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JACOBI FUNCTIONS AND ANALYSIS ON NONCOMPACT SEMISIMPLE LIE GROUPS

CONTENTS

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INTRODUCTION

A Jacobi function $\phi_\lambda^{(\alpha,\beta)}$ $(\alpha,\beta,\lambda\in\mathbb{C},\alpha\neq-1,-2,\ldots)$ is defined as the even C^∞-function on $\mathbb{R}$ which equals 1 at 0 and which satisfies the differential equation

$$(1.1)\qquad (d^2/dt^2+((2\alpha+1)\coth t+(2\beta+1)\operatorname{th} t)d/dt + +\lambda^2+(\alpha+\beta+1)^2)\phi_\lambda^{(\alpha,\beta)}(t) = 0.$$

It can be expressed as an hypergeometric function (cf.(2.4), (2.7)). For $\alpha>-1, |\beta| < \alpha+1$ the Jacobi function occurs as a kernel in the Jacobi transform pair

$$(1.2)\qquad \hat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)(2\operatorname{sh} t)^{2\alpha+1}(2\operatorname{ch} t)^{2\beta+1}dt,$$

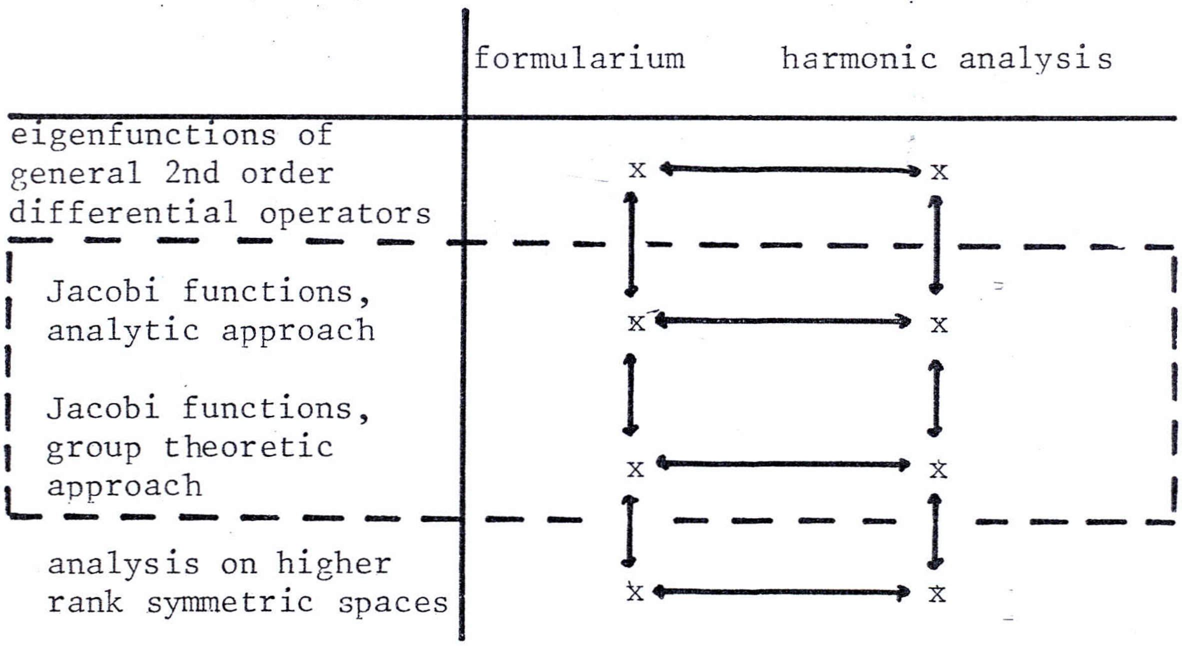
$$(1.3)\qquad f(t) = (2\pi)^{-1}\int_0^\infty \hat{f}(\lambda)\phi_\lambda^{(\alpha,\beta)}(t)|c_{\alpha,\beta}(\lambda)|^{-2}d\lambda,$$

where $c_{\alpha,\beta}(\lambda)$ is a certain quotient of products of gamma functions, cf. (2.18). This transform generalizes the Fourier cosine transform $(\alpha=\beta=-\frac{1}{2})$.

For special α,β there are many group theoretic interpretations of Jacobi functions (cf.§4), first of all as spherical functions on noncompact Riemannian symmetric spaces of rank one, but also as associated spherical functions, intertwining functions, matrix elements of $SL(2,\mathbb{R})$ and spherical functions on certain nonsymmetric Gelfand pairs. This makes Jacobi functions an ideal subject for a case study of the interactions between special functions and group theory: Harmonic analysis can be developed for the Jacobi transform, considered either as a generalization of the Fourier-cosine transform or as a specialization of a group Fourier transform. Both forms of harmonic analysis influence each other. The harmonic analysis applies some "hardware" (a set of meaningful explicit formulas for Jacobi functions), while conversely it raises the need of finding some further formulas. These formulas exist both in analytic and group theoretic form and the two versions of the formulas may be derived from each other.

The interactions already listed form the heart of this paper, but there are two kinds of side interactions which will sometimes be mentioned. First one may consider eigenfunctions of second order ordinary differential operators more general than the one in (1.1) and study the associated transform. For such analysis the Jacobi function case can serve as a prototype, while conversely it makes clear what is general and what is special in the Jacobi case. Second, there is the analysis on noncompact Riemannian symmetric spaces of higher rank, in particular regarding the spherical Fourier transform. This is helpful for putting the rank one analysis in a conceptual framework, while conversely rank one results suggest open problems in the higher rank case.

In the scheme below I summarize these interactions:



The analysis on semisimple symmetric pseudo-Riemannian spaces is a particularly active area in analysis on Lie groups at the moment. I will also discuss some recent developments in this area and the possible relevance of Jacobi functions there.

Many people may have met Jacobi functions without being aware of them, because they were written as hypergeometric functions. Here I want to advertise the use of the Jacobi function notation, because it enables one to make contact with the existing literature on Jacobi functions and because the arrangement of parameters in the Jacobi function notation is better adapted to harmonic analysis than in the hypergeometric notation.

The Contents will give the reader an impression of what is treated inthe various sections. This paper is rather long because it serves a multiple purpose: to provide an introduction to analysis on groups (notably the spherical Fourier transform on rank one spaces), to develop the analytic theory of the Jacobi transform and to discuss the interaction between both. The backbone of this paper is material from a sequence of papers by Flensted-Jensen and the author [41], [48], [81], [49], [50] on the analytic approach to the Jacobi transform with motivation from group theory. For the treatment of analysis on rank one Riemannian symmetric spaces the presentation owes much to work by Helgason [64], [67], [70], [71], [72] and Faraut [39]. The role of the Abel transform, both in the analytic and group theoretic approach, will be emphasized. Possibly new results are given in (5.22), (5.23), §5.4 and §9. Complete proofs will usually not be given, but many proofs are sketched in fairly large detail. Notes with further results and references are often given at the end of a section or subsection.

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How to read this paper

Because of the organization of this paper, it is possible to just read certain parts of it. Here are some suggestions:

(a) One or more subsections of §3 as a tutorial on analysis on groups, eventually to be used in the rest of the paper or elsewhere in this volume.
(b) Only section 2 for main results and history of the Jacobi transform.
(c) Only section 4 (with occasional reference to sections 2,3) for the various group theoretic settings of the Jacobi transform.
(d) Sections 4.1, 5.1, 6, 7, 8.2, 8.3 (with occasional reference to sections 3.1, 3.2) for analysis in the spherical rank one case.
(e) Sections 2, 5.3, 5.4, 6, 7, 8.1, 8.3, 9 for the analytic theory of the Jacobi transform.
(f) Sections 2, 4.1 and 5 for the Abel transform.

2. INVERSION OF THE JACOBI TRANSFORM, STATEMENT OF RESULTS

2.1. Definition of Jacobi functions

The Gaussian hypergeometric function is defined for $|z| < 1$ by the convergent power series

$$(2.1)\qquad {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!}\, z^k,$$

where a,b,c are complex, $c \neq 0,-1,-2,\dots$ and the shifted factorial $(a)_k$ is defined by $(a)_0 := 1$, $(a)_k := a(a+1)\dots(a+k-1)$. It has an analytic continuation to a one-valued analytic function on $\mathbb{C}\backslash[1,\infty)$. Numerous explicit formulas exist for this function of four complex variables a,b,c,z (cf.[33,Ch.2]), which often deter the uninitiated reader. A fruitful way to discover more structure in this formularium is to fix two of the four variables as parameters and to consider the two other variables as the variable and the dual variable in a discrete or continuous orthogonal system. Thus one may consider various families of orthogonal polynomials (Jacobi, Krawtchouk, Meixner and Pollaczek polynomials, see [5, Lecture 2] and [34, 10.21]) and the continuous orthogonal system of Jacobi functions, which will interest us here.

Considered as a function of z, ${}_2F_1(a,b;c;z)$ is the unique solution of the hypergeometric differential equation

$$z(1-z)u''(z)+(c-(a+b+1)z)u'(z)-abu(z) = 0 \tag{2.2}$$

which is regular at 0 and equal to 1 at 0. Now consider c and a+b (or equivalently $\alpha := c-1$ and $\beta := a+b-c$) as parameters and choose the eigenvalue ab in (2.2) as dual variable to z. Depending on the way we restrict the z-variable to an interval connecting two of the three singular points $0,1,\infty$ of (2.2) and on the boundary conditions we get interesting discrete or continuous orthogonal systems. These are in particular the Jacobi polynomials

$$R_n^{(\alpha,\beta)}(x) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(x) := \tag{2.3}$$

$$:= {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\tfrac{1}{2}(1-x)),$$

which, for $n = 0,1,2,\dots$, are orthogonal polynomials on the interval $(-1,1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta(\alpha,\beta>-1)$, and the Jacobi functions

$$\phi_\lambda^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1-i\lambda),\tfrac{1}{2}(\alpha+\beta+1+i\lambda); \alpha+1;-\mathrm{sh}^2 t). \tag{2.4}$$

For $|\beta| < \alpha+1$ the system $\{\phi_\lambda^{(\alpha,\beta)}\}_{\lambda\geq 0}$ is a continuous orthogonal system on $\mathbb{R}_+$ with respect to the weight function

(2.5) $$\Delta_{\alpha,\beta}(t) := (2\operatorname{sh} t)^{2\alpha+1}(2\operatorname{ch} t)^{2\beta+1}, \quad t > 0,$$

cf. Theorem 2.3. Jacobi functions are called so because of their relationship with Jacobi polynomials:

(2.6) $$\phi_\lambda^{(\alpha,\beta)}(t) = R^{(\alpha,\beta)}_{\frac{1}{2}(i\lambda-\alpha-\beta-1)}(\operatorname{ch} 2t).$$

From (2.4) and [33,2.1(22)] one can derive:

(2.7) $$\phi_\lambda^{(\alpha,\beta)}(t) = (\operatorname{ch} t)^{-\alpha-\beta-1-i\lambda} \cdot {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha-\beta+1+i\lambda;\alpha+1;\operatorname{th}^2 t).$$

If no confusion is possible we will suppress the parameters α,β in our notation and we will write

$$\rho := \alpha+\beta+1.$$

Assume that α,β,λ are complex, $\alpha \neq -1,-2,\ldots$. Let

(2.8) $$L := d^2/dt^2+(\Delta'(t)/\Delta(t))d/dt, \quad t > 0,$$

with Δ given by (2.5), so

(2.9) $$L_{\alpha,\beta} = d^2/dt^2+((2\alpha+1)\coth t+(2\beta+1)\operatorname{th} t)d/dt.$$

Then rewriting of (2.2) shows that ϕ_λ is the unique even C^∞-function v on $\mathbb{R}$ such that $v(0) = 1$ and

(2.10) $$(L+\lambda^2+\rho^2)v = 0.$$

It follows from this characterization of ϕ_λ that

(2.11) $$\phi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = \cos\lambda t, \phi_\lambda^{(\frac{1}{2},\frac{1}{2})}(t) = \frac{2\sin\lambda t}{\lambda \operatorname{sh}2t}.$$

