently small, we obtain from (2.4), (2.5) and (4.2),

$$\begin{aligned}
\big(-\epsilon^{-1}q,-\epsilon^{-1}qa;q\big)_\infty\big(\epsilon q^{-\frac{1}{2}}\big)^{m+n}\langle h_m,h_n\rangle_{AW}^{\underline{t}_L(\epsilon)}&\\
=\frac{1}{2\pi\sqrt{-1}}\int_T\big(\epsilon q^{-\frac{1}{2}}\big)^{m+n}h_m(z)h_n(z)\tilde{\Delta}_c^{AW}(z;\epsilon)\frac{dz}{z}&\\
+2\sum_{i=0}^{\infty}\big(\epsilon q^{-\frac{1}{2}}\big)^{m+n}\big(h_mh_n\big)\big(\epsilon^{-1}q^{\frac{1}{2}}q^i\big)\tilde{\Delta}_{d,1}^{AW}(i;\epsilon)&\\
+2\chi\big(a>q^{-\frac{1}{2}}\big)\big(\epsilon q^{-\frac{1}{2}}\big)^{m+n}\big(h_mh_n\big)\big(-aq^{\frac{1}{2}}\big)\tilde{\Delta}_{d,2}^{AW}(\epsilon)&
\end{aligned}\tag{4.7}$$

where $\chi(A)$ is 1 if A is true and 0 if A is false. Here $\tilde{\Delta}_c^{AW}$ is given by

$$\begin{aligned}
\tilde{\Delta}_c^{AW}(z;\epsilon)&=\big(-\epsilon^{-1}q,-\epsilon^{-1}qa;q\big)_\infty\Delta_c^{AW}\big(z;\underline{t}_L(\epsilon)\big)\\
&=\frac{\big(-\epsilon^{-1}q,-\epsilon^{-1}qa;q\big)_\infty}{\big(\epsilon^{-1}q^{\frac{1}{2}}z,\epsilon^{-1}q^{\frac{1}{2}}z^{-1};q\big)_\infty}\\
&\times\frac{\big(z^2,z^{-2};q\big)_\infty}{\big(\epsilon q^{\frac{1}{2}}bz,\epsilon q^{\frac{1}{2}}bz^{-1},-q^{\frac{1}{2}}z,-q^{\frac{1}{2}}z^{-1},-q^{\frac{1}{2}}az,-q^{\frac{1}{2}}az^{-1};q\big)_\infty},
\end{aligned}$$

and the discrete weights are given by

$$\begin{aligned}\tilde{\Delta}_{d,1}^{AW}(i;\epsilon)&=\big(-\epsilon^{-1}q,-\epsilon^{-1}qa;q\big)_{\infty}\Delta_d^{AW}\big(\epsilon^{-1}q^{\frac{1}{2}+i};\epsilon^{-1}q^{\frac{1}{2}};\epsilon q^{\frac{1}{2}}b,-q^{\frac{1}{2}},-q^{\frac{1}{2}}a\big)\\ &=\frac{\big(\epsilon^2q^{-1};q\big)_{\infty}}{\big(q,qb,\epsilon^2b,-\epsilon,-a\epsilon;q\big)_{\infty}}\frac{\big(\epsilon^{-2}q,-\epsilon^{-1}qa,qb;q\big)_i}{\big(\epsilon^{-2}qb^{-1},-\epsilon^{-1}qa^{-1},q;q\big)_i}\frac{\big(\epsilon^{-2}q^{2i+1};q\big)_1}{\big(\epsilon^{-2}q;q\big)_1}\big(qab\big)^{-i}\end{aligned}$$

if $\epsilon < q^{\frac{1}{2}+i}$ and zero otherwise, and

$$\begin{aligned}\tilde{\Delta}_{d,2}^{AW}(\epsilon)&=\big(-\epsilon^{-1}q,-\epsilon^{-1}qa;q\big)_{\infty}\Delta_d^{AW}\big(-q^{\frac{1}{2}}a;-q^{\frac{1}{2}}a;\epsilon^{-1}q^{\frac{1}{2}},\epsilon q^{\frac{1}{2}}b,-q^{\frac{1}{2}}\big)\\ &=\frac{\big(-\epsilon^{-1}q,q^{-1}a^{-2};q\big)_{\infty}}{\big(-\epsilon^{-1}a^{-1},q,-\epsilon qab,-\epsilon ba^{-1},qa,a^{-1};q\big)_{\infty}}.\end{aligned}$$

