Askey-Wilson Polynomials as Zonal Spherical Functions on the SU(2) Quantum Group

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On the SU(2) quantum group the notion of (zonal) spherical element is generalized by considering left and right invariance in the infinitesimal sense with respect to twisted primitive elements of the sl(2) quantized universal enveloping algebra. The resulting spherical elements belonging to irreducible representations of quantum SU(2) turn out to be expressible as a two-parameter family of Askey-Wilson polynomials. For a related basis change of the representation space a matrix of dual q-Krawtchouk polynomials is obtained. Big and little q-Jacobi polynomials are obtained as limits of Askey-Wilson polynomials.

Key words & phrases: quantum groups, SU(2), spherical functions, infinitesimal invariance, Askey-Wilson polynomials, dual q-Krawtchouk polynomials, big q-Jacobi polynomials, little q-Jacobi polynomials.

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1. Introduction

One of the interesting aspects of the rapidly developing subject of quantum groups is that they seem to provide the natural setting for q-hypergeometric functions and orthogonal polynomials. For the relatively simple case of $SU_q(2)$ a first example of this phenomenon was given by Vaksman & Soĭbel'man [21] (see also Masuda e.a. [14] [13] and the author [10]), where it was shown that the matrix elements of the irreducible representations of $SU_q(2)$ can be expressed in terms of little q-Jacobi polynomials. Next Noumi and Mimachi [16] showed that the "spherical harmonics" on quantum homogeneous spaces of $SU_q(2)$ can be expressed in terms of big q-Jacobi polynomials. As a following step, the author [11] gave an interpretation of a two-parameter family of Askey-Wilson polynomials (including continuous q-Legendre polynomials) as (zonal) spherical functions on $SU_q(2)$. Here the notion of spherical function was generalized in the sense that biinvariance with respect to the quantum subgroup $U(1)$ was replaced by "infinitesimal" left and right invariance with respect to twisted primitive elements of the corresponding quantized universal enveloping algebra $\mathcal{U}_q(sl(2, \mathbb{C}))$. Since this paper [11] was meant as a survey paper, the full results were only announced there, while a proof was sketched for the most simple case (parameter values $\sigma = \tau = 0$) corresponding to the continuous q-Legendre polynomials. Nevertheless, the paper [11] has already had some follow-ups by (i) the work of Koelink [8] (appearing as a companion paper to the present paper), which culminates into a quantum group derivation of the continuous q-Legendre case of the Rahman-Verma addition formula [19] for continuous q-ultraspherical polynomials, and (ii) an announcement by Noumi and Mimachi [17], [15], where they extend the author's result to the

expression of all corresponding matrix elements (not just the left and right infinitesimally invariant ones) as Askey-Wilson polynomials.

It is the purpose of the present paper to give full proofs of the results announced in [11]. The contents are as follows. In section 2 we give the preliminaries about q-hypergeometric functions and orthogonal polynomials, mainly referring to Askey & Wilson [5] and Gasper & Rahman [7]. In section 3 we give preliminaries on the quantum group $SL_q(2, \mathbb{C})$. Section 4 introduces the (σ, τ)-spherical elements on $SU_q(2)$ and derives an explicit Fourier series for such elements belonging to irreducible representations. An important tool here is the explicit matrix of dual q-Krawtchouk polynomials for the basis change from the standard basis to a basis of eigenfunctions for an (almost) twisted primitive element. This last result (already announced in [11]) is also crucial in Koelink [8].

In section 5 we prove that the elementary (σ, τ)-spherical matrix elements, when expressed as polynomials, satisfy the same second order q-difference equation as the Askey-Wilson polynomials. This is done by use of the Casimir operator on the quantum group. This way of proving that our polynomials are Askey-Wilson polynomials is different from the proof we had in mind when writing [11]. There we worked with the explicit Fourier series and the knowledge obtained from the quantum group theory that we were dealing with orthogonal polynomials. Then the result could be derived by deriving the three-term recurrence relation. Section 5 contains also the expression of the Haar functional as an Askey-Wilson integral, when applied to (σ, τ)-spherical elements. Finally, in section 6 we examine the limit cases as σ or τ tend to ∞. For the Askey-Wilson polynomials this means a limit transition to big or little q-Jacobi polynomials.

Notation. $\mathbb{Z}_+$ denotes the set of nonnegative integers.

2. Preliminaries on q-hypergeometric orthogonal polynomials

Let $0 < q < 1$. Define *q-shifted factorials*

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

$$(a; q)_\infty := \lim_{n \to \infty} (a; q)_n,$$

$$(a_1, \ldots, a_r; q)_n := \prod_{j=1}^{r} (a_j; q)_n,$$

and the *q-hypergeometric series*

$$_{s+1}\phi_s \left[\begin{matrix} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{s+1}; q)_k \, z^k}{(b_1, \ldots, b_s; q)_k \, (q; q)_k}. \tag{2.1}$$

Usually in this paper we will have the case of a *terminating* series in (2.1), i.e. $a_1 = q^{-n}$ ($n \in \mathbb{Z}_+$), so the series terminates after the term with $k = n$. Then we require that $b_1, \ldots, b_s \notin \{1, q^{-1}, \ldots, q^{-n+1}\}$. See Gasper & Rahman [7, Ch.1] for standard facts about q-hypergeometric series.

Askey-Wilson polynomials are defined by

$$p_n(\cos\theta; a, b, c, d \mid q) := a^{-n} (ab, ac, ad; q)_n \, _4\phi_3 \left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right]. \tag{2.2}$$

See Askey & Wilson [5, (1.15)]. They are symmetric in a, b, c, d (cf. [5, p.6]). Sometimes it will be useful to write the $_4\phi_3$ factor in (2.2) as

$$r_n(\cos\theta; a, b, c, d \mid q) := {_4\phi_3}\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right]. \tag{2.3}$$

The orthogonality properties are stated in [5, Theorems 2.2, 2.5].

Proposition 2.1. Assume a, b, c, d are real, or if complex, appear in conjugate pairs, and that $|a|, |b|, |c|, |d| \leq 1$, but the pairwise products of a, b, c, d have absolute value less than one, then

$$\frac{1}{2\pi} \int_0^\pi p_n(\cos\theta)\, p_m(\cos\theta)\, w(\cos\theta)\, d\theta = \delta_{m,n}\, h_n\,,$$

where

$$p_n(\cos\theta) = p_n(\cos\theta; a, b, c, d \mid q),$$

$$w(\cos\theta) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty}\,,$$

$$\frac{h_n}{h_0} = \frac{(1 - q^{n-1}abcd)\,(q, ab, ac, ad, bc, bd, cd; q)_n}{(1 - q^{2n-1}abcd)\,(abcd; q)_n}\,, \tag{2.4}$$

and

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}\,. \tag{2.5}$$

Proposition 2.2. Assume a, b, c, d are real, or if complex, appear in conjugate pairs, and that the pairwise products of a, b, c, d are not ≥ 1, then

$$\frac{1}{2\pi} \int_0^\pi p_n(\cos\theta)\, p_m(\cos\theta)\, w(\cos\theta)\, d\theta + \sum_k p_n(x_k)\, p_m(x_k)\, w_k = \delta_{m,n}\, h_n\,, \tag{2.6}$$

where $p_n(\cos\theta)$, $w(\cos\theta)$ and h_n are as in Proposition 2.1, while the x_k are the points $(eq^k + e^{-1}q^{-k})/2$ with e any of the parameters a, b, c or d whose absolute value is larger than one, the sum is over the $k \in \mathbb{Z}_+$ with $|eq^k| > 1$ and w_k is $w_k(a; b, c, d)$ as defined by 5, (2.10)] when $x_k = (aq^k + a^{-1}q^{-k})/2$. (Be aware that $(1 - aq^{2k})/(1 - a)$ should be replaced by $(1 - a^2q^{2k})/(1 - a^2)$ in [5, (2.10)].

With notation as in Proposition 2.2 let $dm = dm_{a,b,c,d;q}$ be the normalized orthogonality measure for the Askey-Wilson polynomials. So, for any polynomial p:

$$\int_{-\infty}^\infty p(x)dm(x) = \frac{1}{h_0}\left\{\frac{1}{2\pi} \int_{-1}^1 p(x)\, w(x)\, \frac{dx}{(1 - x^2)^{1/2}} + \sum_k p(x_k)\, w_k\right\}. \tag{2.7}$$

By [5, (5.7)–(5.9)] the Askey-Wilson polynomials, written as

$$R_n(e^{i\theta}) := {_4\phi_3}\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right],$$

are eigenfunctions of a second order q-difference operator:

$$A(-\theta)\left(R_n(q^{-1}e^{i\theta}) - R_n(e^{i\theta})\right) + A(\theta)(R_n(qe^{i\theta}) - R_n(e^{i\theta})) = -(1-q^{-n})(1-q^{n-1}abcd)R_n(e^{i\theta}), \quad (2.8)$$

where

$$A(\theta) := \frac{(1-ae^{i\theta})(1-be^{i\theta})(1-ce^{i\theta})(1-de^{i\theta})}{(1-e^{2i\theta})(1-qe^{2i\theta})}.$$

If $f(e^{i\theta})$ is a polynomial of degree $\leq n$ in $\cos\theta$ and if (2.8) with R_n replaced by f is valid, then f will be a constant multiple of R_n.