2.2. <u>The Jacobi transform</u>

Let us define the <u>(Fourier-) Jacobi transform</u> $f \mapsto \hat{f}$ by

(2.12) $$\hat{f}(\lambda) := \int_0^\infty f(t)\phi_\lambda(t)\Delta(t)dt$$

for all functions f on $\mathbb{R}_+$ and complex numbers λ for which the right hand side is well-defined, possibly by analytic continuation with respect to α. For instance, if

$f \in \mathcal{D}_{even}(\mathbb{R})$ (the space of even C^∞-functions with compact support on $\mathbb{R}$) then, for $n = 0,1,2,\ldots$, (2.12) has an analytic continuation from $\operatorname{Re}\alpha > -1$ to $\operatorname{Re}\alpha > -n-1$, $\alpha \neq -1,-2,\ldots,-n$, in the form

$$(2.13)\qquad \hat{f}(\lambda) = \frac{(-1)^n}{2^{4n}(\alpha+1)_n}\int_0^\infty \left(\frac{1}{\operatorname{sh}2t}\frac{d}{dt}\right)^n f(t)\cdot$$

$$\cdot\ \phi_\lambda^{(\alpha+n,\beta+n)}(t)\Delta_{\alpha+n,\beta+n}(t)dt,$$

where we used [33,2.8(27)]. Thus, in all cases, if $f \in \mathcal{D}_{even}(\mathbb{R})$ then $\hat{f}$ is an even entire analytic function and, by self-adjointness of L with respect to $\Delta(t)dt$, we have

$$(2.14)\qquad (Lf)^\wedge(\lambda) = -(\lambda^2+\rho^2)\hat{f}(\lambda).$$

Note that $f \mapsto \hat{f}$ reduces to the Fourier-cosine transform for $\alpha = \beta = -\frac{1}{2}$ and that it is immediately related to the Fourier-sine transform if $\alpha = \beta = \frac{1}{2}$. In order to give meaning to the notion "continuous orthogonal system" for the functions ϕ_λ we have to invert the transform $f \mapsto \hat{f}$. To formulate the inversion formula we will introduce the function $\lambda \mapsto c(\lambda)$ occuring in the asymptotics of $\phi_\lambda(t)$ as $t \to \infty$.

From [33,§2.9] or by straightforward computation we obtain, for $\lambda \neq -i,-2i,\ldots$, another solution Φ_λ of (2.10) on $(0,\infty)$ given by

$$(2.15)\qquad \Phi_\lambda^{(\alpha,\beta)}(t) :=$$

$$:= (2\operatorname{ch} t)^{i\lambda-\rho}\,{}_2F_1(\tfrac{1}{2}(\rho-i\lambda),\tfrac{1}{2}(\alpha-\beta+1-i\lambda);1-i\lambda;\operatorname{ch}^{-2}t)=$$

$$= (2\operatorname{sh} t)^{i\lambda-\rho}\,{}_2F_1(\tfrac{1}{2}(\rho-i\lambda),\tfrac{1}{2}(-\alpha+\beta+1-i\lambda);1-i\lambda;-\operatorname{sh}^{-2}t).$$

(The second equality is implied by [33,2.1(22)].) Thus

$$(2.16)\qquad \Phi_\lambda(t) = e^{(i\lambda-\rho)t}(1+o(1)) \text{ as } t \to \infty.$$

For $\lambda \notin i\mathbb{Z}$, Φ_λ and $\Phi_{-\lambda}$ are two linearly independent solutions of (2.10), so ϕ_λ is a linear combination of both. In fact, we get from [33,2.10(2) and 2.10(5)] that, for $\lambda \notin i\mathbb{Z}$,

$$(2.17)\qquad \phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda},$$

where

$$(2.18)\qquad c(\lambda) = c_{\alpha,\beta}(\lambda) := \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}.$$

Hence

$$(2.19)\qquad \phi_\lambda(t) = c(\lambda)e^{(i\lambda-\rho)t}(1+o(1)) \text{ as } t\to\infty \text{ if } \mathrm{Im}\lambda < 0.$$

This follows from (2.16) and (2.17) for $\lambda \neq -i,-2i,\ldots$, but it extends to $\mathrm{Im}\,\lambda < 0$. We can now formulate the main theorems about the Jacobi transform. Their proofs will be postponed until §6.

Theorem 2.1 (Paley-Wiener theorem). For all complex α,β with $\alpha \neq -1,-2,\ldots$ the Jacobi transform is a 1-1 map of $\mathcal{D}_{even}(\mathbb{R})$ onto the space of all even entire functions g for which there are positive constants A_g and $C_{g,n}$ $(n=0,1,2,\ldots)$ such that

$$(2.20)\qquad |g(\lambda)| \le C_{g,n}(1+|\lambda|)^{-n}e^{A_g|\mathrm{Im}\lambda|},\quad \lambda\in\mathbb{C},\ n=0,1,\ldots$$

For $\alpha = \beta = -\frac{1}{2}$ this is the classical Paley-Wiener theorem (cf. Rudin [118, Theorem 7.22]). As a refinement of Theorem 2.1 we have that f has support in $[-a,a]$ iff $\hat{f}$ satisfies (2.20) with $A_{\hat{f}} := a$.

Theorem 2.2 (inversion formula, first form). If $\alpha,\beta\in\mathbb{C}$, $\alpha\neq -1,-2,\ldots$, $\mu\ge 0$, $\mu > -\mathrm{Re}(\alpha\pm\beta+1)$, $f\in\mathcal{D}_{even}(\mathbb{R})$, $t > 0$, then

$$(2.21)\qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\lambda+i\mu)\phi_{\lambda+i\mu}(t)(c(-\lambda-i\mu))^{-1}d\lambda.$$

For convenience assume now that $\alpha > -1$, $\beta\in\mathbb{R}$. Then $\lambda\mapsto(c(\lambda))^{-1}$ has only simple poles for $\mathrm{Im}\,\lambda\ge 0$ which lie in the finite set

$$(2.22)\qquad D_{\alpha,\beta} := \{i(|\beta|-\alpha-1-2m)\mid m=0,1,2,\ldots;|\beta|-\alpha-1-2m>0\}.$$

If $|\beta|\le\alpha+1$ then $D_{\alpha,\beta}$ is empty. Put

$$(2.23)\qquad d(\lambda) := -i\,\mathrm{Res}_{\mu=\lambda}(c(\mu)c(-\mu))^{-1}),\quad \lambda\in D_{\alpha,\beta}.$$

The next two theorems are versions of the Plancherel theorem for the Jacobi transform.

Theorem 2.3 (inversion formula, second form). If $\alpha > -1$, $\beta \in \mathbb{R}$, $f \in \mathcal{D}_{even}(\mathbb{R})$, $t \in \mathbb{R}$, then

(2.24) $$f(t) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}d\lambda + \sum_{\lambda\in D_{\alpha,\beta}} \hat{f}(\lambda)\phi_\lambda(t)d(\lambda).$$

If, moreover, $\alpha\pm\beta + 1 \geq 0$ then

(2.25) $$f(t) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}d\lambda.$$

Define the measure ν on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ by

(2.26) $$\int_{\mathbb{R}_+\cup D_{\alpha,\beta}} g(\lambda)d\nu(\lambda) := \frac{1}{2\pi}\int_0^\infty g(\lambda)|c(\lambda)|^{-2}d\lambda + \sum_{\lambda\in D_{\alpha,\beta}} g(\lambda)d(\lambda).$$

Theorem 2.4 (Parseval formula). If $\alpha > -1$, $\beta \in \mathbb{R}$ and $f,g \in \mathcal{D}_{even}(\mathbb{R})$ then

(2.27) $$\int_0^\infty f(t)\overline{g(t)}\Delta(t)dt = \int_{\mathbb{R}_+\cup D_{\alpha,\beta}} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}d\nu(\lambda).$$

The mapping $f \mapsto \hat{f}$ extends to an isometry of $L^2(\mathbb{R}_+, \Delta(t)dt)$ onto $L^2(\mathbb{R}_+ \cup D_{\alpha,\beta}, \nu)$.

In this form these theorems are proved in [41], [44, Appendix 1] and [81], but special cases of Theorems 2.3, 2.4 have a long history. (We will deal with background to Theorem 2.1 in §6.) Mehler [103] (without proof), Heine [63], both in 1881, and Fock [52] in 1943 treated the case $\alpha = \beta = 0$. Consideration of a potential problem for the spherical lens (cf. Mehler [102], Lebedev [94]) brought them to the introduction of what is now known as the Mehler-Fock transform. This is usually formulated in terms of Legendre functions

(2.28) $$P_{i\mu-\frac{1}{2}}(\mathrm{ch}\,2t) := \phi_{2\mu}^{(0,0)}(t)$$

as

$$(2.29)\qquad \begin{aligned} f(\mu) &= \mu\,\mathrm{th}\,\pi\mu\cdot\int_1^\infty P_{i\mu-\frac12}(x)\psi(x)\,dx,\\ \psi(x) &= \int_0^\infty P_{i\mu-\frac12}(x)f(\mu)\,d\mu. \end{aligned}$$

Fock [52] gave precise conditions on f or ψ (in terms of integrability of the function and some of its derivatives) in order that the one integral transform would invert the other.

In the literature one can find four main methods of proving Theorems 2.3, 2.4 or special cases of them, but interaction between the various approaches is rare:

a) Factorization of $f \mapsto \hat f$ as $f = F(F_f)$, where F is the classical Fourier transform and $f \mapsto F_f$ is the so-called Abel transform, which can be explicitly given and explicitly inverted. This fact is exploited in Heine [63,§75], Fock [52], Koornwinder [82] and, in group theoretic context, in Vilenkin [138], [139,Ch.10,§4] and Takahashi [129, 4.2]. We will return to this in section 5.
b) Use of spectral theory of second order o.d.e.'s with singularities at one or both endpoints of the interval. The Jacobi differential equation then usually occurs as an illustration of a quite general theory. This approach was started by Weyl [144] in 1910. He already has the example of the Jacobi o.d.e., including a remark about occurrence of the discrete spectrum for α,β. Later presentations are in Titchmarsh [136,§4.18-4.20], Dunford & Schwartz [28,Ch.13,§18,pp.1520-1526] (see also Flensted-Jensen [41,pp.155-156]) and Faraut [36], where [28],[36] emphasize the functional analytic aspects.
c) Use of function theory and asymptotics. Olevskiĭ [111], [112], van Nostrand [108] and Götze [58] proved that (2.25) implies (2.12) under certain conditions on $\hat f$ by use of

$$(2.30)\qquad \begin{aligned} &\int_0^M \phi_\lambda(t)\phi_\mu(t)\Delta(t)\,dt =\\ &= (\mu^2-\lambda^2)^{-1}\Delta(M)\,(\phi_\lambda'(M)\phi_\mu(M)-\phi_\lambda(M)\phi_\mu'(M)) \end{aligned}$$

and the asymptotics of $\phi_\lambda(t),\phi_\lambda'(t)$ as $t\to\infty$. Inversion formulas in both directions were given in the very thorough paper by Braaksma & Meulenbeld [16].
d) As a special case of the inversion of the spherical Fourier transform on noncompact semisimple Lie groups. This was first done by Harish-Chandra [61]. A considerable

simplification of his proof was given by Rosenberg [1977], see also Helgason [71,Ch.4,§7], [72, 4.2].

Yet order approaches occur in Lebedev [92], [93], Roehner & Valent [116], Stein & Wainger [128] and §9 of the present paper. In general we can say that approach a) connects the case (α,β) with the elementary case $(-\frac{1}{2},-\frac{1}{2})$ by the Abel transform $f \to F_f$, which has the transmutation property

$$F_{Lf}(t) = \frac{d^2}{dt^2} F_f(t), \tag{2.31}$$

while the methods b), c), d) approximate the case (α,β) with the case $(-\frac{1}{2},-\frac{1}{2})$ by observing that, as $t \to \infty$, $L_{\alpha,\beta}$ and its eigenfunctions resemble $d^2/dt^2 + 2\rho d/dt$ and its eigenfunctions.