Since $a\in(0,1/q)$, we have by Lemma 4.1 **(a)** and **(c)** that

$$\lim_{k\to\infty}\tilde{\Delta}_c^{AW}(z;\epsilon_k)=0\qquad(z\in T)$$

and

$$\sup_{k\in\mathbb{Z}_+,z\in T}|\tilde{\Delta}_c^{AW}(z;\epsilon_k)|<\infty,$$

for generic $\epsilon_0>0$. So by the Bounded Convergence Theorem,

$$\lim_{k\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_{z\in T}\big(\epsilon_kq^{-\frac{1}{2}}\big)^{m+n}h_m(z)h_n(z)\tilde{\Delta}_c^{AW}(z;\epsilon_k)\frac{dz}{z}=0 \tag{4.8}$$

for generic $\epsilon_0>0$. Since $a^{-1}>q$, we obtain by Lemma 4.1 **(a)** and **(c)** the limit

$$\lim_{k\to\infty}\tilde{\Delta}_{d,2}^{AW}(\epsilon_k)=0 \tag{4.9}$$

for generic $\epsilon_0>0$. For the sum of the infinite discrete sequence in (4.7) we have for $\epsilon_0>0$ generic,

$$\lim_{k\to\infty}\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k)=\big(q;q\big)_{\infty}^{-2}\Delta^L(q^i;a,b)q^i \tag{4.10}$$

for all $i\in\mathbb{Z}_+$. The limit (4.10) can for instance be checked using Lemma 4.1 **(a)** and **(b)**. As an example, let us calculate the limit $k\to\infty$ of the factor $\big(\epsilon_k^{-2}q;q\big)_i/\big(\epsilon_k^{-2}qb^{-1};q\big)_i$ in $\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k)$,

$$\begin{aligned}\lim_{k\to\infty}\big(\epsilon_k^{-2}q;q\big)_i/\big(\epsilon_k^{-2}qb^{-1};q\big)_i&=\lim_{k\to\infty}\big(\epsilon_{k+i}^{-2}q;q\big)_i/\big(\epsilon_{k+i}^{-2}qb^{-1};q\big)_i\\ &=\lim_{k\to\infty}f_{\{0,i\}}\big(\epsilon_{i+2k};\epsilon_0^{-1},\epsilon_0^{-1}b^{-1}\big)=b^i\end{aligned} \tag{4.11}$$

where the last equality follows from Lemma 4.1**(b)**. The limits of the other ϵ-depending factors in $\tilde{\Delta}_{d,1}^{AW}(i;\epsilon)$ can be calculated in a similar way.

Combining (2.13), (4.7), (4.8), (4.9) and (4.10) we obtain for arbitrary $m,n\in\mathbb{Z}_+$,

$$\begin{aligned}\lim_{k\to\infty}&\big(-\epsilon_k^{-1}q,-\epsilon_k^{-1}qa;q\big)_{\infty}\big(\epsilon_kq^{-\frac{1}{2}}\big)^{r+s}\langle h_m,h_n\rangle_{AW}^{t_L(\epsilon_k)}\\ &=2\lim_{k\to\infty}\sum_{i\in\mathbb{Z}_+}\big(\epsilon_kq^{-\frac{1}{2}}\big)^{m+n}\big(h_mh_n\big)(\epsilon_k^{-1}q^{\frac{1}{2}}q^i\big)\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k)\\ &=2\big(q;q\big)_{\infty}^{-2}(1-q)^{-1}\int_0^1z^{m+n}\Delta^L(z;a,b)d_qz\end{aligned} \tag{4.12}$$