We will need some special Askey-Wilson polynomials which happen to have simple explicit Fourier-cosine expansions: *Chebyshev polynomials of the first kind*

$$p_n(\cos\theta; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}} \mid q) = (2 - \delta_{n,0})(q^n; q)_n \cos(n\theta) \quad (2.9)$$

(cf. [5,(4.25)¿]) and *continuous q-Legendre polynomials*

$$p_n(\cos\theta; q^{1/2}, -q^{1/2}, q^{1/2}, -q^{1/2} \mid q) = (-q;q)_n^2 (q;q)_n \sum_{k=0}^{n} \frac{(q;q^2)_k (q;q^2)_{n-k}}{(q^2;q^2)_k (q^2;q^2)_{n-k}} e^{i(n-2k)\theta}$$

(cf. [3, (3.1)] together with [5, (4.2), (4.20)]).

Lemma 2.3. The connection coefficients $c_{k,n}$ in

$$p_n(x; q^{1/2}a, \beta, qa^2/\beta, q^{1/2}a \mid q) = \sum_{k=0}^{n} c_{k,n}\, p_k(x; a, -a, -q^{1/2}a, q^{1/2}a \mid q) \quad (2.10)$$

can be explicitly written as

$$c_{k,n} = \frac{(qa^2, q; q)_n (q^{n+1}a^4; q)_k}{(qa^2, q, q^k a^4; q)_k (q;q)_{n-k}} \left(\frac{-q^{1/2}}{a^2\beta}\right)^{n-k} q^{(n-k)^2/2} \left\{ {}_2\phi_1\left[\begin{matrix} q^{-n+k}, q^{-n}a^2 \\ q^{k+1}a^2 \end{matrix}; q, q^{n+1/2}a\beta\right]\right\}^2 \quad (2.11).$$

Proof. Askey and Wilson 5, (6.1), (6.2)] gave the connection coefficients between two families of Askey-Wilson polynomials with one common parameter. Their expression involved a terminating balanced ${}_5\phi_4$ of argument q. With our special choice of parameters in (2.10) this ${}_5\phi_4$ has the form occurring in the q-Clausen formula as given by formulas (2.16) (first identity) and (2.4) (with $\alpha = q^{-n}$) in Gasper and Rahman [6]. (Note that both the left and right hand side of this version of the q-Clausen formula are written differently from the usual formulation [6, (1.6)].) Substitution of this q-Clausen formula yields (2.11). $\qquad\square$

Now substitute $a = 1$ in (2.10), (2.11) and apply (2.9). Then

$$p_n(\cos\theta; q^{1/2}, \beta, q/\beta, q^{1/2} \mid q) = \sum_{k=-n}^{n} c_{|k|,n}\, (q^{|k|}; q)_{|k|}\, e^{ik\theta},$$

with $c_{k,n}$ given by (2.11) for $a = 1$. When we switch to base q^2 and put $\beta = -q^{2\sigma+1}$ then we obtain

$$p_n(\cos\theta; q, -q^{2\sigma+1}, -q^{-2\sigma+1}, q \mid q^2) = \sum_{k=-n}^{n} c_{|k|,n}\, (q^{2|k|}; q^2)_{|k|}\, e^{ik\theta} \quad (2.12)$$

with

$$(q^{2k};q^2)_k\, c_{k,n} = \frac{(q^2;q^2)_n^2\,(q^{2n+2};q^2)_k\, q^{(n-k)(n-k+2\sigma)}}{(q^2;q^2)_k^2\,(q^2;q^2)_{n-k}} \left\{ {}_2\phi_1 \left[\begin{matrix} q^{-2n+2k},q^{-2n} \\ q^{2k+2} \end{matrix} ; q^2, -q^{2n-2\sigma+2} \right] \right\}^2. \tag{2.13}$$

Finally, by Jackson's transformation formula, [7, (III.7)], formula (2.13) can be rewritten as

$$(q^{2k};q^2)_k\, c_{k,n} = \frac{(q^2;q^2)_{2n}^2\, q^{(n-k)(n-k+2\sigma)}}{(q^2;q^2)_n\,(q^2;q^2)_{n+k}\,(q^2;q^2)_{n-k}} \left\{ {}_3\phi_2 \left[\begin{matrix} q^{-2n+2k},q^{-2n},-q^{-2n-2\sigma} \\ q^{-4n},0 \end{matrix} ; q^2, q^2 \right] \right\}^2. \tag{2.14}$$

Next we define *little q-Jacobi polynomials*

$$p_n(x;a,b;q) := {}_2\phi_1(q^{-n},abq^{n+1};aq;q,qx) \tag{2.15}$$

(cf. Andrews & Askey [1]) and *big q-Jacobi polynomials*

$$P_n^{(\alpha,\beta)}(x;c,d;q) = {}_3\phi_2 \left[\begin{matrix} q^{-n},q^{n+\alpha+\beta+1},xq^{\alpha+1}/c \\ q^{\alpha+1},-q^{\alpha+1}d/c \end{matrix} ; q,q \right] \tag{2.16}$$

(cf. Andrews & Askey [2]). In this last reference some different normalization is suggested for the big q-Jacobi polynomials, but the authors are not very definite about it. Here we follow the normalization used by Noumi & Mimachi [16]. Little and big q-Jacobi polynomials are orthogonal with respect to discrete measures.

We define *dual q-Krawtchouk polynomials* by

$$R_n(q^{-x} - q^{x-N-c};q^c,N\mid q) := {}_3\phi_2(q^{-n},q^{-x},-q^{x-N-c};0,q^{-N};q,q). \tag{2.17}$$

These are special q-Racah polynomials and satisfy the orthogonality relations

$$\frac{1}{(-q^c;q)_N} \sum_{x=0}^{N} (R_n R_m)(q^{-x} - q^{x-N-c};q^c,N\mid q)$$
$$\times \frac{(1+q^{2x-N-c})(-q^{-N-c},q^{-N};q)_x}{(1+q^{-N-c})(q,-q^{-c+1};q)_x(-q^{x-2N-c})^x} = \delta_{nm}\, \frac{(q;q)_n}{(q^{-N};q)_n}\,(-q^{-N-c})^n, \tag{2.18}$$

where $n,m = 0,\dots,N$. See Askey and Wilson 4] and Stanton [20]. They satisfy the three-term recurrence relation

$$y\, R_n(y;q^c,N\mid q) = (1-q^{n-N})\, R_{n+1}(y;q^c,N\mid q)$$
$$+(q^{-N}-q^{-N-c})\, q^n\, R_n(y;q^c,N\mid q) - (1-q^n)\, q^{-N-c}\, R_{n-1}(y;q^c,N\mid q), \tag{2.19}$$

see [20].

3. Preliminaries on the quantum $SL(2, \mathbb{C})$ group

The reader may use the author's survey [11] and the references given there for further reading in connection with this section. Fix $q \in (0, 1)$. Let $\mathcal{A}_q$ be the complex associative algebra with unit 1, generators $\alpha, \beta, \gamma, \delta$ and relations

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta,$$
$$\alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\beta\gamma = 1. \tag{3.1}$$

It turns out that $\mathcal{A}_q$ becomes a Hopf algebra over $\mathbb{C}$ under the following actions of the comultiplication $\Delta \colon \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{A}_q$, counit $\varepsilon \colon \mathcal{A}_q \to \mathbb{C}$ (unital multiplicative linear mappings) and antipode $S \colon \mathcal{A}_q \to \mathcal{A}_q$ (unital antimultiplicative linear mapping) on the generators:

$$\Delta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

$$\varepsilon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix},$$

where the formula for Δ has to be interpreted in the sense of matrix multiplication: $\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma$, etc. .