2.3. Generalizations of the Jacobi transform

Another limit case of the Jacobi o.d.e. (2.10) is the Bessel equation

$$v''(t)+(2\alpha+1)t^{-1}v'(t)+\lambda^2 v(t) = 0. \tag{2.32}$$

Indeed, replace (t,λ) in (2.10) by $(\varepsilon t,\varepsilon^{-1}\lambda)$ and let $\varepsilon \downarrow 0$. The unique even C^∞-solution of (2.32) being equal to 1 at 0 equals

$$\mathcal{J}_\alpha(\lambda t) := 2^\alpha \Gamma(\alpha+1)(\lambda t)^{-\alpha} J_\alpha(\lambda t), \tag{2.33}$$

where J_α is the usual Bessel function (cf.[34, (7.2(2))]). Then

$$\lim_{\varepsilon\downarrow 0} \phi^{(\alpha,\beta)}_{\varepsilon^{-1}\lambda}(\varepsilon t) = \mathcal{J}_\alpha(\lambda t). \tag{2.34}$$

A full asymptotic expansion of $\phi_\lambda^{(\alpha,\beta)}(t)$, $t \downarrow 0$, in terms of Bessel functions, which is, to a certain extent, uniform in λt, is given by Schindler [120] $(\alpha=\beta)$ and Stanton & Tomas [127]. (See Duistermaat [27] for a new approach in the group case.)

A lot of work (in particular by the Tunis school) has been done in extending the harmonic analysis for the Jacobi transform to the case of more general second order differential operators L of the form (2.8). It is then usually assumed that, for some $\alpha > -\frac{1}{2}$, (2.35) $t \mapsto t^{-2\alpha-1}\Delta(t)$ is even, strictly positive C^∞-function,

while Trimèche [137] and Chébli [23] allow ρ^2 in (2.10) to be replaced by some even C^∞-function on $\mathbb{R}$. Chébli [20], [21] [22], [24] has the additional conditions

(2.36) $$\Delta(t) \uparrow \infty \text{ as } t \to \infty,\ \frac{\Delta'(t)}{\Delta(t)} \downarrow 2\rho \text{ as } t \to \infty.$$

The conditions (2.35), (2.36) are satisfied by the Bessel equation (2.32) ($\alpha>-\frac{1}{2}$) and the Jacobi equation (2.10) with (2.9) ($-\frac{1}{2}\neq\alpha\geq\beta\geq-\frac{1}{2}$). Under condition (2.35) ϕ_λ can again be defined as the even C^∞-solution of (2.10) being equal to 1 at 0 and the transform $f \mapsto \hat{f}$ by (2.12). Under conditions (2.34), (2.35) there are also solutions Φ_λ and a function c such that (2.16), (2.17) hold. Chébli [20], [21], [22], using approach b), obtains the analogue of the inversion formula (2.25). Trimèche [137], having only condition (2.35), uses approach a) and obtains an inversion formula in the form

(2.37) $$f(t) = \int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)d\nu_1(\lambda)+\int_0^\infty \hat{f}(i\lambda)\phi_{i\lambda}(t)d\nu_2(\lambda),$$

where ν_1 and ν_2 are positive measures, ν_1 being tempered and ν_2 such that $\int_0^\infty e^{a\lambda}d\nu_2(\lambda) < \infty$ for all $a > 0$. Both Chébli [24] and Trimèche [137] have asymptotic expansions of $\phi_\lambda(t)$ in terms of Bessel functions which are reminiscent to Stanton & Tomas [127]. Chébli uses results of Langer [91], Trimèche of Olver [113,Ch.12].

3. SOME GROUP THEORETIC PRELIMINARIES (A TUTORIAL)

In this section we put together some background material from the general theory of analysis on Lie groups. The five subsections (see contents) are rather disconnected. Each subsection may be used as a reference in reading other parts of this paper or as a tutorial to this whole volume or it may just be read for its own interest.

3.1. Structure theory of noncompact semisimple Lie groups

References for this subsection are [70] and [86,Ch.1,2,3]. Let $\mathfrak{g}$ be a real <u>semisimple</u> Lie algebra, i.e., $\mathfrak{g}$ is a real Lie algebra on which the <u>Killing form</u> $B(X,Y) := \mathrm{tr}(\mathrm{ad}X\,\mathrm{ad}Y)$ $(X,Y\in\mathfrak{g})$ is nondegenerate as bilinear form on $\mathfrak{g}$. Let θ be a

Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$, i.e., θ is an automorphism of $\mathfrak{g}$, $\theta^2 = \mathrm{id}$, $\mathfrak{k}$ and $\mathfrak{p}$ are eigenspaces of θ for eigenvalues 1 and -1, respectively, and B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Such involutions do exist and they are all conjugate under inner automorphisms. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G is called a semisimple Lie group. It can be shown that the Cartan involution θ is the differential of a unique involutive automorphism of G, also denoted by θ, and that the fixed point subgroup $K := \{g \in G \mid \theta(g) = g\}$ is precisely the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. The subgroup K is compact iff G has finite center; then K is also a maximal compact subgroup. Assume that G has finite center (such a choice of G is possible).

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. All such subspaces are conjugate under Ad(K). The (real) rank of $\mathfrak{g}$ is defined as the dimension of $\mathfrak{a}$. We will exclude the rank zero case, i.e., we assume that $\theta \neq \mathrm{id}$. For α in $\mathfrak{a}^*$ (the real linear dual of $\mathfrak{a}$) put $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X$ for all H in $\mathfrak{a}\}$. Then $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. Put $\mathfrak{m} := \mathfrak{g}_0 \cap \mathfrak{k}$, $\Sigma := \{\alpha \in \mathfrak{a}^*\backslash\{0\} \mid \dim \mathfrak{g}_\alpha > 0\}$. Then $\mathfrak{g}$, as a linear space, has the direct sum decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \Sigma_{\alpha\in\Sigma}\, \mathfrak{g}_\alpha$. Since B, restricted to $\mathfrak{a}$, is an inner product, it canonically gives rise to an inner product on $\mathfrak{a}^*$, denoted by $\langle .,. \rangle$. The triple $\{\mathfrak{a}^*, \langle .,. \rangle, \Sigma\}$ satisfies the axioms of a root system: Span $\Sigma = \mathfrak{a}^*$ and for all α,β in Σ, $\beta - 2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle\, \alpha \in \Sigma$ and $2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle \in \mathbb{Z}$. This root system is not necessarily reduced: if $\alpha, c\alpha \in \Sigma$ then $c = \pm\frac{1}{2}, \pm 1$ or ± 2; in a reduced root system only ± 1 would be possible. Introduce some linear vector space ordering $<$ on $\mathfrak{a}^*$. Put $\Sigma^+ := \{\alpha \in \Sigma \mid \alpha > 0\}$, $\mathfrak{n} := \Sigma_{\alpha\in\Sigma^+}\, \mathfrak{g}_\alpha$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. Put $m_\alpha := \dim \mathfrak{g}_\alpha$, $\rho := \frac{1}{2} \Sigma_{\alpha\in\Sigma^+}\, m_\alpha \alpha$.

Let $N := \exp \mathfrak{n}$, $A := \exp \mathfrak{a}$. Then N,A are closed subgroups of G and diffeomorphic images of $\mathfrak{n},\mathfrak{a}$ under exp. Let $\bar{N} := \theta(N)$. Let $M := Z_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)H = H$ for all H in $\mathfrak{a}\}$, $M' := N_K(\mathfrak{a}) = \{k \in K \mid \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\}$. Then M,M' are compact subgroups of K, both with Lie algebra $\mathfrak{m}$, M is a normal subgroup of M', the group $W := M'/M$ is finite. Via Ad the group M'/M acts as a group of orthogonal transformations on $\mathfrak{a}$, and hence on $\mathfrak{a}^*$. Under this identification W can be shown to be isomorphic to the Weyl group of the root system Σ, i.e. the group generated by the reflections $\lambda \to \lambda - 2\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\, \alpha$ of $\mathfrak{a}^*$ $(\alpha\in\Sigma)$.

If $\mathfrak{g}$ has rank one then, for some α in $\mathfrak{a}^*$, $\Sigma = \{\alpha,-\alpha\}$ or $\{\alpha,-\alpha,2\alpha,-2\alpha\}$. Let H_1 in $\mathfrak{a}$ be such that $\alpha(H_1) = 1$ and write $\mathfrak{g}_{\pm 1}$, $\mathfrak{g}_{\pm 2}$ instead of $\mathfrak{g}_{\pm\alpha}$, $\mathfrak{g}_{\pm 2\alpha}$ with dimension $m_{\pm 1}$, $m_{\pm 2}$, respectively. Choose the ordering on $\mathfrak{a}^*$ such that α is positive. Then $\mathfrak{n} = \mathfrak{g}_1 + \mathfrak{g}_2$. Put $H_t := tH_1$, $a_t := \exp H_t$. We will always keep to these conventions in the rank one case.

We mention some decompositions of G:

(a) $G = K \exp \mathfrak{p}$ (polar decomposition), where $(k,X) \mapsto k \exp X$ is an analytic diffeomorphism of $K \times \mathfrak{p}$ onto G.

(b) $G = KAK$ (Cartan decomposition), where $KaK = KbK (a,b \in A)$ iff $b = kak^{-1}$ for some k in M'. In the rank one case: $Ka_sK = Ka_tK$ iff $s = \pm t$.

(c) $G = KAN$ (Iwasawa decomposition), where $(k,a,n) \mapsto kan$ is an analytic diffeomorphism of $K \times A \times N$ onto G. In the rank one case: if $g \in G$ and $g = ka_tn$ according to this decomposition then write $u(g) := k$, $H(g) := t$.

(d) NMAN is open and dense in G (part of Bruhat decomposition).

In the rank one case we finally need the result that the Ad(K)-orbits on $\mathfrak{p}$ are the spheres $\{X \in \mathfrak{p} \mid B(X,X) = \text{const.}\}$. Up to local isomorphisms the different rank one cases are:

G	$SO_0(1,n)$	$SU(1,n)$	$Sp(1,n)$	$F_{4(-20)}$
K	$SO(n)$	$S(U(1)\times U(n))$	$Sp(1)\times Sp(n)$	$Spin(9)$
n	2,3,4,...	2,3,4,...	2,3,4,...	

Table 1

The first three columns in this Table can be treated in a uniform way as follows (cf.[39]). Let $\mathbb{F} := \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ ($\mathbb{H}$ denotes the skew field of quaternions) with real dimension $d = 1$, 2 or 4. Let $U(p,q;\mathbb{F})$ be the Lie group of (right) linear operators on $\mathbb{F}^{p+q}$ which leave invariant the hermitian form

$$\bar{y}_1 x_1 + \ldots + \bar{y}_p x_p - \bar{y}_{p+1} x_{p+1} - \ldots - \bar{y}_{p+q} x_{p+q}, \quad x,y \in \mathbb{F}^{p+q}.$$

In particular, we will meet the groups $U(n,\mathbb{F}) := U(n,o;\mathbb{F})$ and $U(1,n;\mathbb{F})$. In case of the latter group we label coordinates on $\mathbb{F}^{n+1}$ such that the group elements leave invariant the form

$$\bar{y}_0 x_0 - \bar{y}_1 x_1 - \ldots - \bar{y}_n x_n, \quad x,y \in \mathbb{F}^{n+1}.$$

If $G := U(1,n;\mathbb{F})$ and $\theta g := (g^*)^{-1}$ ($g \in G$, g^* denoting $\mathbb{F}$-hermitian adjoint of g) then θ is an involutive automorphism of G and $K := U(1,\mathbb{F}) \times U(n,\mathbb{F}) := \{\begin{pmatrix} u & 0 \\ 0 & V \end{pmatrix} \mid u \in U(1,\mathbb{F}), V \in U(n,\mathbb{F})\}$ is the subgroup of elements fixed under θ. For $\mathbb{F} := \mathbb{H}$ this pair (G,K) matches the third column of Table 1, but for $\mathbb{F} := \mathbb{R}$ or $\mathbb{C}$ the groups $U(1,n;\mathbb{F})$ are reductive groups, bigger than $SO_0(1,n)$ and $SU(1,n)$, respectively, but with semisimple parts equal to the latter groups. This difference is rather harmless since G/K remains the same. All structure theory presented in this subsection will also hold for the groups $G = U(1,n;\mathbb{F})$. The remaining rank one group $F_{4(20)}$, which is related to the octonions, needs individual care. We will not treat it here, but refer to Takahashi [132]. From now on both the groups G from Table 1 and the groups $U(1,n;\mathbb{F})$ will be called <u>rank one groups</u>.