provided that we may interchange limit and summation in (4.12). We show that for generic $\epsilon_0>q^{\frac{1}{2}}$ it is allowed to interchange limit and summation in (4.12), which will complete the proof of the theorem. Since the weight $\Delta^L(q^i;a,b)$ are positive and the infinite sum $\sum_{i=0}^{\infty}\Delta^L(q^i;a,b)q^i$ is absolutely convergent, it suffices to prove that for generic $\epsilon_0 > q^{\frac{1}{2}}$ and for $r \in \mathbb{Z}_+$,

$$\sup_{k\in\mathbb{Z}_+} |\big(\epsilon_k q^{-\frac{1}{2}}\big)^r h_r(\epsilon_k^{-1}q^{\frac{1}{2}}q^i)\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k)| \leq M\Delta^L(q^i;a,b)q^i \tag{4.13}$$

for some $M > 0$ independent of $i \in \mathbb{Z}_+$. Since $\tilde{\Delta}_{d,1}^{AW}(i;\epsilon) = 0$ for $\epsilon \geq q^{\frac{1}{2}+i}$, we have for $\epsilon_0 > q^{\frac{1}{2}}$

$$\begin{aligned}\sup_{k\in\mathbb{Z}_+} & |\big(\epsilon_k q^{-\frac{1}{2}}\big)^r h_r(\epsilon_k^{-1}q^{\frac{1}{2}}q^i)\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k)| \\ &= \sup_{k\in\mathbb{Z}_+} |\big(\epsilon_k q^i q^{-\frac{1}{2}}\big)^r h_r(\epsilon_k^{-1}q^{\frac{1}{2}})\tilde{\Delta}_{d,1}^{AW}(i;q^i\epsilon_k)| \\ &\leq M' \sup_{k\in\mathbb{Z}_+} |\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k q^i)|\end{aligned} \tag{4.14}$$

with M' independent of i, and the required estimate (4.13) follows from the estimates of Lemma 4.1 **(a)** and **(b)**. For instance, we have seen that the factor $\big(\epsilon_k^{-2}q;q\big)_i/\big(\epsilon_k^{-2}qb^{-1};q\big)_i$ in $\tilde{\Delta}_{d,1}^{AW}(i;\epsilon_k)$ tends to b^i for $k \to \infty$ (cf. (4.11)). The corresponding estimate, needed for (4.13), is then provided by

$$\sup_{k\in\mathbb{Z}_+} |\big((\epsilon_k q^i)^{-2}q;q\big)_i/\big((\epsilon_k q^i)^{-2}qb^{-1};q\big)_i| = \sup_{k\in\mathbb{Z}_+} |f_{\{0,i\}}\big(\epsilon_{i+2k};\epsilon_0^{-1},\epsilon_0^{-1}b^{-1}\big)| \leq M_1|b|^i$$

with $M_1 > 0$ independent of $i \in \mathbb{Z}_+$, in view of Lemma 4.1**(b)**. Estimates for the other ϵ-depending factors in $\tilde{\Delta}_{d,1}^{AW}(i;\epsilon)$ can be obtained in a similar way. ☐

Note that $\underline{t}_L(\epsilon) \in V_{AW}$ for $\epsilon > 0$ sufficiently small if the parameters a and b satisfy the assumptions of Theorem 4.2 ($\underline{t}_L(\epsilon)$ given by (2.7)). So in the proof of Theorem 4.2 we obtain the positive orthogonality measure for the little q-Jacobi polynomials as limit case of the positive (partly discrete, partly continuous) orthogonality measure (4.2) for the Askey-Wilson polynomials. In particular, the proof of Theorem 4.2 shows that the only part of the (rescaled) orthogonality measure (4.2) which survives in the limit from Askey-Wilson polynomials to little q-Jacobi polynomials (2.8) is a sum of an infinite sequence of discrete weights coming from residues of $\Delta_c^{AW}(z)/z$ at $z = \epsilon^{-1}q^{\frac{1}{2}}q^i$, where $\epsilon^{-1}q^{\frac{1}{2}}$ is the parameter in $\underline{t}_L(\epsilon)$ which tends to infinity in the limit $\epsilon \downarrow 0$. This infinite sequence of weights is, up to a positive constant, exactly the set of weights which occur in the orthogonality measure for the little q-Jacobi polynomials.