A *Hopf $*$-algebra* is a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ with an involution $a \mapsto a^*$ such that $\mathcal{A}$ becomes a unital $*$-algebra and $\Delta \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and $\varepsilon \colon \mathcal{A} \to \mathbb{C}$ are $*$-homomorphisms. Then it can be shown that, if S is invertible (which is the case in our example $\mathcal{A}_q$), we have

$$S \circ * \circ S \circ * = \mathrm{id}.$$

We can make $\mathcal{A}_q$ into a Hopf $*$-algebra by taking for the involution the unital antimultiplicative anti-linear mapping $a \mapsto a^*$ such that

$$\begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} = \begin{pmatrix} \delta & -q\gamma \\ -q^{-1}\beta & \alpha \end{pmatrix}.$$

Let $\mathcal{U}_q$ be the complex associative algebra with unit 1, generators A, B, C, D and relations

$$AD = DA = 1, \quad AB = qBA, \quad AC = q^{-1}CA, \quad BC - CB = \frac{A^2 - D^2}{q - q^{-1}}. \tag{3.2}$$

We can make $\mathcal{U}_q$ into a Hopf $*$-algebra with comultiplication $\Delta \colon \mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$, counit $\varepsilon \colon \mathcal{U}_q \to \mathbb{C}$, antipode $S \colon \mathcal{U}_q \to \mathcal{U}_q$ and involution $* \colon \mathcal{U}_q \to \mathcal{U}_q$ by requiring that

$$\Delta(A) = A \otimes A, \quad \Delta(D) = D \otimes D,$$
$$\Delta(B) = A \otimes B + B \otimes D, \quad \Delta(C) = A \otimes C + C \otimes D,$$
$$\varepsilon(A) = \varepsilon(D) = 1, \quad \varepsilon(B) = \varepsilon(C) = 0,$$
$$S(A) = D, \quad S(D) = A, \quad S(B) = -q^{-1}B, \quad S(C) = -qC, \tag{3.3}$$
$$A^* = A, \quad D^* = D, \quad B^* = C, \quad C^* = B. \tag{3.4}$$

Two Hopf algebras $\mathcal{U}, \mathcal{A}$ are said to be *in duality* if there is a doubly nondegenerate bilinear form $(u, a) \mapsto \langle u, a \rangle \colon \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ such that, for $u, v \in \mathcal{U}$, $a, b \in \mathcal{A}$:

$$\langle \Delta(u), a \otimes b \rangle = \langle u, ab \rangle, \quad \langle u \otimes v, \Delta(a) \rangle = \langle uv, a \rangle, \tag{3.5}$$
$$\langle 1_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a), \quad \langle u, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(u), \quad \langle S(u), a \rangle = \langle a, S(u) \rangle.$$

If $\mathcal{U}, \mathcal{A}$ are moreover Hopf $*$-algebras then they are said to be *Hopf $*$-algebras in duality* if the above pairing satisfies in addition that

$$\langle u^*, a \rangle = \overline{\langle u, (S(a))^* \rangle}. \tag{3.6}$$

Instead of $\langle u, a \rangle$ we will also write $u(a)$ or $a(u)$.

It can be shown that $\mathcal{U}_q$ and $\mathcal{A}_q$ become Hopf $*$-algebras in duality with the following pairing between the generators:

$$\left\langle A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad \left\langle D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix},$$

$$\left\langle B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \left\langle C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The pairing between products of generators then follows by the rules (3.5).

The element

$$\Omega := \frac{q^{-1} A^2 + q D^2 - 2}{(q^{-1} - q)^2} + BC = \Omega^* \tag{3.7}$$

is a *Casimir element* of $\mathcal{U}_q$: it commutes with any $X \in \mathcal{U}_q$.

In a Hopf algebra $\mathcal{U}$ an element u is called *group-like* if $u \neq 0$ and $\Delta(u) = u \otimes u$, *primitive* if $\Delta(u) = 1 \otimes u + u \otimes 1$ and *twisted primitive* (with respect to a group-like element g) if $\Delta(u) = g \otimes u + u \otimes S(g)$. In $\mathcal{U}_q$ the group-like elements are all elements A^n ($n \in \mathbb{Z}$) and (cf. Masuda e.a. [12]):

Lemma 3.1. The twisted primitive elements with respect to A are the elements X in the linear span of $A - D$, B and C. They satisfy

$$\Delta(X) = A \otimes X + X \otimes D.$$

For $t \neq 1$ the twisted primitive elements with respect to A^t are the constant multiples of $A^t - A^{-t}$.

Let $\mathcal{U}$ and $\mathcal{A}$ be Hopf algebras in duality. For $u \in \mathcal{U}$ and $a \in \mathcal{A}$ define elements $u.a$ and $a.u$ of $\mathcal{A}$ by

$$u.a := (\mathrm{id} \otimes u)(\Delta(a)), \quad a.u := (u \otimes \mathrm{id})(\Delta(a)). \tag{3.8}$$

Hence, if $v \in \mathcal{U}$,

$$(u.a)(v) = a(vu), \quad (a.u)(v) = a(uv).$$

The operations defined in (3.8) are left respectively right algebra actions of $\mathcal{U}_q$ on $\mathcal{A}_q$:

$$(uv).a = u.(v.a), \quad a.(uv) = (a.u).v. \tag{3.9}$$

If $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$ ($u \in \mathcal{U}$) and $a, b \in \mathcal{A}$ then

$$u.(ab) = \sum_{(u)} (u_{(1)}.a)(u_{(2)}.b), \quad (ab).u = \sum_{(u)} (a.u_{(1)})(b.u_{(2)}). \tag{3.10}$$

Furthermore, if $u \in \mathcal{U}$, $a \in \mathcal{A}$ then

$$(\mathrm{id} \otimes u.)(\Delta(a)) = \Delta(u.a) \tag{3.11}$$

and if, moreover, $v \in \mathcal{U}$ then

$$\langle v, u.a^* \rangle = \overline{\langle S(v)^*, S(u)^*.a \rangle}. \tag{3.12}$$

We call an element $a \in \mathcal{A}$ left (*right*) *invariant* with respect to an element $u \in \mathcal{U}$ if $u.a = \varepsilon(u)a$ respectively $a.u = \varepsilon(u)a$. Note that the unit 1 of $\mathcal{A}$ is biinvariant with respect to all $u \in \mathcal{U}$. If u is twisted primitive then $\varepsilon(u) = 0$ and:

$$u.a = 0 \quad \& \quad b.u = 0 \Longrightarrow u.(ab) = 0,$$
$$a.u = 0 \quad \& \quad b.u = 0 \Longrightarrow (ab).u = 0.$$

Hence:

Lemma 3.2. The left (or right) invariant elements of $\mathcal{A}$ with respect to some twisted primitive element of $\mathcal{U}$ form a unital subalgebra of $\mathcal{A}$. In particular, if $X \in \mathrm{Span}\{A - D, B, C\}$ then the set of all $a \in \mathcal{A}_q$ satisfying $X.a = 0$ (respectively $a.X = 0$) forms a unital subalgebra of $\mathcal{A}_q$.

Let $\mathcal{U}$ and $\mathcal{A}$ be Hopf algebras in duality. A *matrix corepresentation* of $\mathcal{A}$ is a square matrix $t = (t_{i,j})$ of elements of $\mathcal{A}$ such that

$$\Delta(t_{i,j}) = \sum_k t_{i,k} \otimes t_{k,j}, \quad \varepsilon(t_{i,j}) = \delta_{i,j}. \tag{3.13}$$

To a matrix corepresentation t of $\mathcal{A}$ corresponds a matrix representation of $\mathcal{U}$, also denoted by t and defined by

$$(t(u))_{i,j} := t_{i,j}(u) = \langle u, t_{i,j} \rangle, \qquad u \in \mathcal{U}.$$

The matrix entries of a corepresentation of $\mathcal{A}$ (elements of $\mathcal{A}$) are completely determined by the matrix entries of the corresponding representation of $\mathcal{U}$ (linear functionals on $\mathcal{U}$). A matrix corepresentation t of $\mathcal{A}$ is called *unitary* if $t_{i,j}^* = S(t_{j,i})$ and a representation t of $\mathcal{U}$ is called a *$*$-representation* if $t_{i,j}(u^*) = \overline{t_{j,i}(u)}$ ($u \in \mathcal{U}$). Note that a matrix corepresentation of $\mathcal{A}$ is unitary if and only if the corresponding matrix representation of $\mathcal{U}$ is a $*$-representation.