For $G := U(1,n;\mathbb{F})$ we already specified K and θ. Let us list some of the other structural elements of such G (cf. [39]):

$$\mathfrak{p} = \left\{\begin{pmatrix} 0 & z^* \\ z & 0 \end{pmatrix} \;\middle|\; z \in \mathbb{F}^n\right\};$$

$$H_1 := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ \vdots & & 0 & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ 1 & 0 & & 0 \end{pmatrix}, \quad a_t = \begin{pmatrix} \operatorname{ch} t & 0 \ldots 0 & \operatorname{sh} t \\ 0 & & 0 \\ \vdots & I_{n-1} & \vdots \\ 0 & & 0 \\ \operatorname{sh} t & 0 \ldots 0 & \operatorname{ch} t \end{pmatrix};$$

$$M = \left\{\begin{pmatrix} u & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & u \end{pmatrix} \;\middle|\; u \in U(1,\mathbb{F}),\ V \in U(n-1,\mathbb{F})\right\};$$

$$N = \{n_{z,w} \mid z \in \mathbb{F}^{n-1}, \ w \in \operatorname{Im}\mathbb{F}\}, \text{ where}$$

$$(3.1) \quad n_{z,w} := \begin{pmatrix} 1+\frac{1}{2}|z|^2+w & z^* & -\frac{1}{2}|z|^2-w \\ z & I_{n-1} & -z \\ \frac{1}{2}|z|^2+w & z^* & 1-\frac{1}{2}|z|^2-w \end{pmatrix}$$

and $\mathrm{Im}\mathbb{F} := \{w \in \mathbb{F} \mid w + \bar{w} = 0\}$;

(3.2) $m_1 = d(n-1), \quad m_2 = d-1,$

(3.3) $\rho := \frac{1}{2}(m_1+2m_2) = \frac{1}{2}d(n+1)-1.$

We will also use parameters

(3.4) $\alpha := \frac{1}{2}(m_1+m_2-1) = \frac{1}{2}dn-1, \quad \beta := \frac{1}{2}(m_2-1) = \frac{1}{2}d-1.$

3.2. Spherical functions on Gelfand pairs

In this subsection we summarize the facts needed from the general theory of spherical functions. References are Helgason [64,Ch.X], [72,Ch.IV], Faraut [39,Ch.I], Godement [57].

Let G be a locally compact group G with left Haar measure dg. The convolution product on G is defined by

$$(f_1*f_2)(x) := \int_G f_1(y)f_2(y^{-1}x)\,dy, \quad f_1,f_2 \in L^1(G).$$

Let K be a compact subgroup of G with normalized Haar measure dk. The space $C_c(G//K)$ of K-biinvariant continuous functions on G with compact support is an algebra under convolution. The pair (G,K) is called a Gelfand pair if the algebra $C_c(G//K)$ is commutative. This property implies that the group G is unimodular. A sufficient condition in order that (G,K) is a Gelfand pair is the existence of a continuous involutive automorphism θ of G such that $\theta(KxK) = Kx^{-1}K$ for all x in G. In view of the polar decomposition $G = K \exp \mathfrak{p}$ (cf.§3.1) this criterium shows that for rank one groups G the pairs (G,K) are Gelfand pairs.

Let (G,K) be a Gelfand pair and provide $C_c(G//K)$ with the usual topology. Then each nonzero continuous character ω on the commutative topological algebra $C_c(G//K)$ determines a unique ϕ in $C(G//K)$ such that

(3.5) $\omega(f) = \int_G f(x)\phi(x)\,dx$

for all f in $C_c(G//K)$. Such functions ϕ are called spherical functions for the pair (G,K). They satisfy $\phi(e) = 1$. They can also be characterized as the nonzero functions ϕ in $C(G)$ which satisfy the functional equation (product formula)

(3.6) $\phi(x)\phi(y) = \int_K \phi(xky)\,dk, \quad x,y \in G.$

The Banach algebra $L^1(G//K)$ is also commutative. Via (3.5) its nonzero characters are in 1-1 correspondence with the bounded spherical functions. Equip the set of bounded spherical functions with the Gelfand topology.

A continuous function ϕ on a locally compact group G is called <u>positive definite</u> if for each finite subset $\{x_1,\ldots,x_n\}$ of G and for all complex $c_1,\ldots,c_n$

$$\Sigma_{i,j=1}^{n} c_i \bar{c}_j \, \phi(x_i^{-1}x_j) \geq 0$$

or if, equivalently,

$$(3.7) \qquad \int_G \phi(x^{-1}y)f(x)\overline{f(y)}\, dxdy \geq 0$$

for each f in $C_c(G)$. If ϕ is positive definite then

$$(3.8) \qquad |\phi(x)| \leq \phi(e), \quad \phi(x^{-1}) = \overline{\phi(x)}$$

for all x in G. The formula

$$(3.9) \qquad \phi(x) = (\pi(x)e,e), \qquad x \in G,$$

establishes a 1-1 correspondence between the nonzero positive definite functions ϕ on G and the equivalence classes of pairs (π,e), where π is a unitary representation of G and e is a cyclic vector in the representation space $\mathcal{H}(\pi)$ of π. By use of tensor products of representations it follows from this correspondence that the product of two positive definite functions on G is again positive definite.

Let (G,K) be a Gelfand pair. This property can be shown to be equivalent to the fact that the representation 1 of K occurs at most once in each irreducible unitary representation of G. Let $(G/K)^\wedge$ consist of all π in $\hat{G}$ in which the representation 1 of K occurs with multiplicity 1. Now (3.9) gives in particular a 1-1 correspondence between the positive definite spherical functions ϕ and the elements π of $(G/K)^\wedge$, where e is chosen as a K-fixed unit vector in $\mathcal{H}(\pi)$. Via this correspondence and in view of (3.8), $(G/K)^\wedge$ is included in the set of bounded spherical functions. Let $(G/K)^\wedge$ inherit the Gelfand topology of this set.

If $f \in C_c(G//K)$ or $L^1(G//K)$ and ϕ is a spherical function or bounded spherical function, respectively, then write

$$(3.10) \qquad \hat{f}(\phi) := \int_G f(x)\phi(x)dx.$$

The transform $f \mapsto \hat{f}$ is called the spherical Fourier transform associated with the pair (G,K). The Plancherel-Godement theorem states that there is a unique positive measure ν on $(G/K)^\wedge$ (the Plancherel measure) such that

$$(3.11)\qquad f(x) = \int_{(G/K)^\wedge} \hat{f}(\phi)\phi(x)d\nu(\phi), \qquad x \in G,$$

for all continuous and positive definite functions f in $L^1(G//K)$. Moreover, if $f \in L^1(G//K) \cap L^2(G//K)$ then the Parseval formula

$$(3.12)\qquad \int_G |f(x)|^2 dx = \int_{(G/K)^\wedge} |\hat{f}(\phi)|^2 d\nu(\phi)$$

holds and the mapping $f \mapsto \hat{f}$ extends to an isometry of $L^2(G//K)$ onto $L^2(\Omega,\nu)$.

Let (G,K) be a Gelfand pair such that G is a Lie group and G/K is connected. Identify functions on G/K with right-K-invariant functions on G in the obvious way. Then the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K is commutative, the spherical functions ϕ are C^∞-functions and they are joint eigenfunctions of all D in $\mathbb{D}(G/K)$ for certain complex eigenvalues λ_D:

$$(3.13)\qquad D\phi = \lambda_D\phi, \qquad D \in \mathbb{D}(G/K).$$

Hence, since $\mathbb{D}(G/K)$ contains the Laplace-Beltrami operator Ω on G/K, which is elliptic, all spherical functions are analytic. Conversely, if $\phi \in C^\infty(G//K)$, $\phi(e) = 1$ and ϕ satisfies (3.13) for certain eigenvalues λ_D then ϕ is a spherical function uniquely determined by these eigenvalues.

As an application of this last property let G be a rank one group, cf. §3.1. Choose an orthonormal basis $X_1,\ldots,X_r$ of $\mathfrak{p}$. Since the Ad(K)-orbits on $\mathfrak{p}$ are spheres (cf. §3.1), the Ad(K)-invariant polynomials on $\mathfrak{p}$ are polynomial functions of the polynomial $x_1X_1 + \ldots + x_rX_r \mapsto x_1^2 + \ldots + x_r^2$ on $\mathfrak{p}$. Now, by identification of $\mathfrak{p}$ with the tangent space to G/K at eK, B_θ induces a G-invariant Riemannian structure on G/K. Hence the corresponding Laplace-Beltrami operator Ω given by

$$(3.14)\quad (\Omega f)(x) := \left(\frac{\partial^2}{\partial t_1^2} + \ldots + \frac{\partial^2}{\partial t_r^2}\right) f(x \exp(t_1X_1 + \ldots + t_rX_r))\Big|_{t_i=0}$$

($f \in C^\infty(G/K)$) generates the algebra $\mathbb{D}(G/K)$, so to each complex eigenvalue of Ω there corresponds one and only one spherical function as an eigenfunction.

3.3. Associated spherical functions and addition formulas

Let (G,K) be a Gelfand pair. Let A be a subset of G such that $G = KAK$. Let M be the centralizer of A in K, i.e., $M := \{k \in K \mid ka = ak \ \forall a \in A\}$. Suppose that (K,M) is also a Gelfand pair. For instance, these conditions hold if $G = U(1,n;\mathbb{F})$ and K,A,M are as in §3.1. Then (K,M) is a Gelfand pair because the sufficient condition of §3.2 holds with $\theta((g_{ij})) := (\bar{g}_{ij})((g_{ij}) \in U(1,\mathbb{F}) \times U(n,\mathbb{F}))$. (Kostant [87] showed for all rank one cases that (K,M) is a Gelfand pair.)

Let π be a strongly continuous, not necessarily unitary representation of G on Hilbert space $\mathcal{H}(\pi)$ such that $\pi|_K$ is a unitary representation of K which is the direct sum of certain representations in $(K/M)^\wedge$, each occurring with multiplicity one, and which contains the trivial representation of K, so $\pi|_K = \oplus_{\delta \in \mathcal{M}(\pi)} \delta$ for some subset $\mathcal{M}(\pi)$ of $(K/M)^\wedge$ and $1 \in \mathcal{M}(\pi)$. If $\delta \in \mathcal{M}(\pi)$ then let e_δ be an M-fixed unit vector in $\mathcal{H}(\pi)$ behaving under K according to δ. Then

$$\phi_\pi(x) := (\pi(x)e_1, e_1), \quad x \in G, \tag{3.15}$$

defines a spherical function ϕ_π for (G,K) and

$$\psi_\delta(k) := (\pi(k)e_\delta, e_\delta), \ k \in K, \ \delta \in \mathcal{M}(\pi), \tag{3.16}$$

defines a spherical function ψ_δ for (K,M). Let d_δ be the degree of δ. The function $\phi_{\pi,\delta}(\delta \in \mathcal{M}(\pi))$ defined by

$$\phi_{\pi,\delta}(x) := d_\delta^{-\frac{1}{2}}(\pi(x)e_1, e_\delta), \quad x \in G, \tag{3.17}$$

will be called an <u>associated spherical function</u> for (G,K). Note that

$$\phi_{\pi,\delta}(ka\ell) = \phi_{\pi,\delta}(a)\psi_\delta(k), \ k,\ell \in K, \ a \in A. \tag{3.18}$$

Corresponding to (3.17) there is the expansion

$$\pi(a)e_1 = \sum_{\delta \in \mathcal{M}(\pi)} d_\delta^{\frac{1}{2}} \, \phi_{\pi,\delta}(a) e_\delta, \quad a \in A. \tag{3.19}$$

Let $\tilde{\pi}$ be the <u>conjugate contragredient</u> representation to π, that is, the representation $\tilde{\pi}$ of G on $\mathcal{H}(\pi)$ which satisfies

$$(\pi(x)v, w) = (v, \tilde{\pi}(x^{-1})w), \ v,w \in \mathcal{H}(\pi), \ x \in G.$$

For $a,b \in A$, $k \in K$ we can expand:

$$\phi(a^{-1}kb) = (\pi(a^{-1}kb)e_1,e_1) = (\pi(k)\pi(b)e_1,\tilde{\pi}(a)e_1) =$$
$$= \sum_{\gamma,\delta\in M(\pi)} (\pi(b)e_1,e_\gamma)(e_\delta,\tilde{\pi}(a)e_1)(\pi(k)e_\gamma,e_\delta).$$

Hence,

$$(3.20)\quad \phi(a^{-1}kb) = \sum_{\delta\in M(\pi)} d_\delta \phi_{\pi,\delta}(b)\overline{\phi_{\tilde{\pi},\delta}(a)}\psi_\delta(k), a,b\in A, k\in K.$$

The convergence in (3.20) is absolute, uniform for (a,b,k) in compact subsets of $A\times A\times K$. We call (3.20) the addition formula for the spherical function ϕ. The right hand side expands $\phi(a^{-1}kb)$ as a function of k on the compact group K. Integration of both sides with respect to k over K yields the product formula (3.6).