The little q-Jacobi polynomials were first observed by Hahn [H]. A detailed discussion of the orthogonality relations and norm evaluations was given by Andrews and Askey [AA1]. The orthogonality relations and norm evaluations were derived from the q-binomial formula [AA1, (3.6)], [GR, (II.3),p.236] and the q-Pfaff-Saalschütz formula [AA1, (3.7)], [GR, (II.12),p.237]. The evaluation of the q-Jackson integral over the weight function

$$\int_0^1 \Delta^L(z;a,b)d_qz = \frac{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}{\Gamma_q(2+\alpha+\beta)} \qquad (a = q^\alpha, b = q^\beta) \tag{4.15}$$

is a well known q-analogue of the beta integral, and is equivalent with the q-binomial formula [GR, (II.3),p.236].

5. Limit to big q-Jacobi polynomials.

In this section, we prove the orthogonality relations and norm evaluations for the big q-Jacobi polynomials by extending the limit (2.16) to the level of the orthogonality measure (4.2). The methods are analogous to the little q-Jacobi polynomials case which we have treated in the previous section.

The orthogonality relations and norm evaluations for the monic big q-Jacobi polynomials can be stated as follows.

Theorem 5.1. *(*[AA3, section 3]*) Let $c, d > 0$ and $-c/dq < a < 1/q$, $-d/cq < b < 1/q$ or $a = cu$, $b = -d\overline{u}$ with $u \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\int_{-d}^{c} \big(p_m^B p_n^B\big)(z; a, b, c, d)\Delta^B(z; a, b, c, d)d_q z = \delta_{m,n}\mathcal{N}^B(n; a, b, c, d), \tag{5.1}$$

with

$$\Delta^B(z; a, b, c, d) := \frac{\big(qz/c, -qz/d; q\big)_\infty}{\big(qaz/c, -qbz/d; q\big)_\infty}.$$

The quadratic norms $\mathcal{N}^B(n)$ of the monic big q-Jacobi polynomials are explicitly given by

$$\begin{aligned}\mathcal{N}^B(n; a, b, c, d) \quad &:= \quad \frac{\Gamma_q(n+1)\Gamma_q(n+1+\alpha)\Gamma_q(n+1+\beta)\Gamma_q(n+1+\alpha+\beta)}{\Gamma_q(2n+1+\alpha+\beta)\Gamma_q(2n+2+\alpha+\beta)}\\ &\times \quad \frac{(cd)^{n+1}q^{\binom{n}{2}}\big(-c/d, -d/c; q\big)_\infty}{(c+d)\big(-q^{n+1}bc/d, -q^{n+1}ad/c; q\big)_\infty}\end{aligned}$$

where $a = q^\alpha$ and $b = q^\beta$.

Proof. We assume throughout the proof that $a, b \neq 0$. This assumption can be removed at the end of the proof by continuity. For given ϵ_0, we set $\epsilon_k := \epsilon_0 q^k$. We claim that there exists an $\epsilon_0 > 0$ such that

$$\begin{aligned}&\lim_{k\to\infty}\big(-\epsilon_k^{-2}q; q\big)_\infty\big(\epsilon_k(cd/q)^{\frac{1}{2}}\big)^{m+n}\langle h_m, h_n\rangle_{AW}^{\underline{t}_B(\epsilon_k)}\\ &\qquad = \frac{2(c+d)}{(1-q)cd\big(q, q, -c/d, -d/c; q\big)_\infty}\int_{-d}^{c} z^{m+n}\Delta^B(z; a, b, c, d)d_q z\end{aligned} \tag{5.2}$$

for all $m, n \in \mathbb{Z}_+$. Since

$$\begin{aligned}&\lim_{k\to\infty}\big(-\epsilon_k^{-2}q; q\big)_\infty\big(\epsilon_k(cd/q)^{\frac{1}{2}}\big)^{2n}\mathcal{N}^{AW}(n; \underline{t}_B(\epsilon_k)) =\\ &\qquad\qquad \mathcal{N}^B(n; a, b, c, d)\frac{2(c+d)}{(1-q)cd\big(q, q, -c/d, -d/c; q\big)_\infty}\end{aligned}$$

the theorem follows from (2.13), (2.17) and (5.2) by similar arguments as in the little q-Jacobi case (see the proof of Theorem 4.2).