Up to equivalence, there is for each finite dimension precisely one irreducible matrix corepresentation of $\mathcal{A}_q$, which can be chosen to be unitary. The corresponding irreducible $*$-representation of $\mathcal{U}_q$ is realized as a representation $t^l = (t_{i,j}^l)_{i,j=-l,-l+1,\ldots,l}$ ($l \in \frac{1}{2}\mathbb{Z}_+$) on a $(2l+1)$-dimensional vector space with orthonormal basis $\{e_n^l\}_{n=-l,-l+1,\ldots,l}$ such that

$$t^l(A)\, e_n^l = q^{-n}\, e_n^l, \quad t^l(D)\, e_n^l = q^n\, e_n^l,$$
$$t^l(B)\, e_n^l = \frac{(q^{-l+n-1} - q^{l-n+1})^{\frac{1}{2}}\,(q^{-l-n} - q^{l+n})^{\frac{1}{2}}}{q^{-1} - q}\, e_{n-1}^l,$$
$$t^l(C)\, e_n^l = \frac{(q^{-l+n} - q^{l-n})^{\frac{1}{2}}\,(q^{-l-n-1} - q^{l+n+1})^{\frac{1}{2}}}{q^{-1} - q}\, e_{n+1}^l, \tag{3.14}$$

with the convention that e_{-l-1}^l and e_{l+1}^l are zero. The $t_{i,j}^l$, being elements of $\mathcal{A}_q$, can be expressed in terms of the generators by expressions involving little q-Jacobi polynomials. The lowest dimensional cases are particularly simple:

$$t^0 = (t_{0,0}^0) = (1), \quad t^{\frac{1}{2}} = \begin{pmatrix} t_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}} & t_{\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}} \\ t_{-\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}} & t_{-\frac{1}{2},-\frac{1}{2}}^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix},$$

$$t^1 = \begin{pmatrix} t^1_{1,1} & t^1_{1,0} & t^1_{1,-1} \\ t^1_{0,1} & t^1_{0,0} & t^1_{0,-1} \\ t^1_{-1,1} & t^1_{-1,0} & t^1_{-1,-1} \end{pmatrix} = \begin{pmatrix} \delta^2 & (1+q^2)^{\frac{1}{2}}\delta\gamma & \gamma^2 \\ (1+q^2)^{\frac{1}{2}}\delta\beta & 1+(q+q^{-1})\beta\gamma & (1+q^2)^{\frac{1}{2}}\gamma\alpha \\ \beta^2 & (1+q^2)^{\frac{1}{2}}\beta\alpha & \alpha^2 \end{pmatrix}. \qquad (3.15)$$

Define

$$\mathcal{A}^l_q := \mathrm{Span}\{t^l_{i,j} \mid i,j = -l, -l+1, \ldots, l\}. \qquad (3.16)$$

It is known that:

Proposition 3.3. The $t^l_{i,j}$ $(i,j = -l, -l+1, \ldots, l)$ form a basis of $\mathcal{A}^l_q$ and

$$\mathcal{A}_q = \bigoplus_{l \in \frac{1}{2}\mathbb{Z}_+} \mathcal{A}^l_q. \qquad (3.17)$$

Let $a := \sum_{i,j=-l}^l \gamma_{i,j}\, t^l_{i,j} \in \mathcal{A}^l_q$ $(\gamma_{i,j} \in \mathbb{C})$. Let $X \in \mathcal{U}_q$. It follows from (3.8) and (3.13) that

$$X.a = \sum_{i,k} \left(\sum_j t^l_{k,j}(X)\, \gamma_{i,j} \right) t^l_{i,k}, \qquad (3.18)$$

$$a.X = \sum_{k,j} \left(\sum_i t^l_{i,k}(X)\, \gamma_{i,j} \right) t^l_{k,j}. \qquad (3.19)$$

Lemma 3.4. With a and X as above we have:

(i) $\quad X.a = 0 \iff t^l(X)\,(\sum_{j=-l}^l \gamma_{ij}\, e^l_j) = 0$ for all i;

(ii) $\quad a.X = 0 \iff t^l(X^*)\,(\sum_{i=-l}^l \overline{\gamma_{ij}}\, e^l_i) = 0$ for all j.

Proof. (i) By (3.18) and Proposition 3.3:

$X.a = 0 \iff \sum_j t^l_{k,j}(X)\,\gamma_{i,j} = 0$ for all $i,k \iff t^l(X)\,(\sum_{j=-l}^l \gamma_{ij}\, e^l_j) = 0$ for all i.

(ii) By (3.19): $a.X = 0 \iff \sum_i t^l_{i,k}(X)\,\gamma_{i,j} = 0$ for all k,j

$\iff \sum_i t^l_{k,i}(X^*)\,\overline{\gamma_{i,j}} = 0$ for all $k,j \iff t^l(X^*)\,(\sum_i \overline{\gamma_{i,j}}\, e^l_i) = 0$ for all j,

where we used that t^l is a $*$-representation of $\mathcal{U}_q$. $\qquad\square$

For the Casimir element Ω (cf. (3.7)) we compute from (3.14) that

$$t^l(\Omega) = \left(\frac{q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}}}{q^{-1} - q} \right)^2 \mathrm{id}\,.$$

Hence, by (3.18), (3.19):

$$\Omega.a = \left(\frac{q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}}}{q^{-1} - q} \right)^2 a = a.\Omega \qquad \text{if } a \in \mathcal{A}^l_q\,. \qquad (3.20)$$

The *tensor product* $t^l \otimes t^{l'}$ is defined as the matrix corepresentation of $\mathcal{A}_q$ with matrix entries

$$(t^l \otimes t^{l'})_{i,i';j,j'} := t^l_{i,j}\, t^{l'}_{i',j'} \qquad (i,j = -l, -l+1, \ldots, l;\ i',j' = -l', -l'+1, \ldots, l').$$

Then:

Proposition 3.5. $t^l \otimes t^{l'}$ is equivalent to the direct sum of the corepresentations t^k $(k = l+l', l+l'-1, \ldots, |l-l'|)$.

There is a unique linear functional $h: \mathcal{A}_q \to \mathbb{C}$, called the *Haar functional* on $\mathcal{A}_q$, with the properties

(i) $h(1) = 1$,
(ii) $h(aa^*) \geq 0$ for all $a \in \mathcal{A}_q$,
(iii) $(h \otimes \mathrm{id})(\Delta(a)) = h(a)1 = (\mathrm{id} \otimes h)(\Delta(a))$.

Then $h(aa^*) > 0$ if $a > 0$. It can also be shown that

$$h((t^{l'}_{i',j'})^* \, t^l_{i,j}) = \delta_{l,l'} \, \delta_{i,i'} \, \delta_{j,j'} \, q^{2(l-i)} \, \frac{1-q^2}{1-q^{2(2l+1)}} \, . \tag{3.21}$$

For $\theta \in \mathbb{C}$ let $\pi^1_\theta: \mathcal{A}_q \to \mathbb{C}$ be the unital algebra homomorphism (one-dimensional representation of $\mathcal{A}_q$) defined by

$$\pi^1_\theta \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} . \tag{3.22}$$

In particular, if $\theta \in \mathbb{R}$, π^1_θ is a $*$-representation of $\mathcal{A}_q$. We have

$$\pi^1_\theta(t^l_{n,m}) = e^{-2in\theta} \, \delta_{n,m} \, . \tag{3.23}$$

4. (σ, τ)-spherical elements

Define, for $\sigma \in \mathbb{R}$,

$$X_\sigma := i \, q^{1/2} \, B - i \, q^{-1/2} \, C - \frac{q^{-\sigma} - q^\sigma}{q^{-1} - q} \, (A - D). \tag{4.1}$$

Then

$$X_\sigma^* = i \, q^{-1/2} \, B - i \, q^{1/2} \, C - \frac{q^{-\sigma} - q^\sigma}{q^{-1} - q} \, (A - D) = -S(X_\sigma) \tag{4.2}$$

by (3.3), (3.4) and X_σ is twisted primitive with respect to A (cf. Lemma 3.1):

$$\Delta(X_\sigma) = A \otimes X_\sigma + X_\sigma \otimes D.$$

Define also the twisted primitive element

$$X_\infty := D - A = \lim_{\sigma \to \infty} (q^{-1} - q) \, q^\sigma \, X_\sigma = \lim_{\sigma \to -\infty} (q - q^{-1}) \, q^{-\sigma} \, X_\sigma.$$

Left or right invariance of an element of $\mathcal{A}_q$ with respect to X_∞ is the same as left or right invariance with respect to the diagonal quantum subgroup of $SL_q(2, \mathbb{C})$.

We will call an element $a \in \mathcal{A}_q$ (σ, τ)-*spherical* if a is left invariant with respect to X_σ and right invariant with respect to X_τ:

$$X_\sigma . a = 0 \quad \& \quad a . X_\tau = 0.$$

The nonzero (σ, τ)-spherical elements in the subspaces $\mathcal{A}^l_q$ (defined by (3.16)) will be called *elementary (σ, τ)-spherical*. It follows from Lemma 3.2 that the (σ, τ)-spherical elements form a subalgebra with 1 of $\mathcal{A}_q$. Since the subspaces $\mathcal{A}^l_q$ are invariant under left and right action of $\mathcal{U}_q$ (cf. (3.18), (3.19)), it follows from the direct sum decomposition (3.17) that any (σ, τ)-spherical element in $\mathcal{A}_q$ will be a sum of elementary (σ, τ)-spherical elements.