Now let G be a rank one group and K,A,M as in §3.1. For π we take the representation $\pi_\lambda (\lambda\in\mathbb{C})$ of G induced by the one-dimensional representation $ma_tn \mapsto e^{-i\lambda t}$ of the subgroup MAN (cf. Wallach [142,§8.3]). Then π_λ has a realization on $L^2(K/M)$ given by

$$(3.21)\quad (\pi_\lambda(x)f)(kM) = e^{(i\lambda-\rho)H(x^{-1}k)} f(u(x^{-1}k)M),$$
$$x \in G,\ k \in K,\ f \in L^2(k/M),$$

where $u(x)$ and $H(x)$ are as in §3.1(c). The series of representations π_λ is called the spherical principal series for G. If $\lambda \in \mathbb{R}$ then π_λ is unitary. Restriction of π_λ to K yields the regular representation of K on $L^2(K/M)$, which is unitary. By Frobenius reciprocity: $\pi_\lambda|_K = \oplus_{\delta\in(K/M)^\wedge}\delta$. Thus all conditions are satisfied in order to have an addition formula of the form (3.20). Observe that, with ψ_δ defined by (3.16), we can now take for $e_\delta (\delta\in(K/M)^\wedge)$ the element

$$kM \mapsto d_\delta^{\frac{1}{2}}\ \overline{\psi_\delta(k)} \text{ of } L^2(K/M).$$

Thus (3.17) becomes

$$(3.22)\qquad \phi_{\lambda,\delta}(a_t) = \int_K e^{(i\lambda-\rho)H(a_{-t}k)}\psi_\delta(k)dk,$$

where we replaced the subscript π_λ by λ (cf. Helgason's [69, 4] definition of generalized spherical function). In particular, for $\delta = 1$, we get for the spherical function ϕ_λ corresponding to π_λ:

(3.23) $$\phi_\lambda(a_t) = \int_K e^{(i\lambda-\rho)H(a_{-t}k)}\,dk.$$

Formules (3.19) and (3.20) can now be written as:

(3.24) $$e^{(-i\lambda-\rho)H(a_{-t}k)} = \sum_{\delta\in(K/M)^\wedge} d_\delta \overline{\phi_{\bar\lambda,\delta}(a_t)}\,\psi_\delta(k),$$

(3.25) $$\phi_\lambda(a_{-s}ka_t) = \sum_{\delta\in(K/M)^\wedge} d_\delta \phi_{\lambda,\delta}(a_s)\overline{\phi_{\bar\lambda,\delta}(a_t)}\,\psi_\delta(k).$$

For the derivation of (3.25) we used that $\tilde\pi_\lambda = \pi_{\bar\lambda}$. See also [50] for a treatment of addition formulas in group theoretic form.

3.4. Generalized Gelfand pairs and spherical distributions

References for this subsection are Faraut [37,§I], Thomas [135], van Dijk [30], Benoist[10],[11] and Flensted-Jensen [47]. Let G be a unimodular Lie group. If $\phi \in \mathcal{D}(G)$, $x,y \in G$ then write $(\lambda(x)\phi)(y) := \phi(x^{-1}y)$, $(\rho(x)\phi)(y) := \phi(yx)$. A distribution vector of a unitary representation π of G is a continuous linear mapping $u\colon \mathcal{D}(G) \to \mathcal{H}(\pi)$ such that $\pi(x)(u(\phi)) = u(\lambda(x)\phi)$ $(x\in G, \phi\in\mathcal{D}(G))$. Let $\mathcal{H}_{-\infty}(\pi)$ denote the space of all distribution vectors of π. There is an embedding $v \mapsto u\colon \mathcal{H}(\pi) \hookrightarrow \mathcal{H}_{-\infty}(\pi)$ defined by $u(\phi) := \pi(\phi)\,v$ $(\phi\in\mathcal{D}(G))$. The representation π extends to a representation $\pi_{-\infty}$ of G on $\mathcal{H}_{-\infty}(\pi)$:

$$(\pi_{-\infty}(x)u)(\phi) := u(\rho(x^{-1})\phi),\ (\pi_{-\infty}(\psi)u)(\phi) := u(\phi*\psi)$$

$(x\in G, \phi,\psi\in\mathcal{D}(G), u\in\mathcal{H}_{-\infty}(\pi))$. If $\psi \in \mathcal{D}(G)$, $u \in \mathcal{H}_{-\infty}(\pi)$ then $\pi(\psi)u$ can be shown to lie in $\mathcal{H}(\pi)$. A distribution vector u in $\mathcal{H}_{-\infty}(\pi)$ is called cyclic if $\pi(\mathcal{D}(G))u$ is dense in $\mathcal{H}(\pi)$.

A distribution T on G is called positive definite if $T(\tilde\phi*\phi) \geq 0$ for all ϕ in $\mathcal{D}(G)$ ($\tilde\phi$ defined by (3.2.9)). The formula

(3.26) $$T(\tilde\psi*\phi) = (\pi_{-\infty}(\phi)u, \pi_{-\infty}(\psi)u),\ \phi,\psi \in \mathcal{D}(G),$$

establishes a 1-1 correspondence between the nonzero positive definite distributions T on G and the equivalence classes of pairs (π,u), where π is a unitary representation of G and u is a cyclic element of $\mathcal{H}_{-\infty}(\pi)$ (cf. the corresponding statement for positive definite functions in §3.2).

Let H be a closed unimodular subgroup of G. The pair (G,H) is called a generalized Gelfand pair if, for each

π in $\hat{G}$, the dimension of the space of H-invariant distribution vectors in $\mathcal{H}_{-\infty}(\pi)$ is at most one-dimensional. (There are many equivalent definitions, cf. [30],[135].) Assume that (G,H) has this property. Let $(G/H)^\wedge$ denote the set of all π in $\hat{G}$ for which there is a nonzero H-invariant distribution vector. A positive definite spherical distribution for (G,H) is a positive definite distribution T on G such that (3.26) holds for some π in $(G/H)^\wedge$ and some H-fixed nonzero u. Then $T \in \mathcal{D}'(G//H)$, i.e., T is H-biinvariant. If H is compact then the notions of generalized Gelfand pair, positive definite spherical distribution are equivalent to Gelfand pair, positive definite spherical function, respectively.

A potential source of generalized Gelfand pairs is given by the symmetric pairs, i.e. pairs (G,H) with an involutive automorphism σ of G such that the Lie algebra of H is precisely the 1-eigenspace of $d\sigma$. If H is compact then a symmetric pair is always a Gelfand pair, but, in general, symmetric pairs are not always generalized Gelfand pairs (cf. the end of this subsection). A particular class of symmetric pairs are the pairs $(G\times G, G^*)$, where G is a unimodular Lie group, G^* is the diagonal subgroup of $G\times G$ and $\sigma(x,y) := (y,x)$ $(x,y\in G)$. Then the homogeneous space $G \times G/G^*$ can be identified with G, with the action of $G \times G$ on G given by $(x,y).z := xzy^{-1}$ $(x,y,z\in G)$. If G is a type I group (for instance a semisimple Lie group) then $(G\times G,G^*)$ is a generalized Gelfand pair. Then $(G\times G/G^*)^\wedge = \{\pi\otimes\pi^* \mid \pi\in\hat{G}\}$. Let T be a spherical distribution corresponding to $\pi\otimes\pi^*$ $(\pi\in\hat{G})$ and consider T as G^*-invariant distribution on $G\times G/G^*$, i.e. as central distribution on G. Then, up to a constant factor, $T(\phi) = \mathrm{tr}(\phi)$, $\phi \in \mathcal{D}(G)$. ($\pi(\phi) := \int_G \phi(x)\pi(x)dx$ can be shown to be a trace class operator on $\mathcal{H}(\pi)$.)

Next consider a symmetric pair (G,H) with G being a connected semisimple Lie group with finite center. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the corresponding Lie algebra decomposition with respect to $\sigma(=d\sigma)$. The rank of (G,H) is the dimension of a maximal abelian subspace of $\mathfrak{q}$ consisting of semisimple elements. Infinitesimally, the rank one cases can be obtained from Berger's [14] classification. They fall into two classes depending on whether the space G/H is isotropic or not. The isotropic cases are (up to local isomorphisms) the pairs $(U(p,q;\mathbb{F}), U(1)\times U(p-1,q;\mathbb{F}))$ $(\mathbb{F}=\mathbb{R}, \mathbb{C}$ or $\mathbb{H})$ considered by Faraut [37] (and other authors mentioned in the references to [37]) and the pair $(F_{4(-20)}, Spin(1,8))$ considered by Kosters [88],[89,Ch.3]. (Note that, like in §3.1, $U(p,q;\mathbb{F})$ is taken reductive in order to get a uniform presentation.) The non-isotropic cases are $(SL(n,\mathbb{R}), GL(n-1,\mathbb{R}))$ (considered by Kosters [89,Ch.4]), $(Sp(n,\mathbb{R}), Sp(1,\mathbb{R})\times Sp(n-1,\mathbb{R}))$ and $(F_{4(4)}, Spin(4,5))$.

In the structure theory of these rank one cases it is convenient to take a Cartan involution θ of G commuting with σ. Let K be the fixed point group of θ (a maximal compact subgroup of G) and let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be the corresponding Lie algebra decomposition with respect to $\theta(=d\theta)$. Choose a one-dimensional subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$ and a non-zero element H_1 of $\mathfrak{a}$. With $\mathfrak{g}_\lambda$ defined as in the rank one case of §3.1, H_1 can be chosen such that $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_{-1} + \mathfrak{g}_{-2}$. Put $A := \{a_t := \exp tH_1 \mid t \in \mathbb{R}\}$, $M := Z_H(A)$, $\mathfrak{n} := \mathfrak{g}_1 + \mathfrak{g}_2$, $N := \exp \mathfrak{n}$. There is a generalized Cartan decomposition $G = KAH$, where Ka_tH determines $|t|$ completely.

The isotropic rank one pairs (G,H) in the form given above as well as the pairs $(SL(n,\mathbb{R}), GL(n-1,\mathbb{R}))$ $(n\geq 3)$ can be shown to be generalized Gelfand pairs. But this is not true for the pairs $(SL(2,\mathbb{R}), GL(1,\mathbb{R}))$ and $(O(1,n),O(1,n-1))$ (van Dijk, yet to be published).

If (G,H) is a semisimple symmetric pair then G/H naturally becomes a pseudo-Riemannian symmetric space with G-invariant metric. H-biinvariant distributions on G can be identified with H-invariant distributions on G/H. For a generalized Gelfand pair (G,H) of rank one all positive definite spherical distributions are eigendistributions of the Laplace-Beltrami operator Ω on G/H. More generally, Faraut [37] defines a spherical distribution on a rank one space G/H to be a H-biinvariant eigendistribution of Ω.

3.5. Plancherel theorems, general theory

Let (G,K) be a Gelfand pair and assume that G is a Lie group. Then it can be shown that the spherical Plancherel theorem (cf. (3.11), (3.12)) is equivalent to

$$(3.27)\qquad f(e) = \int_{(G/K)^\wedge} \hat{f}(\phi)\,d\nu(\phi),\quad f \in \mathcal{D}(G//K),$$

where $f \mapsto \hat{f}$ is the spherical Fourier transform defined by (3.10), ν is the spherical Plancherel measure and $\mathcal{D}(G//K)$ is the space of K-biinvariant C^∞-functions on G with compact support. For f in $\mathcal{D}(G)$ and for a spherical function ϕ put

$$(3.28)\qquad \phi(f) = \hat{f}(\phi) := \int_G f(x)\phi(x)\,dx.$$

Put

$$(3.29)\qquad \tilde{f}(x) := \overline{f(x^{-1})}, \qquad x \in G,$$

for a function f on G. Now two versions of the spherical Plancherel theorem equivalent to (3.27) are

$$(3.30)\qquad \int_K f(k)dk = \int_{(G/K)^\wedge} \phi(f)d\nu(\phi), \quad f \in \mathcal{D}(G),$$

$$(3.31)\qquad \int_{G/K} |f(x)|^2 d(xK) = \int_{(G/K)^\wedge} \phi(\tilde{f}*f)d\nu(\phi), \quad f \in \mathcal{D}(G/K).$$

The equivalence of (3.30), (3.31) follows from the fact that

$$(3.32)\qquad \mathcal{D}(G) = \mathrm{span}(\mathcal{D}(G)*\mathcal{D}(G)),$$

which was proved by Dixmier & Malliavin [26].