For the proof of (5.2), note that the parameters $\epsilon^{-1}(qc/d)^{\frac{1}{2}}$ and $-\epsilon^{-1}(qd/c)^{\frac{1}{2}}$ of $\underline{t}_B(\epsilon)$ cause a contribution to the discrete part of $\langle .,. \rangle_{AW}^{\underline{t}_B(\epsilon)}$ for $\epsilon > 0$ sufficiently small. In fact, we have for $\epsilon > 0$ sufficiently small,

$$
\begin{aligned}
&\big(-\epsilon^{-2}q;q\big)_\infty \big(\epsilon(cd/q)^{\frac{1}{2}}\big)^{m+n}\langle h_m, h_n\rangle_{AW}^{\underline{t}_B(\epsilon)}\\
&\qquad= \frac{1}{2\pi\sqrt{-1}}\int_T \big(\epsilon(cd/q)^{\frac{1}{2}}\big)^{m+n} h_m(z)h_n(z)\hat{\Delta}_c^{AW}(z;\epsilon)\frac{dz}{z}\\
&\qquad+ 2\sum_{i=0}^{\infty}\big(\epsilon(cd/q)^{\frac{1}{2}}\big)^{m+n}\big(h_m h_n\big)\big(\epsilon^{-1}(q/cd)^{\frac{1}{2}}cq^i\big)\hat{\Delta}_{d,1}^{AW}(i;\epsilon)\\
&\qquad+ 2\sum_{i=0}^{\infty}\big(\epsilon(cd/q)^{\frac{1}{2}}\big)^{m+n}\big(h_m h_n\big)\big(-\epsilon^{-1}(q/cd)^{\frac{1}{2}}dq^i\big)\hat{\Delta}_{d,2}^{AW}(i;\epsilon)
\end{aligned}
\tag{5.3}
$$

with $\hat{\Delta}_c^{AW}(z;\epsilon) := \big(-\epsilon^{-2}q;q\big)_\infty \Delta_c^{AW}(z;\underline{t}_B(\epsilon))$ and with discrete weights

$$
\begin{aligned}
\hat{\Delta}_{d,1}^{AW}(i;\epsilon) &= \frac{\big(\epsilon^2 d/qc;q\big)_\infty}{\big(q, qa, \epsilon^2 ad/c, -d/c, -qbc/d, -\epsilon^2 b;q\big)_\infty}\\
&\times \frac{\big(\epsilon^{-2}qc/d, -\epsilon^{-2}q, qa, -qbc/d;q\big)_i}{\big(\epsilon^{-2}qc/ad, -\epsilon^{-2}q/b, q, -qc/d;q\big)_i}\frac{\big(\epsilon^{-2}q^{2i+1}c/d;q\big)_1}{\big(\epsilon^{-2}qc/d;q\big)_1}(qab)^{-i}
\end{aligned}
$$

if $\epsilon < (qc/d)^{\frac{1}{2}}q^i$ and zero otherwise,

$$
\begin{aligned}
\hat{\Delta}_{d,2}^{AW}(i;\epsilon) &= \frac{\big(\epsilon^2 c/qd;q\big)_\infty}{\big(q, -qad/c, -\epsilon^2 a, -c/d, qb, \epsilon^2 bc/d;q\big)_\infty}\\
&\times \frac{\big(\epsilon^{-2}qd/c, -\epsilon^{-2}q, -qad/c, qb;q\big)_i}{\big(-\epsilon^{-2}q/a, \epsilon^{-2}qd/bc, q, -qd/c;q\big)_i}\frac{\big(\epsilon^{-2}q^{2i+1}d/c;q\big)_1}{\big(\epsilon^{-2}qd/c;q\big)_1}(qab)^{-i}
\end{aligned}
$$