From now on assume that σ and τ are finite. From (3.14) we obtain:

Lemma 4.1. Let $l \in \frac{1}{2}\mathbb{Z}_+$, $a_{-l}, a_{-l+1}, \ldots, a_l \in \mathbb{C}$. Let $v := \sum_{m=-l}^{l} a_m e_m^l$. Then

$$t^l \left(iq^{\pm\frac{1}{2}} B - iq^{\mp\frac{1}{2}} C - \frac{q^{-\sigma} - q^\sigma}{q^{-1} - q} (A - D) \right) v = 0 \tag{4.3}$$

iff

$$i\, q^{\pm\frac{1}{2}} \left(q^{-l+m} - q^{l-m} \right)^{\frac{1}{2}} \left(q^{-l-m-1} - q^{l+m+1} \right)^{\frac{1}{2}} a_{m+1} - i\, q^{\mp\frac{1}{2}} \left(q^{-l+m-1} - q^{l-m+1} \right)^{\frac{1}{2}}$$
$$\times \left(q^{-l-m} - q^{l+m} \right)^{\frac{1}{2}} a_{m-1} - \left(q^{-\sigma} - q^\sigma \right) \left(q^{-m} - q^m \right) a_m = 0, \qquad m = -l, -l+1, \ldots, l, \tag{4.4}$$

with the convention that $a_{-l-1} = 0 = a_{l+1}$.

By Lemma 4.1 one easily finds the general solution to (4.3) for low l:

if $l = 0$ then $v = \text{const.}$,

if $l = \frac{1}{2}$ then $v = 0$,

if $l = 1$ then

$$v = \text{const.} \left(-i\, q^{\mp\frac{1}{2}} e_1^1 + \frac{q^{-\sigma} - q^\sigma}{(q^{-1} + q)^{\frac{1}{2}}} e_0^1 - iq^{\pm\frac{1}{2}} e_{-1}^1 \right). \tag{4.5}$$

Also, if $\sigma = 0$ then the coefficient of a_m in (4.4) vanishes, so (4.4) becomes a two-term recurrence relation with solution $v = 0$ if $l \in \mathbb{Z}_+ + \frac{1}{2}$, and

$$v = \text{const.} \sum_{m=-l,-l+2,\ldots,l} q^{\mp\frac{1}{2}m} \left(\frac{(q^2; q^4)_{(l-m)/2} \, (q^2; q^4)_{(l+m)/2}}{(q^4; q^4)_{(l-m)/2} \, (q^4; q^4)_{(l+m)/2}} \right)^{\frac{1}{2}} e_m^l$$

if $l \in \mathbb{Z}_+$.

From Lemma 3.4 we derive:

Lemma 4.2. Let, for some $l \in \frac{1}{2}\mathbb{Z}_+$ and $\sigma, \tau \in \mathbb{R}$, $t^l(X_\sigma)$ and $t^l(X_\tau^*)$ have both one-dimensional zero-space spanned by $\sum_{m=-l}^{l} a_m e_m^l$ and $\sum_{m=-l}^{l} b_m e_m^l$, respectively. Then the (σ, τ)-spherical elements in $\mathcal{A}_q^l$ form a one-dimensional subspace spanned by

$$\sum_{i,j=-l}^{l} \overline{b_i}\, a_j\, t_{i,j}^l. \tag{4.6}$$

In view of (3.15), (4.5) and (4.6), the (σ, τ)-spherical elements in $\mathcal{A}_q^1$ are just the constant multiples of

$$2\rho_{\sigma,\tau} + \frac{(q^{-\sigma} - q^\sigma)(q^{-\tau} - q^\tau)}{q^{-1} + q}, \tag{4.7}$$

where

$$\rho_{\sigma,\tau} := \tfrac{1}{2}\big(\alpha^2 + \delta^2 + q\gamma^2 + q^{-1}\beta^2 + i(q^{-\sigma} - q^\sigma)(q\delta\gamma + \beta\alpha)$$
$$- i(q^{-\tau} - q^\tau)(\delta\beta + q\gamma\alpha) + (q^{-\sigma} - q^\sigma)(q^{-\tau} - q^\tau)\beta\gamma \big). \tag{4.8}$$

Note that

$$\rho_{\sigma,\tau}^* = \rho_{\sigma,\tau}.$$

In order to compute the null space of $t^l(X_\sigma)$ in general, we consider the more general problem of finding the eigenvectors of $t^l(-DX_\sigma)$. Note that

$$(DX_\sigma)^* = DX_\sigma,$$

hence $t^l(-DX_\sigma)$ is self-adjoint. Clearly, $t^l(X_\sigma)$ and $t^l(-DX_\sigma)$ have the same zero space.

Let $\lambda \in \mathbb{R}$, $v = \sum_{m=-l}^l a_m e_m^l$. Then

$$t^l(-DX_\sigma)\,v = \lambda\,v$$

iff

$$
\begin{aligned}
-i\,q^{\frac{1}{2}}\,(q^{-l+m} - q^{l-m})^{\frac{1}{2}}\,(q^{-l-m-1} - q^{l+m+1})^{\frac{1}{2}}\,a_{m+1} + i\,q^{-\frac{1}{2}}\,(q^{-l+m-1} - q^{l-m+1})^{\frac{1}{2}} \\
\times\,(q^{-l-m} - q^{l+m})^{\frac{1}{2}}\,a_{m-1} + (q^{-\sigma} - q^\sigma)\,(q^{-m} - q^m)\,a_m = q^{-m}\,\lambda\,(q^{-1} - q)\,a_m
\end{aligned}
\tag{4.9}
$$

for $m = -l, -l+1, \ldots, l$. Put

$$R_n := i^n\,q^{-\frac{1}{2}n(n-1)}\,q^{-n\sigma}\,(q^2; q^2)_n^{\frac{1}{2}}\,(q^{4l}; q^{-2})_n^{-\frac{1}{2}}\,a_{-l+n},$$

$$x := q^{-2l-\sigma}\,((q^{-1} - q)\lambda + q^\sigma - q^{-\sigma}).$$

Then (4.9) can be rewritten as

$$(1 - q^{2n-4l})\,R_{n+1} - (1 - q^{2n})\,q^{-4l-2\sigma}\,R_{n-1} + (q^{-4l} - q^{-4l-2\sigma})\,q^{2n}\,R_n = x\,R_n, \qquad n = 0, 1, \ldots, 2l,
\tag{4.10}$$

with the convention that $R_{-1} = 0 = R_{2l+1}$. We recognize (4.10) as the three-term recurrence relation (2.19) for the dual q-Krawtchouk polynomials (2.17). Thus the possible eigenvalues in (4.10) are

$$x_j := q^{-2j-2l} - q^{2j-2l-2\sigma}, \qquad j = -l, -l+1, \ldots, l,
\tag{4.11}$$

and the corresponding eigenvectors, up to a constant factor, are given by

$$R_n = R_n(x_j; q^{2\sigma}, 2l \mid q^2).$$

When we translate this back to (4.9) we obtain

Theorem 4.3. $t^l(-DX_\sigma)$ has simple spectrum consisting of eigenvalues

$$\lambda_j := \frac{-q^{2j-\sigma} + q^{\sigma-2j} - q^\sigma + q^{-\sigma}}{q^{-1} - q}, \qquad j = -l, -l+1, \ldots, l,$$

with corresponding eigenvectors

$$\text{const.} \sum_{n=0}^{2l} i^{-n}\,q^{n\sigma}\,q^{n(n-1)/2}\,(q^2; q^2)_n^{-1/2}\,(q^{4l}; q^{-2})_n^{1/2}\,R_n(x_j; q^{2\sigma}, 2l \mid q^2)\,e_{n-l}^l,$$

where x_j is given by (4.11).

Similarly, in order to compute the null space of $t^l(X^*_\sigma)$ in general, we consider the more general problem of finding the eigenvectors of $t^l(AX^*_\sigma)$. Note that

$$(AX^*_\sigma)^* = AX^*_\sigma,$$

so $t^l(AX^*_\sigma)$ is self-adjoint and has the same zero space as $t^l(X^*_\sigma)$. But also,

$$AX^*_\sigma = X_\sigma A = J(-X^*_\sigma D) = J(-DX_\sigma),$$

where $J:\mathcal{U}_q \to \mathcal{U}_q$ is the involutive algebra isomorphism generated by

$$J(A) = D, \quad J(D) = A, \quad J(B) = C, \quad J(C) = B$$

(well-defined in view of (3.2)). Also observe from (3.14) that

$$t^l_{m,n}(J(X)) = t^l_{-m,-n}(X), \qquad X \in \mathcal{U}_q.$$

Hence

Lemma 4.4. $\sum_{m=-l}^l a_m e^l_m$ is eigenvector of $t^l(-DX_\sigma)$ with eigenvalue λ iff $\sum_{m=-l}^l a_m e^l_{-m}$ is eigenvector of $t^l(AX^*_\sigma)$ with eigenvalue λ.