Next, let G be a unimodular Lie group which is type I (for instance a semisimple Lie group). Then the Plancherel theorem for G (cf. Dixmier [25,§18]) states that there is a unique measure ν on $\hat{G}$ such that the two following equivalent statements hold:

$$(3.33)\qquad f(e) = \int_{\hat{G}} \mathrm{tr}(\pi(f))d\nu(\pi), \quad f \in \mathcal{D}(G),$$

$$(3.34)\qquad \int_G |f(x)|^2 dx = \int_{\hat{G}} \mathrm{tr}(\pi(f)^*\pi(f))d\nu(\pi), \quad f \in \mathcal{D}(G).$$

Equivalence of (3.33), (3.34) follows from (3.32). For f in $\mathcal{D}(G/K)$ formulas (3.33), (3.34) imply their spherical analogues (3.30), (3.31).

Finally, let (G,H) be a generalized Gelfand pair (cf. §3.4). Define

$$(3.35)\qquad f^0(xH) := \int_H f(xh)dh, \quad f \in \mathcal{D}(G), \ x \in G.$$

Then $f \mapsto f^0$ is a continuous surjection of $\mathcal{D}(G)$ onto $\mathcal{D}(G/H)$. Attach to each π in $(G/H)^\wedge$ a corresponding spherical distribution T_π in a measurable way (with respect to the canonical Borel structure of $(G/H)^\wedge$). Note that, in general, there is no canonical normalization of spherical distributions. Now there is a unique positive measure ν on $(G/H)^\wedge$ (cf. Thomas [135, Theorem A]) such that the two following equivalent statements hold:

$$(3.36) \qquad \int_H f(h)dh = \int_{(G/H)^\wedge} T_\pi(f)d\nu(\pi) \forall f \in \mathcal{D}(G),$$

$$(3.37) \qquad \int_{G/H} |f^0(xH)|^2 d(xH) = \int_{(G/H)^\wedge} T_\pi(\tilde{f}*f)d\nu(\pi), \forall f \in \mathcal{D}(G).$$

If $(G,H) = (G_1 * G_1, G_1^*)$ (cf. §3.4) then (3.36), (3.37) reduce to (3.33), (3.34) and if H is compact then (3.36), (3.37) imply (3.30), (3.31).

In the case that G is semisimple, K a maximal compact subgroup, the Plancherel measure in (3.27) was explicitly determined by work of Harish-Chandra [61] and Gindikin & Karpelevic [56]. The Plancherel measure in the group case (3.33) (G noncompact semisimple Lie group) was obtained in a number of papers by Harish-Chandra, cf. the survey paper by Schmid [121]. The Plancherel measure for G/H in (3.36) was only determined in a number of special cases, we mention Faraut [37], Kosters [88], [89,Ch.3], Benoist [10], [11].

We now want to emphasize one particular method to obtain the Plancherel measure, namely the method of K-finite functions. In special cases, to be treated in §4, this method allows reduction to the Jacobi transform.

Let G be a semisimple Lie group, K a maximal compact subgroup and, in the case (G,H), let K,H correspond to commuting involutions θ,σ. Let $\gamma,\delta \in \hat{K}$ and let χ_δ denote the character of δ in $\hat{K}$. We call f in $\mathcal{D}(G)$ a <u>K-finite function of double K-type γ,δ</u> if

$$(3.38) \qquad f(x) = \int_K \int_K d_\gamma \chi_\gamma(k^{-1}) f(kx\ell) d_\delta \chi_\delta(\ell^{-1}) dk d\ell, x \in G.$$

Similarly, f in $\mathcal{D}(G/H)$ is a <u>K-finite function of K-type δ</u> if

$$(3.39) \qquad f(\xi) = \int_K d_\delta \chi_\delta(k^{-1}) f(k\xi) dk, \ \xi \in G/H.$$

Now, by density properties of K-finite functions, each of the Plancherel formulas (3.30), (3.31), (3.33), (3.34), (3.36), (3.37) will be valid for all C_c^∞-functions iff it is valid for all such functions which are K-finite. Observe that (3.30), (3.33), (3.36) become trivial for most K-types. More concretely, (3.30) only needs to be verified for double K-type (1,1), i.e., it is implied by (3.27). For (3.33) we can restrict ourselves to functions f of double K-type (δ,δ) $(\delta \in \hat{K})$ which are moreover K-central, i.e. $f(kxk^{-1}) = f(x)$ for x in G, k in K. Finally, for (3.36) it is sufficient to consider functions f in $\mathcal{D}(G)$ for which f^0 is K-finite in $\mathcal{D}(G/H)$ and moreover $K \cap H$-invariant. On the other hand, the versions (3.31), (3.34), (3.37) will imply Plancherel-type formulas for functions of many more K-types than were needed in order to verify (3.30), (3.33), (3.36). These things will become more clear by examples in §4.

4. THE JACOBI TRANSFORM IN GROUP THEORY

In this section we treat a number of cases where Jacobi functions appear in the context of semisimple Lie groups: as spherical or intertwining functions, as associated spherical or intertwining functions and as matrix elements of irreducible representations. Correspondingly, the Jacobi differential operator arises by separation of variables of the Casimir operator on the group and the Jacobi transform can be interpreted as the group Fourier transform acting on certain function classes which possess special symmetries.

4.1. Jacobi functions as spherical functions

Let G be a rank one group and use the notation of §3.1. The parameters α,β will be as in (3.4). In view of the Iwasawa and Cartan decompositions (§3.1), the restriction $f \mapsto f|_A$ identifies $C^\infty(N\backslash G/K)$, with $C^\infty(A)$ and $C^\infty(G//K)$ with $C^\infty_{\mathrm{even}}(A) := \{f \in C^\infty(A) \mid f(a_t) = f(a_{-t}),\ t \in \mathbb{R}\}$. Note also that the Laplace-Beltrami operator Ω sends $C^\infty(N\backslash G/K)$ and $C^\infty(G//K)$ into itself. It can be shown (cf. [72,Ch.2], [39, Ch.3]) that Ω has the following A-radial parts with respect to these two decompositions:

$$(4.1)\qquad (\Omega f)(a_t) = \left(\frac{d^2}{dt^2} - 2\rho\,\frac{d}{dt}\right)f(a_t),\quad f \in C^\infty(N\backslash G/K),$$

$$(4.2)\qquad (\Omega f)(a_t) = (L_{\alpha,\beta} f)(a_t),\quad f \in C^\infty(G//K),$$

where $L_{\alpha,\beta}$ is the Jacobi differential operator (2.9).

With $H\colon G \to \mathbb{R}$ defined as in §3.1(c) the function $x \mapsto \exp((i\lambda-\rho)H(x^{-1}))$ $(\lambda\in\mathbb{C})$ is in $C^\infty(N\backslash G/K)$. Hence, by (4.1):

$$(4.3)\qquad (\Omega+\lambda^2+\rho^2)\, e^{(i\lambda-\rho)H(x^{-1})} = 0.$$

Let

$$(4.4)\qquad \phi_\lambda(x) := \int_K e^{(i\lambda-\rho)H(x^{-1}k)}\,dk,\quad x \in G,\ \lambda \in \mathbb{C}.$$

Then $\phi_\lambda \in C^\infty(G//K)$, $\phi_\lambda(e) = 1$ and, because of (4.2) and the G-invariance of Ω, ϕ_λ satisfies the differential equation

$$(4.5) \qquad (\Omega+\lambda^2+\rho^2)\phi_\lambda = 0.$$

It follows from the results at the end of §3.2 that ϕ_λ is a spherical function, that the set $\{\phi_\lambda \mid \lambda \in \mathbb{C}\}$ equals the set of all spherical functions for (G,K) and that $\phi_\lambda = \phi_\mu$ iff $\lambda = \pm\mu$. In view of (3.23), (4.4), ϕ_λ equals the matrix element (3.15) of the principal series representation π_λ. In view of (4.5), (4.2),

$$(4.6) \qquad \phi_\lambda(a_t) = \phi_\lambda^{(\alpha,\beta)}(t),$$

where $\phi_\lambda^{(\alpha,\beta)}$ is the Jacobi function (2.4) and α,β are as in (3.4). The integral representation (4.4) is due to Harish-Chandra [61] (in the case of general rank).

For the groups G under consideration the spherical Fourier transform (3.10) can be rewritten as a transform $f \mapsto \hat{f}$ defined by

$$(4.7) \qquad \hat{f}(\lambda) := \int_G f(x)\phi_\lambda(x)dx,$$

where $f \in C_c(G//K)$, $\lambda \in \mathbb{C}$. It can be shown that, up to a positive constant factor, the Haar measure on G satisfies

$$(4.8) \qquad \int_G f(x)dx = \int_0^\infty f(a_t)\Delta(t)dt, \quad f \in C_c(G//K),$$

where Δ is given by (2.9). Normalize dx on G such that (4.8) holds exactly. Then combination of (4.7) and (4.8) shows that (2.12) holds with f(t) replaced by $f(a_t)$, i.e., the spherical Fourier transform of f equals the Jacobi transform of $t \mapsto f(a_t)$.

By use of this identification we can now apply Theorems 2.3, 2.4 (which will be proved in section 6) in order to obtain the Plancherel measure ν (cf. (3.11), (3.12)) for the present groups. Note that α,β in (3.4) are such that $\alpha \geq \beta \geq -\frac{1}{2}$, so we get from Theorems 2.3, 2.4 and from (4.8) that

$$(4.9) \qquad f(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|c(\lambda)|^{-2}d\lambda, \quad f\in\mathcal{D}(G//K),\ x\in G,$$

$$(4.10) \qquad \int_G f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\overline{\hat{g}(\lambda)}|c(\lambda)|^{-2}d\lambda, \quad f,g\in\mathcal{D}(G//K),$$

and (4.10) extends to an isometry of L^2-spaces. The spherical functions ϕ_λ ($\lambda\in\mathbb{R}$) occurring in (4.9), (4.10) are positive definite because (3.15) holds with $\pi = \pi_\lambda$ ($\lambda\in\mathbb{R}$) being a unitary principal series representation. Thus the Plancherel measure ν (cf. (3.11), (3.12)) becomes the measure $(2\pi)^{-1}|c(\lambda)|^{-2}d\lambda$ on $\mathbb{R}_+$ in the present case.

The above interpretation of the Jacobi transform as a spherical Fourier transform was first observed by Olevskiĭ [111] for the real hyperbolic spaces and by Harish-Chandra [61,§13] in the general rank one case. Harish-Chandra recognized the radial part of (4.5) as a hypergeometric differential equation and he obtained the explicit value (2.18) of $c(\lambda)$ by identifying (2.17) with an identity for hypergeometric functions. But he obtained the inversion formula by specialization of his general rank result.