if $\epsilon < (qd/c)^{\frac{1}{2}}q^i$ and zero otherwise. Now note that

$$
\big(-\epsilon^{-2}q;q\big)_\infty = \big(\epsilon^{-1}q^{\frac{1}{2}}\sqrt{-1}, -\epsilon^{-1}q^{\frac{1}{2}}\sqrt{-1}, \epsilon^{-1}q\sqrt{-1}, -\epsilon^{-1}q\sqrt{-1};q\big)_\infty,
$$

so it follows from Lemma 4.1 **(a)**, **(c)** and the Bounded Convergence Theorem that

$$
\lim_{k\to\infty}\frac{1}{2\pi\sqrt{-1}}\int_T \big(\epsilon_k(cd/q)^{\frac{1}{2}}\big)^{m+n}h_m(z)h_n(z)\hat{\Delta}_c^{AW}(z;\epsilon_k)\frac{dz}{z} = 0
\tag{5.4}
$$

for generic $\epsilon_0 > 0$ (compare with the little q-Jacobi case (proof of Theorem 4.2)). By a straightforward calculation, using Lemma 4.1**(a)** and **(b)**, we obtain for generic $\epsilon_0 > 0$,

$$
\lim_{k\to\infty}\hat{\Delta}_{d,1}^{AW}(i;\epsilon_k) = \frac{(c+d)}{cd\big(q, q-c/d, -d/c;q\big)_\infty}\Delta^B(cq^i;a,b,c,d)cq^i
\tag{5.5}
$$

$$
\lim_{k\to\infty}\hat{\Delta}_{d,2}^{AW}(i;\epsilon_k) = \frac{(c+d)}{cd\big(q, q-c/d, -d/c;q\big)_\infty}\Delta^B(-dq^i;a,b,c,d)dq^i
\tag{5.6}
$$

for $i \in \mathbb{Z}_+$. For generic $\epsilon_0 > K := \max\big((qc/d)^{\frac{1}{2}}, (qd/c)^{\frac{1}{2}}\big)$ we furthermore have the estimates

$$
\sup_{k\in\mathbb{Z}_+}|\hat{\Delta}_{d,1}^{AW}(i;\epsilon_k)| = \sup_{k\in\mathbb{Z}_+}|\hat{\Delta}_{d,1}^{AW}(i;q^i\epsilon_k)| \le M_1\Delta^B(cq^i;a,b,c,d)cq^i
\tag{5.7}
$$

$$
\sup_{k\in\mathbb{Z}_+}|\hat{\Delta}_{d,2}^{AW}(i;\epsilon_k)| = \sup_{k\in\mathbb{Z}_+}|\hat{\Delta}_{d,2}^{AW}(i;q^i\epsilon_k)| \le M_2\Delta^B(-dq^i;a,b,c,d)dq^i
\tag{5.8}
$$

for $i \in \mathbb{Z}_+$, where $M_1, M_2 > 0$ are independent of i. The first equality in (5.7) respectively (5.8) follows from the fact that $\hat{\Delta}_{d,1}^{AW}(i;\epsilon) = 0$ for $\epsilon \ge (qc/d)^{\frac{1}{2}}q^i$, respectively $\hat{\Delta}_{d,2}^{AW}(i;\epsilon) = 0$ for $\epsilon \geq (qd/c)^{\frac{1}{2}}q^i$. The second inequality in (5.7) respectively (5.8) follows from Lemma 4.1 **(a)**, **(b)** and the fact that the weights Δ^B are positive for the parameter values a,b,c,d under consideration (compare with the little q-Jacobi case (proof of Theorem 5.1)). Now we subsitute $\epsilon = \epsilon_k$ in (5.3) and take the limit $k \to \infty$. The infinite sums and limits may be interchanged by the estimates above and the fact that the infinite sums

$$\sum_{i=0}^{\infty} \Delta^B(cq^i;a,b,c,d)cq^i, \qquad \sum_{i=0}^{\infty} \Delta^B(-dq^i;a,b,c,d)dq^i$$

are absolutely convergent, so the limit (5.2) follows for generic $\epsilon_0 > K$ by (5.4), (5.5) and (5.6). □