Since $X^*_\sigma D = DX_\sigma$, we see also that $t^l(X_\sigma) v = 0$ iff $t^l(X^*_\sigma)(t^l(D) v) = 0$. In combination with Lemma 4.4 this yields:

Lemma 4.5. Let $l \in \frac{1}{2}\mathbb{Z}_+$ and $c_{-l}, c_{-l+1}, \ldots, c_l \in \mathbb{C}$. Then:

$$t^l(X_\sigma)\,\Big(\sum_{m=-l}^l q^{-\frac{1}{2}m} c_m e^l_m\Big) = 0 \Longleftrightarrow t^l(X^*_\sigma)\,\Big(\sum_{m=-l}^l q^{\frac{1}{2}m} c_m e^l_m\Big) = 0$$

$$\Longleftrightarrow t^l(X_\sigma)\,\Big(\sum_{m=-l}^l q^{-\frac{1}{2}m} c_{-m} e^l_m\Big) = 0.$$

So, by Theorem 4.3 and Lemma 4.2:

Lemma 4.6. $t^l(X_\sigma)$ and $t^l(X^*_\sigma)$ have zero-dimensional null space if $l \in \mathbb{Z}_+ + \frac{1}{2}$ and one-dimensional null space spanned by

$$\sum_{m=-l}^l q^{-\frac{1}{2}m} c^{l,\sigma}_m e^l_m \qquad \text{resp.} \qquad \sum_{m=-l}^l q^{\frac{1}{2}m} c^{l,\sigma}_m e^l_m$$

if $l \in \mathbb{Z}_+$. Here

$$c^{l,\sigma}_m := \frac{i^m\, q^{-(l+\sigma)m}\, q^{m^2/2}}{(q^2;q^2)^{1/2}_{l+m}\,(q^2;q^2)^{1/2}_{l-m}}\; {}_3\phi_2\left(\begin{matrix} q^{-2l+2m}, q^{-2l}, -q^{-2l-2\sigma} \\ q^{-4l}, 0 \end{matrix}; q^2, q^2\right) = c^{l,\sigma}_{-m}. \tag{4.12}$$

Furthermore, the subspace of (σ,τ)-spherical elements in $\mathcal{A}^l_q$ is zero-dimensional if $l \in \mathbb{Z}_+ + \frac{1}{2}$ and one-dimensional if $l \in \mathbb{Z}_+$. For $l \in \mathbb{Z}_+$, the (σ,τ)-spherical elements are spanned by

$$\sum_{n,m=-l}^l q^{(n-m)/2} c^{l,\sigma}_m\, \overline{c^{l,\tau}_n}\, t^l_{n,m}. \tag{4.13}$$

The symmetry $c_m^{l,\sigma} = c_{-m}^{l,\sigma}$ in (4.12) follows from Lemma 4.5, but this symmetry can also be shown for the $_3\phi_2$ in (4.12) by iteration of [7, (3.2.3)].

We already found that the (σ,τ)-spherical elements in $\mathcal{A}_q^1$ were spanned by the element given by (4.7). Since, by Proposition 3.5, the l-fold tensor product of the representation t^1 will be a direct sum of irreducible representations equivalent to t^k, $k = 0, 1, \ldots, l$, the polynomials of degree $\leq l$ in $\rho_{\sigma,\tau}$ will certainly be (σ,τ)-spherical elements contained in $\oplus_{k=0}^l \mathcal{A}_q^l$. On the other hand, the algebra homomorphism $\pi_{\theta/2}^1 \colon \mathcal{A}_q \to \mathbb{C}$ (cf. (3.22)) sends $(\rho_{\sigma,\tau})^k$ to $(\cos\theta)^k$, so the monomials $(\rho_{\sigma,\tau})^k$ will be linearly independent in $\mathcal{A}_q$. So the element given by (4.13) must be a polynomial of degree l in $\rho_{\sigma,\tau}$. Thus we can state:

Proposition 4.7. The algebra of (σ,τ)-spherical elements in $\mathcal{A}_q$ is generated by $\rho_{\sigma,\tau}$ (given by (4.8)) and is, as a linear space, the direct sum of the (σ,τ)-spherical elements in $\mathcal{A}_q^l$ ($l = 0, 1, 2, \ldots$), which are spanned by

$$\sum_{n,m=-l}^{l} q^{(n-m)/2} c_m^{l,\sigma} \overline{c_n^{l,\tau}} \, t_{n,m}^l = P_l^{\sigma,\tau}(\rho_{\sigma,\tau}), \tag{4.14}$$

where $P_l^{\sigma,\tau}$ is a certain polynomial of degree l.

Apply $\pi_{-\theta/2}^1$ to both sides of (4.14). Then, by (3.23):

$$\sum_{n=-l}^{l} c_n^{l,\sigma} \overline{c_n^{l,\tau}} \, e^{in\theta} = P_l^{\sigma,\tau}(\cos\theta). \tag{4.15}$$

Remark 4.8. Consider (4.15) with $\sigma = \tau$, together with (4.12). Compare it with (2.12) together with (2.14). Then we obtain

$$P_l^{\sigma,\sigma} = \frac{|c_l^{l,\sigma}|^2}{(q^{2l+2}; q^2)_l} \, p_l(\,.\,; -q^{2\sigma+1}, -q^{-2\sigma+1}, q, q \mid q^2), \tag{4.16}$$

where p_l is an Askey-Wilson polynomial (2.2).

Lemma 4.9. We have

$$h\big(P_{l'}^{\sigma,\tau}(\rho)^* P_l^{\sigma,\tau}(\rho)\big) = \delta_{l,l'} \, \frac{(1-q^2)q^{2l}}{1 - q^{2(2l+1)}} \, P_l^{\sigma,\sigma}(\tfrac{1}{2}(q+q^{-1})) \, P_l^{\tau,\tau}(\tfrac{1}{2}(q+q^{-1})). \tag{4.17}$$

Proof. Apply (3.21) and (4.14). The case $l \neq l'$ is clear. For $l = l'$ we have:

$$\begin{aligned}
h\big(P_l^{\sigma,\tau}(\rho)^* P_l^{\sigma,\tau}(\rho)\big) &= \sum_{n,m=-l}^{l} q^{n-m} c_n^{l,\tau} \overline{c_m^{l,\sigma}} \overline{c_n^{l,\tau}} c_m^{l,\sigma} \, q^{2(l-n)} \, \frac{1-q^2}{1 - q^{2(2l+1)}} \\
&= \frac{(1-q^2)q^{2l}}{1 - q^{2(2l+1)}} \left(\sum_{m=-l}^{l} q^{-m} |c_m^{l,\sigma}|^2 \right) \left(\sum_{n=-l}^{l} q^{-n} |c_n^{l,\tau}|^2 \right) \\
&= \frac{(1-q^2)q^{2l}}{1 - q^{2(2l+1)}} \, P_l^{\sigma,\sigma}(\tfrac{1}{2}(q+q^{-1})) \, P_l^{\tau,\tau}(\tfrac{1}{2}(q+q^{-1})). \qquad \square
\end{aligned}$$

5. The action of the Casimir operator

Let $\lambda \in \mathbb{Z}_+$, let Ω be the Casimir element given by (3.7), let $\rho_{\sigma,\tau}$ be given by (4.8) and $P_l^{\sigma,\tau}$ by (4.15). We have

$$
\begin{aligned}
\langle A^\lambda \Omega, P_l^{\sigma,\tau}(\rho_{\sigma,\tau}) \rangle &= \langle A^\lambda, \Omega.P_l^{\sigma,\tau}(\rho_{\sigma,\tau}) \rangle \\
&= \left(\frac{q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}}}{q^{-1}-q} \right)^2 \langle A^\lambda, P_l^{\sigma,\tau}(\rho_{\sigma,\tau}) \rangle \\
&= \left(\frac{q^{-l-\frac{1}{2}} - q^{l+\frac{1}{2}}}{q^{-1}-q} \right)^2 P_l^{\sigma,\tau}(\tfrac{1}{2}(q^\lambda + q^{-\lambda})),
\end{aligned} \tag{5.1}
$$

where the second identity follows from (3.20). Let X_σ be given by (4.1).