4.2. Jacobi functions as associated spherical functions

Let G be as in §4.1. Since (3.11) implies (3.31), (4.9) will imply that, for F in $\mathcal{D}(G/K)$,

$$\int_{G/K} |F(x)|^2 d(xK) = (2\pi)^{-1}\int_0^\infty (\tilde{F}*F)^\wedge(\lambda)\,|c(\lambda)|^{-2}d\lambda. \tag{4.11}$$

In particular, this identity will hold for F in $\mathcal{D}(G/K)$ of the form

$$F(ka_tK) = f(t)Y_\delta(k),\quad k\in K,\ t\in\mathbb{R}, \tag{4.12}$$

where Y_δ is in $L^2(K/M)$ with norm 1 and is of K-type δ ($\delta\in(K/M)^\wedge$). (Any K-finite function is a finite sum of functions of the form (4.12).) It follows from (4.7), (4.8), (4.12), (3.25) that

$$(\tilde{F}*F)^\wedge(\lambda) = \int_K\int_K\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \overline{F(\ell a_s)}F(ka_t)\cdot \tag{4.13}$$

$$\cdot\ \phi_\lambda(a_{-s}\ell^{-1}ka_t)\Delta(s)\Delta(t)ds\,dt\,dk\,d\ell =$$

$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \overline{f(s)}f(t)\phi_{\lambda,\delta}(a_s)\overline{\phi_{\bar{\lambda},\delta}(a_t)}\Delta(s)\Delta(t)ds\,dt.$$

Hence (4.11) becomes

$$(4.14)\qquad \int_0^\infty |f(t)|^2 \Delta(t)\,dt =$$

$$= (2\pi)^{-1}\int_0^\infty \left|\int_0^\infty f(t)\phi_{\lambda,\delta}(a_t)\Delta(t)\,dt\right|^2 |c(\lambda)|^{-2}\,d\lambda.$$

The set $(K/M)^\wedge$ was determined by Kostant [87], Johnson & Wallach [77], Johnson [76], while Helgason [68] computed the functions $\phi_{\lambda,\delta}$ in terms of hypergeometric functions. It turns out (see also §8.1) that the functions $t \mapsto \phi_{\lambda,\delta}(a_t)$ coincide with the associated Jacobi function

$$(4.15)\qquad \phi_{\lambda,k,\ell}^{(\alpha,\beta)}(t) := (c_{\alpha,\beta}(-\lambda)/c_{\alpha+k+\ell,\beta+k-\ell}(-\lambda))\cdot$$

$$\cdot\ (2\operatorname{sh} t)^{k-\ell}(2\operatorname{ch} t)^{k+\ell}\phi_\lambda^{(\alpha+k+\ell,\beta+k-\ell)}(t),$$

where the ϕ_λ-function and c-functions at the right hand side are defined by (2.4), (2.18), α,β are as in (3.4) and k,ℓ run over all integers with $k \ge \ell \ge 0$ if $\mathbb{F} = \mathbb{C},\mathbb{H}$ or $\mathbb{O}$ (octonions) and $\ell = 0,1,2,\ldots$, $k = \ell$ or $\ell-1$ if $\mathbb{F} = \mathbb{R}$. Thus, if we put

$$f(t) := (2\operatorname{sh} t)^{k-\ell}(2\operatorname{ch} t)^{k+\ell}g(t),\quad g \in \mathcal{D}_{\text{even}}(\mathbb{R})$$

in (4.14) then we have obtained a group theoretic interpretation of (2.27) with α,β replaced by $\alpha+k+\ell,\beta+k-\ell$ and α,β,k,ℓ having values as above. (Note that for these values $\alpha+k+\ell \ge \beta+k-\ell \ge -\frac{1}{2}$). An analogue of (4.14) in the case of general rank was proved by Helgason [69,Cor.10.2].

The explicit expressions for the associated spherical functions in the case of rank one were exploited by Helgason [66,pp.140,141], [68, sections 6,7], Lewis [95] and some other authors mentioned in [68,§1] in order to characterize the image of the Poisson transform on a rank one space G/K. This transform $T \mapsto f$, defined for λ in $\mathbb{C}$ by

$$(4.16)\qquad f(x) = \int_{K/M} e^{(i\lambda-\rho)H(x^{-1}k)}\,dT(kM),\quad x \in G,$$

maps the space of analytic functionals T on K/M into the space of all C^∞-functions f on G/K such that $\Omega f = -(\lambda^2+\rho^2)f$. This mapping is surjective, as Helgason showed in the case $\mathbb{F} = \mathbb{R}$ for $\operatorname{Im}\lambda \ge 0$ and in the other rank one cases for $i\lambda \le 0$. In the case of general rank one and general complex λ Helgason's elementary method using estimates in k,ℓ for the associated Jacobi functions (4.15) failed. Here the proof for general rank given by Kashiwara e.a. [78] and using the full machinery of hyperfunction theory is the only available proof until now. Lewis [95] shows for all rank one cases and for generic λ that the Poisson transform maps $\mathcal{D}'(K/M)$ onto the space of eigenfunctions of Ω of at most exponential growth.

See §8 for other applications of associated spherical functions.

In the following subsections we will consider some group theoretic interpretations of Jacobi transforms for which the Plancherel measure has a discrete part.

4.3. Jacobi functions as matrix elements of irreducible representations

The Plancherel measure in (3.33) with $G = SL(2,\mathbb{R})$ was first determined by Bargmann [8]. Here we will sketch an approach which uses K-finite functions. This approach was followed earlier by Takahashi [132] and (for the universal covering of $SL(2,\mathbb{R})$) by Flensted-Jensen [43].

For the following facts about $SL(2,\mathbb{R})$ the reader may consult, for instance, [85]. We will work with $G = SU(1,1)$, which is isomorphic to $SL(2,\mathbb{R})$. Consider an Iwasawa decomposition $G = KAN$ with

$$K := \{u_\theta := \mathrm{diag}(e^{\frac{1}{2}i\theta}, e^{-\frac{1}{2}i\theta}) \mid 0 \le \theta < 4\pi\},$$

$$A := \left\{a_t := \begin{pmatrix} \mathrm{ch}\, t & \mathrm{sh}\, t \\ \mathrm{sh}\, t & \mathrm{ch}\, t \end{pmatrix} \;\middle|\; t \in \mathbb{R}\right\}.$$

Then $M = \{u_0, u_{2\pi}\}$. $\hat{K}$ consists of all δ_n, $n \in \frac{1}{2}\mathbb{Z}$, given by $\delta_n(u_\theta) := e^{in\theta}$ and $\hat{M}$ consist of $\delta_\xi|_M$, $\xi = 0$ or $\frac{1}{2}$. The principal series representation $\pi_{\xi,\lambda}$ ($\lambda \in \mathbb{C}, \xi = 0$ or $\frac{1}{2}$) of G is the representation of G induced by the representation $ma_tn \mapsto e^{-i\lambda t}\delta_\xi(m)$ of MAN. Then $\pi_{\xi,\lambda}|_K = \oplus_{n\in\mathbb{Z}}\, \delta_{n+\xi}$. In particular, each K-type occurs in $\pi_{\xi,\lambda}$ at most once. For a suitable orthonormal basis $\{e_n\}_{n\in\mathbb{Z}+\xi}$ of $\mathcal{H}(\pi_{\xi,\lambda})$ with $\pi_{\xi,\lambda}(u_\theta)e_n = \delta_n(u_\theta)e_n$, the matrix elements

$$\pi_{\xi,\lambda;m,n}(x) := (\pi_{\xi,\lambda}(x)e_n, e_m), \quad x \in G,\ m,n \in \mathbb{Z}+\xi$$

can be expressed in terms of Jacobi functions (cf.[85, Theorem 2.1]):

$$(4.17) \qquad \pi_{\xi,\lambda;m,n}(a_t) = (c_{\xi,\lambda;m,n}/(|m-n|)!)\cdot$$
$$\cdot(\mathrm{sh}\ t)^{|m-n|}(\mathrm{ch}\ t)^{m+n}\phi_\lambda^{(|m-n|,m+n)}(t),$$

where

$$c_{\xi,\lambda;m,n} := \begin{cases} (-\frac{1}{2}i\lambda+n+\frac{1}{2})_{m-n} & \text{if } m \geq n, \\ (-\frac{1}{2}i\lambda-n+\frac{1}{2})_{n-m} & \text{if } n \geq m. \end{cases}$$

These explicit expressions (in terms of hypergeometric functions) were already obtained in [8].

The elements of $\hat{G}$ can all be obtained by unitarizing suitable subquotients of representations $\pi_{\xi,\lambda}$. They are (cf.[85]):

(a) unitary principal series: $\pi_{\xi,\lambda}$ $(\lambda>0, \xi=0$ or $\frac{1}{2})$,

$\pi_{0,0}$, $\pi^+_{\frac{1}{2},0}$, $\pi^-_{\frac{1}{2},0}$ $(\pi_{\frac{1}{2},0} = \pi^+_{\frac{1}{2},0} \oplus \pi^-_{\frac{1}{2},0})$.

(b) complementary series: $\pi_{0,i\mu}$ $(0<\mu<1)$ up to unitarization.

(c) discrete series: $\pi^+_{\xi,\lambda}, \pi^-_{\xi,\lambda}$ $(\xi=0$ or $\frac{1}{2}$, $\lambda = i(2\xi+1)$, $i(2\xi+3)$, $i(2\xi+5),\ldots)$, where, up to unitarization,

$$\pi^+_{\xi,\lambda;m,n} = \pi_{\xi,\lambda;m,n} \left(m,n = \tfrac{1}{2}|\lambda|+\tfrac{1}{2}, \tfrac{1}{2}|\lambda|+\tfrac{3}{2},\ldots\right),$$
$$\pi^-_{\xi,\lambda;m,n} = \pi_{\xi,\lambda;m,n} \left(m,n = -\tfrac{1}{2}|\lambda|-\tfrac{1}{2}, -\tfrac{1}{2}|\lambda|-\tfrac{3}{2},\ldots\right).$$

(d) identity representation.

Now we determine the Plancherel measure in (3.33) by the method of K-finite functions. It follows from (2.24), (2.18), (2.23) that, for f in $\mathcal{D}_{\text{even}}(\mathbb{R})$ and $n = 0,1,2,\ldots$:

$$(4.18) \qquad f(0) = \int_0^\infty \hat{f}_{0,2n}(\lambda)2^{-4n-2}\lambda\ \mathrm{th}\tfrac{1}{2}\pi\lambda\ d\lambda + $$
$$+ \textstyle\sum_{k=0}^{n-1} 2^{-4n-1}(2k+1)\hat{f}_{0,2n}((2k+1)i),$$

$$(4.19) \qquad f(0) = \int_0^\infty \hat{f}_{0,2n+1}(\lambda)2^{-4n-4}\lambda\ \mathrm{coth}\tfrac{1}{2}\pi\lambda\ d\lambda + $$
$$+ \textstyle\sum_{k=1}^{n} 2^{-4n-3}(2k)\ \hat{f}_{0,2n+1}(2ki),$$

where $\hat{f}_{\alpha,\beta}(\lambda) = \hat{f}(\lambda)$ as defined by (2.12). Now let $F \in \mathcal{D}(G)$ such that

$$(4.20) \qquad F(u_\theta a_t u_\eta) = (\mathrm{ch}\ t)^{2n} f(t) e^{in(\theta+\eta)}$$

with $f \in \mathcal{D}_{even}(\mathbb{R})$, $n \in \frac{1}{2}\mathbb{Z}$, i.e., F is K-central of K-type δ_n. Then (4.18), (4.19), (4.20), (2.12), (4.17), (4.8) imply that

$$F(e) = \int_0^\infty \left(\int_G F(x)\pi_{0,\lambda;n,n}(x)dx\right)\tfrac{1}{4}\lambda \mathrm{th}\tfrac{1}{2}\pi\lambda\ d\lambda\ +$$

$$+ \sum_{k=0}^{|n|-1} (k+\tfrac{1}{2}) \int_G F(x)\pi_{0,(2k+1)i;n,n}(x)dx, n \in \mathbb{Z},$$

$$F(e) = \int_0^\infty \left(\int_G F(x)\pi_{\frac{1}{2},\lambda;n,n}(x)dx\right)\tfrac{1}{4}\lambda \coth\tfrac{1}{2}\pi\lambda\ d\lambda\ +$$

$$+ \sum_{k=1}^{|n|} k \int_G F(x)\pi_{\frac{1}{2},2ki;n,n}(x)dx,\ n \in \mathbb{Z} + \tfrac{1}{2}.$$

Hence, for all F of the form (4.20) we have

$$(4.21) \qquad F(e) = \int_0^\infty \mathrm{tr}\pi_{0,\lambda}(F)\tfrac{1}{4}\lambda\mathrm{th}\tfrac{1}{2}\pi\lambda\ d\lambda\ +$$

$$+ \int_0^\infty \mathrm{tr}\pi_{\frac{1}{2},\lambda}(F)\tfrac{1}{4}\lambda\coth\tfrac{1}{2}\pi\lambda\ d\lambda\ +$$

$$\sum_{k=0}^\infty \sum_{\xi=0,\frac{1}{2}} (k+\xi+\tfrac{1}{2})\mathrm{tr}(\pi^+_{\xi,i(2k+2\xi+1)}(F)+\pi^-_{\xi,i(2k+2\xi+1)}(F))$$

In view of §3.5, formula (4.21) is now valid for all F in $\mathcal{D}(G)$. This yields the Plancherel measure we looked for. Now (3.34) holds with the same measure ν. Specialization of this formula to functions f of arbitrary double K-type yields, in view of (4.17), Theorem 2.4 for all $\alpha,\beta \in \mathbb{Z}$ with $\alpha \geq 0$.