Note that $\underline{t}_B(\epsilon) \in V_{AW}$ for $\epsilon > 0$ sufficiently small if the parameters a,b,c and d satisfy the assumptions of Theorem 5.1 ($\underline{t}_B(\epsilon)$ given by (2.15)). So in the proof of Theorem 5.1 we obtain the positive orthogonality measure for the big q-Jacobi polynomials as limit case of the positive (partly discrete, partly continuous) orthogonality measure (4.2) for the Askey-Wilson polynomials. In particular, the proof of Theorem 5.1 shows that the only part of the (rescaled) orthogonality measure which survives in the limit from Askey-Wilson polynomials to big q-Jacobi polynomials are sums of two infinite sequences of discrete weights coming from residues of $\Delta_c^{AW}(z)/z$ at $z = \epsilon^{-1}(qc/d)^{\frac{1}{2}}q^i$ and $z = -\epsilon^{-1}(qd/c)^{\frac{1}{2}}q^i$, where $\epsilon^{-1}(qc/d)^{\frac{1}{2}}$ respecively $-\epsilon^{-1}(qd/c)^{\frac{1}{2}}$ is the parameter in $\underline{t}_B(\epsilon)$ which tends to infinity respectively minus infinity in the limit $\epsilon \downarrow 0$. The two infinite sequences of weights are, up to a positive constant, exactly the set of weights which occur in the orthogonality measure for the big q-Jacobi polynomials.

The big q-Jacobi polynomials were first hinted at by Hahn [H]. A detailed discussion of the orthogonality relations and norm evaluations was given by Andrews and Askey [AA3]. The orthogonality relations and norm evaluations were derived using the q-Vandermonde formula [AA3, (3.29)], [GR, (II.6),p.236] and the evaluation of the q-Jackson integral over the weight function

$$\begin{aligned}\int_{-d}^{c} \Delta^B(z;a,b,c,d)d_qz &= \frac{\Gamma_q(1+\alpha)\Gamma_q(1+\beta)}{\Gamma_q(2+\alpha+\beta)} \frac{\big(-c/d,-d/c;q\big)_\infty cd}{\big(-qbc/d,-qad/c;q\big)_\infty (c+d)} \\ &= (1-q)c\frac{\big(q,-d/c,-qc/d,q^2ab;q\big)_\infty}{\big(qa,qb,-qbc/d,-qad/c;q\big)_\infty}\end{aligned} \tag{5.9}$$

where $a = q^\alpha$, $b = q^\beta$. The summation formula (5.9) is a q-analogue of the beta integral which first appeared in [AA2, Theorem 1].

6. Concluding remarks.

The orthogonality relations and norm evaluations for the little q-Jacobi polynomials (cf. Theorem 4.2) can also be obtained from the orthogonality relations and norm evaluations of the big q-Jacobi polynomials (Theorem 5.1) by considering the little q-Jacobi polynomials as limit cases of the big q-Jacobi polynomials,

$$\lim_{d\downarrow 0} P_n^B(z;b,a,1,d) = P_n^L(z;a,b) \qquad (n \in \mathbb{Z}_+). \tag{6.1}$$

See [K3] for details.

The proof of the orthogonality relations and norm evaluations for the q-Racah polynomials, the big and the little q-Jacobi polynomials we have presented in this paper has the advantage that summation formulas and transformation formulas for basic hypergeometric series, which were used in the original proofs of the orthogonality relations and norm evaluations, are now no longer needed. In fact, with this method, various types of summation formulas may be seen as special cases of integral type formulas. For instance the three summation formulas (3.3), (4.15) and (5.9) are obtained from the evaluation of the Askey-Wilson q-beta integral (2.2) by calculating the residues of the integrand $\Delta_c^{AW}(z)/z$ in (2.2) and taking suitable limits.

The first author has recently extended the methods of this paper to the multivariable setting. The orthogonality relations and norm evaluations for the multivariable q-Racah polynomials (defined in [DS]) and the multivariable big resp. little q-Jacobi polynomials (defined in [S]) can then be obtained by taking suitable limit transitions in the orthogonality relations and norm evaluations for the multivariable Askey-Wilson polynomials (defined in [M] and [K1]). A paper on this subject is in preparation.

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