Lemma 5.1. We have

$$
q(q^{-1}-q)^2 A^\lambda \Omega \ \in \ f(q^\lambda)\,(A^{\lambda+2} - A^\lambda) + f(q^{-\lambda})\,(A^{\lambda-2} - A^\lambda) + (1-q)^2\,A^\lambda + \mathcal{U}_q\,X_\sigma + X_\tau\,\mathcal{U}_q\,, \tag{5.2}
$$

where

$$
f(q^\lambda) := \frac{(1+q^{\sigma+\tau+1+\lambda})\,(1+q^{-\sigma-\tau+1+\lambda})\,(1-q^{\sigma-\tau+1+\lambda})\,(1-q^{-\sigma+\tau+1+\lambda})}{(1-q^{2\lambda})\,(1-q^{2\lambda+2})}. \tag{5.3}
$$

Proof. If $Y, Z \in \mathcal{U}_q$ then $Y \sim Z$ will mean that

$$
Y \ \in \ Z + \mathcal{U}_q X_\sigma + X_\tau \mathcal{U}_q\,.
$$

First observe that

$$
A^\lambda BC = q^\lambda(B - q^{-1}C)A^\lambda C + q^\lambda C A^\lambda(q^{-1}C - B) + q^{2\lambda} A^\lambda CB.
$$

Substitute

$$
A^\lambda CB = A^\lambda BC - A^\lambda(A^2 - D^2)/(q - q^{-1})
$$

and

$$
q^{\frac{1}{2}}B - q^{-\frac{1}{2}}C = -i\big(X_\tau + (q^{-\tau} - q^\tau)(A - D)/(q^{-1} - q)\big) \tag{5.4}
$$

and similarly for X_σ. Then

$$
\begin{aligned}
iq^{\frac{1}{2}}(1-q^{2\lambda})(q^{-1}-q)\,A^\lambda BC \sim{}& \big(q^{-1}(q^{-\tau} - q^\tau) - q^\lambda(q^{-\sigma} - q^\sigma)\big)CA^{\lambda+1} \\
&- \big(q(q^{-\tau} - q^\tau) - q^\lambda(q^{-\sigma} - q^\sigma)\big)CA^{\lambda-1} + iq^{2\lambda+\frac{1}{2}}(A^{\lambda+2} - A^{\lambda-2}).
\end{aligned} \tag{5.5}
$$

Observe that

$$
CA^\mu = (C - qB)A^\mu + q^{1-\mu} A^\mu(B - q^{-1}C) + q^{-2\mu}CA^\mu.
$$

Substitute again (5.4) and its analogue for X_σ. We obtain

$$
(q^{-2\mu} - 1)(q^{-1} - q)\,CA^\mu \sim i\big(q^{\frac{1}{2}-\mu}(q^{-\sigma} - q^\sigma) - q^{\frac{1}{2}}(q^{-\tau} - q^\tau)\big)(A^{\mu+1} - A^{\mu-1}).
$$

Substitute this last equivalence in (5.5). We obtain

$$
\begin{aligned}
(1 - q^{2\lambda})&(q^{-1} - q)^2\,A^\lambda BC \\
\sim{}& (q^{-2\lambda-2} - 1)^{-1}\big(q^{-1}(q^{-\tau} - q^\tau) - q^\lambda(q^{-\sigma} - q^\sigma)\big)\big(q^{-1-\lambda}(q^{-\sigma} - q^\sigma) - (q^{-\tau} - q^\tau)\big)(A^{\lambda+2} - A^\lambda) \\
&- (q^{-2\lambda+2} - 1)^{-1}\big(q(q^{-\tau} - q^\tau) - q^\lambda(q^{-\sigma} - q^\sigma)\big)\big(q^{1-\lambda}(q^{-\sigma} - q^\sigma) - (q^{-\tau} - q^\tau)\big)(A^\lambda - A^{\lambda-2}) \\
&+ q^{2\lambda}(q^{-1} - q)(A^{\lambda+2} - A^{\lambda-2}).
\end{aligned}
$$

Now add $(1 - q^{2\lambda})\,A^\lambda(q^{-1}A^2 + qD^2 - 2)$ to both sides and next multiply both sides with $q(1 - q^{2\lambda})^{-1}$. Then the left hand side becomes $q(q^{-1} - q)^2\,A^\lambda \Omega$ and the right hand side can be rewritten as

$$
f(q^\lambda)\,(A^{\lambda+2} - A^\lambda) + f(q^{-\lambda})\,(A^{\lambda-2} - A^\lambda) + (1-q)^2\,A^\lambda
$$

with f given by (5.3). $\qquad\square$

Substitute (5.2) into the left hand side of (5.1). With the notation

$$R_l(q^\lambda) := P_l^{\sigma,\tau}(\tfrac{1}{2}(q^\lambda + q^{-\lambda}))$$

we obtain

$$f(q^\lambda)\,(R_l(q^{\lambda+2}) - R_l(q^\lambda)) + f(q^{-\lambda})\,(R_l(q^{\lambda-2}) - R_l(q^\lambda)) = -(1 - q^{-2l})(1 - q^{2l+2})\,R_l(q^\lambda). \quad (5.6)$$

Since this is an identity of rational functions in q^λ which is valid for infinitely many values of q^λ, it will remain valid if q^λ is arbitrarily complex, in particular if q^λ is replaced by $e^{i\theta}$. Then we recognize (5.6) as the second order q-difference equation for Askey-Wilson polynomials, cf. (2.8). Hence

$$R_l(e^{i\theta}) = \text{const.}\, p_l(\cos\theta; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1} \mid q^2),$$

where p_l is an Askey-Wilson polynomial (2.2). We can compute the constant by comparing the coefficient of $e^{il\theta}$ at both sides (use (4.15)). The result generalizing (4.16) is:

Theorem 5.2. The polynomial $P_l^{\sigma,\tau}$ occurring in (4.14) and (4.15) equals

$$P_l^{\sigma,\tau} = \frac{c_l^{l,\sigma}\,\overline{c_l^{l,\tau}}}{(q^{2l+2};q^2)_l}\, p_l(\,.\,; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+1} \mid q^2). \quad (5.7)$$

Theorem 5.3. Let $dm(x) = dm_{a,b,c,d;q}(x)$ be the normalized orthogonality measure for the Askey-Wilson polynomials $p_n(x; a, b, c, d \mid q)$ as in (2.7). Let p be any polynomial. Then

$$h(p(\rho_{\sigma,\tau})) = \int p(x)\,dm_{a,b,c,d;q^2}(x), \quad (5.8)$$

where $a = -q^{\sigma+\tau+1}$, $b = -q^{-\sigma-\tau+1}$, $c = q^{\sigma-\tau+1}$, $d = q^{-\sigma+\tau+1}$.

Proof. By (4.17) (for $l' = 0$) and (5.7) the theorem is valid for $p = P_l^{\sigma,\tau}$ ($l \in \mathbb{Z}_+$). $\qquad\square$

In the proof of Theorem 5.3 we only used the case $l' = 0$ of (4.17). Substitution of (5.7) and (5.8) into (4.17) for general l, l' should yield the full orthogonality relations for the Askey-Wilson polynomials of these particular parameters. We can indeed check that this is true. For $l = l'$ the left hand side of (4.17) becomes

$$\frac{|c_l^{l,\sigma}|^2\,|c_l^{l,\tau}|^2}{(q^{2l+2};q^2)_l^2} \int p_l(x; a, b, c, d; q^2)^2\,dm_{a,b,c,d;q^2}(x),$$

where a, b, c, d are as in Theorem 5.3, while the right hand side becomes

$$\frac{|c_l^{l,\sigma}|^2\,|c_l^{l,\tau}|^2\,(q^2;q^2)_l^2}{(q^{2l+2};q^2)_l^2\,q^{4l}}\,(-q^{2\sigma+2}, -q^{-2\sigma+2}, -q^{2\tau+2}, -q^{-2\tau+2}; q^2)_l\,.$$

These two expressions are equal because of (2.4), (2.5) and (2.6).

Remark 5.4. It follows from (4.15), (4.12) and (5.7) that

$$p_n(\cos\theta; -q^{(\sigma+\tau+1)/2}, -q^{(-\sigma-\tau+1)/2}, q^{(\sigma-\tau+1)/2}, q^{(-\sigma+\tau+1)/2} \mid q) = \sum_{k=-n}^{n} \frac{(q^{n+1};q)_n (q;q)_{2n}}{(q;q)_{n+k}(q;q)_{n-k}}$$

$$\times q^{(n-k)(n-k+\sigma+\tau)/2} {}_3\phi_2 \left[\begin{matrix} q^{-n+k}, q^{-n}, -q^{-n-\sigma} \\ q^{-2n}, 0 \end{matrix} ; q, q \right] {}_3\phi_2 \left[\begin{matrix} q^{-n+k}, q^{-n}, -q^{-n-\tau} \\ q^{-2n}, 0 \end{matrix} ; q, q \right] e^{ik\theta}.$$

$$(5.9)$$

This formula, obtained from the quantum group interpretation, cannot be found in the literature in the case of general σ, τ. For $\sigma = \tau$ we already gave an analytic proof of (5.9) in (2.12), (2.14). In a forthcoming paper [9] we will give an analytic proof for (5.9) in general and even for an extension of it with one more parameter. There it will turn out that the addition formula for classical ultraspherical polynomials (for Legendre polynomials in the case of (5.9)) is a limit case of our result. So it may be considered as an alternative to the Rahman-Verma [19] addition formula. In fact its derivation will be similar as for the Rahman-Verma formula.

6. Little and big q-Jacobi polynomials as limit cases of Askey-Wilson polynomials

Propositions 6.1 and 6.3 in this section are limit results for special functions, motivated by quantum group theory, but independent of quantum groups in formulation and proof. Before the author's paper [11] these limits have not been mentioned in literature, although R. Askey told me that he had been aware of them already some years ago.

Let X_σ ($\sigma \in \mathbb{R}$) be given by (4.1). Let

$$\mathcal{B}_\sigma := \{a \in \mathcal{A}_q \mid X_\sigma.a = 0\}.$$

By Lemma 3.2, $\mathcal{B}_\sigma$ is a subalgebra of $\mathcal{A}_q$ and, by (3.12) and (4.2), $\mathcal{B}_\sigma$ is moreover a $*$-subalgebra. It follows from (3.11) that $\Delta(\mathcal{B}_\sigma) \subset \mathcal{A}_q \otimes \mathcal{B}_\sigma$. Thus the quantum group $SU_q(2)$ corresponding to the Hopf $*$-algebra $\mathcal{A}_q$ acts on the quantum space corresponding to the $*$-algebra $\mathcal{B}_\sigma$. Thus it is natural to conjecture that this quantum action of $SU_q(2)$ coincides with its action on some quantum sphere as considered by Podleś [18]. According to Noumi & Mimachi [17, §5] this is indeed the case and they have made a precise identification between the two models.

Here we will restrict ourselves to the question of finding the elementary (σ, ∞)-spherical elements in $\mathcal{A}_q$. These can also be characterized as the elements of $\mathcal{B}_\sigma$ belonging to irreducible subspaces (with respect to $SU_q(2)$) and being invariant with respect to the diagonal quantum subgroup of $SU_q(2)$. We will obtain these (σ, ∞)-spherical elements as limit cases for $\tau \to \infty$ of the corresponding (σ, τ)-spherical elements.

It follows from Proposition 4.7, Theorem 5.2 and (2.3) that the (σ, ∞)-spherical elements in $\mathcal{A}_q^l$ ($l = 0, 1, 2, \ldots$) are spanned by

$$\lim_{\tau \to \infty} r_l(\rho_{\sigma,\tau}; q^{\tau-\sigma+1}, q^{-\tau+\sigma+1}, -q^{-\tau-\sigma+1}, -q^{\tau+\sigma+1} \mid q^2), \qquad (6.1)$$

provided this limit exists and is nonzero. The limit can be obtained from the following limit transition from general Askey-Wilson polynomials to general big q-Jacobi polynomials.

Proposition 6.1. Let Askey-Wilson polynomials and big q-Jacobi polynomials be denoted by (2.3) and (2.16), respectively. Then

$$\lim_{a \to 0} r_n \left(\frac{q^{1/2}x}{2a(cd)^{1/2}}; q^{\alpha+1/2}a(d/c)^{1/2}, q^{1/2}a^{-1}(c/d)^{1/2}, -q^{1/2}a^{-1}(d/c)^{1/2}, -q^{\beta+1/2}a(c/d)^{1/2} \mid q \right)$$
$$= P_n^{(\alpha,\beta)}(x; c, d; q). \tag{6.2}$$

Proof. The left hand side of (6.2) can be written as

$$\sum_{k=0}^{n} \frac{(q^{-n}, q^{n+\alpha+\beta+1}; q)_k \, q^k}{(q^{\alpha+1}, -q^{\alpha+1}d/c, -q^{\alpha+\beta+1}a^2, q; q)_k} \prod_{j=0}^{k-1} \left(1 - \frac{q^{\alpha+1}x}{c} q^j + \frac{q^{2\alpha+1}a^2d}{c} q^{2j} \right). \qquad \square$$

From (6.1), (6.2) and (4.7) we now obtain:

Theorem 6.2. The (σ, ∞)-spherical elements in $\mathcal{A}_q^l$ $(l = 0, 1, 2, \ldots)$ are spanned by

$$P_l^{(0,0)}(q^{-1}(1 - q^{2\sigma})\beta\gamma - iq^{\sigma-1}(\delta\beta + q\gamma\alpha); q^{2\sigma}, 1; q^2),$$

where $P_l^{(0,0)}$ is a big q-Jacobi polynomial.

The above theorem corresponds nicely with the interpretation of big q-Jacobi polynomials on quantum spheres by Noumi & Mimachi [16].

We can also try to get the (∞, ∞)-spherical elements in $\mathcal{A}_q^l$ by the limit

$$\lim_{\tau \to \infty} r_l(\rho_{-\tau,\tau}; q^{2\tau+1}, q^{-2\tau+1}, -q, -q \mid q^2). \tag{6.3}$$

For this we need:

Proposition 6.3. Let Askey-Wilson polynomials and little q-Jacobi polynomials be denoted by (2.3) and (2.15), respectively. Then

$$\lim_{a \to 0} r_n \left(\frac{q^{1/2}x}{2a^2}; q^{\alpha+1/2}a^2, q^{1/2}a^{-2}, -q^{1/2}, -q^{\beta+1/2} \mid q \right) = \frac{(q^{\beta+1}; q)_n}{(q^{-n-\alpha}; q)_n} p_n(x; q^\beta, q^\alpha; q). \tag{6.4}$$

Proof. Put $d := a^2$ in the proof of Proposition 6.1. Then we obtain for the limit in (6.4)

$$_3\phi_2 \left[\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\alpha+1}x \\ q^{\alpha+1}, 0 \end{matrix}; q, q \right].$$

Now the proposition follows from [7, (III.7)] and (2.15). $\qquad \square$

From (6.3), (6.4) and (4.7) we now obtain:

Theorem 6.4. The (∞, ∞)-spherical elements in $\mathcal{A}_q^l$ $(l = 0, 1, 2, \ldots)$ are spanned by

$$p_l(-q^{-1}\beta\gamma; 1, 1; q^2),$$

where p_l is a little q-Jacobi polynomial.

The (∞, ∞)-spherical elements in $\mathcal{A}_q^l$ coincide with the biinvariant elements in $\mathcal{A}_q^l$ with respect to the diagonal quantum subgroup of $SU_q(2)$. These last ones are well-known, cf. for instance [10]. where we find the same explicit expression as in Theorem 6.3.

Remark 6.5.   Askey-Wilson polynomials, with q fixed but with dilation of the argument admitted, form a five-parameter family of orthogonal polynomials. When these parameters are chosen as the α, β, a, c, d in the left hand side of (6.2) then, for each choice of α, β, we obtain a three-parameter family of orthogonal polynomials which contain on the one hand the continuous q-Jacobi polynomials in Rahman's notation $P_n^{(\alpha,\beta)}(x; q)$ (cf. [5, (4.17)]) and on the other hand big and little q-Jacobi polynomials as limit cases.

Remark 6.6.   When we compare Proposition 6.1 with Proposition 2.2 we see that the orthogonal polynomials in x after the limit sign in the left hand side of (6.2) will have continuous mass on the interval $[-2a(cd/q)^{1/2}, 2a(cd/q)^{1/2}]$ and discrete mass points on the two sets

$$\{cq^k + a^2 dq^{-k-1} \mid k \in \mathbb{Z}_+, \ q^k > a(qc/d)^{-1/2}\}$$

and

$$\{-dq^k - a^2 cq^{-k-1} \mid k \in \mathbb{Z}_+, \ q^k > a(qd/c)^{-1/2}\}.$$

Clearly, when a tends to 0, the continuous mass interval shrinks to $\{0\}$, while the discrete mass points tend two the two infinite sets $\{cq^k \mid k \in \mathbb{Z}_+\}$ and $\{-dq^k \mid k \in \mathbb{Z}_+\}$, just the location for the mass points of the big q-Jacobi polynomials. A similar remark can be made about the limit transition in Proposition 6.3.

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