The explicit knowledge of the matrix elements of the principal series representations of $SL(2,\mathbb{R})$ in a K-basis, cf. (4.17), was exploited by Koornwinder [84], [85], Takahashi [133] in order to treat the representation theory of $SL(2,\mathbb{R})$ in a global, i.e. non-infinitesimal way.

For the universal covering group $\tilde{G}$ of $SL(2,\mathbb{R})$ the subgroup $\tilde{K}$ in the Iwasawa composition is no longer compact but isomorphic to $\mathbb{R}$. The principal series representations of $\tilde{G}$ restricted to $\tilde{K}$ still decompose as a multiplicity-free direct sum of irreducible $\tilde{K}$-types and the matrix-elements of these representations can again be expressed in terms of Jacobi functions. In particular, for the diagonal matrix elements restricted to A we get the function $a_t \mapsto (\mathrm{ch}\ t)^\beta \phi_\lambda^{(0,\beta)}(t)$ with β arbitrarily real. Flensted-Jensen [43], [44] used (2.24) for these Jacobi functions in order to obtain the Plancherel measure for $\tilde{G}$. This measure was earlier determined by Pukanszky [114]. Another derivation using K-finite functions is given by Matsushita [99].

4.4. Jacobi functions occurring in spherical functions on nonsymmetric Gelfand pairs

In §3.3 we pointed out that, for each rank one group G,(K,M) is a Gelfand pair. Then the spherical functions for (K,M) can be expressed in terms of Jacobi polynomials (cf. §8.1). Case-by-case inspection shows that noncompact duals (in a certain sense) of these pairs (K,M) are also Gelfand pairs and that the corresponding spherical functions involve Jacobi functions. If $\mathbb{F} = \mathbb{R}$ this yields nothing new, for $\mathbb{F} = \mathbb{C}$ this has been considered by Flensted-Jensen [44], while the quaternionic and octonionic cases were treated by Takahashi [130], [131].

In the complex case we take $G = SU(1,n)$ or its universal covering group $\tilde{G}$, and $K = SU(n)$ $(n\geq 2)$. Then (G,K) and $(\tilde{G},K)$ are Gelfand pairs. The K-biinvariant functions on G are completely determined by their restriction to some abelian subgroup $L\times A$ isomorphic to $\mathbb{T}\times\mathbb{R}$ or its universal covering $\mathbb{R}\times\mathbb{R}$. The spherical functions for (G,K) restricted to this subgroup are

$$(\ell_\theta, a_t) \mapsto e^{i\beta\theta}(\mathrm{ch}\ t)^\beta\ \phi_\lambda^{(n-1,\beta)}(t),$$

where ϕ_λ is a Jacobi function, $\lambda \in \mathbb{C}$ and β runs over $\mathbb{Z}$ or $\mathbb{C}$, respectively. In [44] the spherical Plancherel measure for this Gelfand pair is obtained from the Plancherel formula for the Jacobi functions involved. The associated spherical functions can also be expressed in terms of Jacobi functions. In the special case $n = 1$ we can work with $G = SU(1,1)\times S(U(1)\times U(1))$ and K the diagonal of $S(U(1)\times U(1))$ in this direct product. Then the spherical Plancherel formula for (G,K) yields the group Plancherel formula for SU(1,1). A similar result holds for $\widetilde{SU(1,1)}$.

In the quaternionic case we take $G = Sp(1,n)\times Sp(1)$, $K = Sp(n)\times Sp(1)^*$, where $Sp(1)^*$ is the diagonal in the direct product $Sp(1,n)\times Sp(1)$. Then Takahashi [130] shows that

(G,K) is a Gelfand pair and that the spherical functions, in suitable coordinates, have the form

$$(\theta,t) \mapsto (\mathrm{ch}\ t)^k\ \phi_\lambda^{(2n-1,k+1)} R_k^{(\frac12,\frac12)}(\cos\theta),$$

where $\lambda \in \mathbb{C}$, $k = 0,1,2,\ldots$, and $R_k^{(\frac12,\frac12)}$ is a Jacobi polynomial (cf.(2.3)). (Note that $R_k^{(\frac12,\frac12)}(\cos\theta) = \sin(k+1)\theta/((k+1)\sin\theta)$ and that it has an interpretation as a character on Sp(1).) For n = 1 this situation was earlier met by Takahashi [129] in the context of the representation theory of $SO_0(4,1)$.

Finally, in the octonionic case Takahashi [131] points out that ($Spin_0(1,8)$, Spin(7)) is a Gelfand pair with spherical functions

$$(\theta,t) \mapsto (\mathrm{ch}\ t)^k\ \phi_\lambda^{(3,k+3)}(t) R_k^{(3/2,3/2)}(\cos\theta).$$

4.5. Jacobi functions as K-finite functions on G/H

For preliminaries to this subsection we refer to §3.4, §3.5. We restrict ourselves to Faraut's [37] generalized Gelfand pairs $(G,H) = (U(p,q;\mathbb{F}), U(1,\mathbb{F})\times U(p-1,q;\mathbb{F}))$.

$$\text{Let } B := \left\{ a_{i\theta} := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & \\ 0 & & I_{p+q-2} \end{pmatrix} \right\}.$$

Then HAH ∪ HBH is dense in G and the H-biinvariant distributions on G are in 1-1 correspondence with a certain class of generalized functions on $\{a_t \mid t \geq 0\} \cup \{a_{i\theta} \mid 0 \leq \theta \leq \frac12\pi\}$ (i.e. on $\mathbb{R}_+ \cup i[0,\frac12\pi]$) with respect to a suitable class of test functions on $\mathbb{R}_+ \cup i[0,\frac12\pi]$, having a certain singularity at 0. The "radial" part of the Laplace-Beltrami operator on G/H becomes, up to a constant factor, the Jacobi differential operator $L_{\alpha,\beta}$ (cf.(2.9)) on $\mathbb{R}_+ \cup i[0,\frac12\pi]$ with $\alpha = \frac12 d(p+q-1)-1$, $\beta = \frac12 d-1$ ($d=\dim_{\mathbb{R}}\mathbb{F}$). Thus the spherical distributions are essentially distributional solutions T_λ of the Jacobi differential equation (2.10). Faraut shows that, for λ outside a specific discrete countable set, (2.10) has only a one-dimensional eigenspace, for the exceptional values λ the eigenspace is two-dimensional. The eigendistributions are regular on $(0,\infty)$ and on $i(0,\frac12\pi)$, but they are not necessarily regular at 0. In particular, they do not necessarily coincide with the Jacobi functions $\phi_\lambda^{(\alpha,\beta)}$ on $(0,\infty)$, but for certain d,p,q they do, as follows from [37,§III]. We normalize the eigendistribution T_λ such that it depends analytically on λ and is nonzero for all λ.

In §3.4 we introduced subgroups K,M,N of G. (These M,N are different from M,N in §3.1.) For the present G we have $K = U(p,\mathbb{F})\times U(q,\mathbb{F})$, $M \simeq U(1,\mathbb{F})\times U(p-1,\mathbb{F})\times U(q-1,\mathbb{F})$. Let π_λ ($\lambda\in\mathbb{C}$) be the representation of G induced by the representation $ma_tn \mapsto e^{-i\lambda t}$ of the subgroup MAN (a maximal parabolic subgroup, while the subgroup MAN used for obtaining (3.21) is minimal parabolic). Then, as shown in [37], an H-invariant distribution vector u_λ and the spherical distribution T_λ can be associated with π_λ. The representation π_λ is unitary if $\lambda \geq 0$ (H-spherical unitary principal series) and $\pi_{i\mu}$ may contain unitarizable subquotients belonging to $(G/H)^\wedge$ if $\mu > 0$. This happens if $0 < \mu \leq \mu_0$, where μ_0 depending on p,q,d can be explicitly given (H-spherical complementary series) and, possibly, if $\mu > \mu_0$, $\mu - \rho \in \mathbb{Z}$ (H-spherical discrete series). It can only occur in the discrete series case that two distinct elements of $(G/H)^\wedge$ correspond to one π_λ. All of $(G/H)^\wedge$ can be obtained in the above way. Faraut obtains these results by using K-finite functions (cf. an analogous approach for the K-spherical case in §8.2).

We will now give some more details about the K-finite functions because they involve Jacobi functions and can be used for deriving the Plancherel formula for G/H. Since $G = KAH$, a K-finite function f on G/H can be written as a function $(kM,t) \mapsto f(ka_tH)$ on $K/M \times \mathbb{R}$. Futhermore, K/M can be identified with a space of orbits of $U(1,\mathbb{F})$ on $S(\mathbb{F}^p)\times S(\mathbb{F}^q)$ ($S(\mathbb{F}^p)$ is unit sphere in $\mathbb{F}^p$). Denote by $\mathcal{Y}_\ell^{dp}$ the space of spherical harmonics of degree ℓ on $S(\mathbb{F}^p)$. Then any K-finite function of certain K-type on K/M is in particular contained in the space $\mathcal{Y}_{\ell,m} := \mathcal{Y}_\ell^{dp} \times \mathcal{Y}_m^{dq}$ for certain ℓ,m in $\mathbb{Z}_+$. Faraut obtains the expression of the Laplace-Beltrami operator Ω as a differential operator on $K/M \times \mathbb{R}$. In this way it can be shown that the K-finite solutions f of $(\Omega+\lambda^2+\rho^2)f = 0$ which are of certain K-type are given by the functions

$$(4.22)\qquad ka_tH \mapsto (\operatorname{sh} t)^m (\operatorname{ch} t)^\ell \phi_\lambda^{(\frac{1}{2}dq-1+m,\frac{1}{2}dp-1+\ell)}(t)\, Y_{\ell,m}(kM),$$

where $Y_{\ell,m} \in \mathcal{Y}_{\ell,m}$. Now let f in $\mathcal{D}(G)$ be such that f^0 (cf. (3.35)) is of the form

$$(4.23)\qquad f^0(ka_tH) = (\operatorname{sh} t)^m (\operatorname{ch} t)^\ell F(t)\, Y_{\ell,m}(kM),$$

where $F \in \mathcal{D}_{\mathrm{even}}(\mathbb{R})$, $Y_{\ell,m} \in \mathcal{Y}_{\ell,m}$. Then it follows that

$$(4.24)\qquad T_\lambda(f) = \delta_{m,0}\, b_{\lambda,\ell}\, Y_{\ell,0}(eK)\cdot \cdot\int_0^\infty F(t)\,\phi_\lambda^{(\frac{1}{2}dq-1,\frac{1}{2}dp-1+\ell)}(t)\,\Delta_{\frac{1}{2}dq-1,\frac{1}{2}dp-1+\ell}(t)\,dt,$$

for certain constants $b_{\lambda,\ell}$ which are explicitly evaluated by Faraut. A similar formula holds for $S_\lambda f$, where S_λ is a spherical distribution for λ in the discrete spectrum of Ω.

Now, in view of (4.23), (4.24) the Plancherel measure ν in (3.36) is obtained from inversion of the Jacobi transform in the cases $(\alpha,\beta) = (\frac{1}{2}dq-1,\frac{1}{2}dp-1+\ell)$, where ℓ runs over a certain subset of $\mathbb{Z}_+$. Thus Theorem 2.3 can be applied again. In a similar way, the version (3.37) of the Plancherel theorem can be reduced to Theorem 2.4, where now $(\alpha,\beta) = (\frac{1}{2}dq-1+m,\frac{1}{2}dp-1+\ell)$. This is Faraut's second proof of his Plancherel theorem in [37,§10]. (His first proof uses direct spectral decomposition of $L_{\alpha,\beta}$ on $\mathbb{R}_+ \cup i[0,\frac{1}{2}\pi]$.) Observe that in Faraut's second proof more cases of the Plancherel theorem for the Jacobi transform are used than is strictly needed: m can be put zero.

Kosters [88], [89,Ch.3] derived the Plancherel formula for $(F_{4(-20)}, \mathrm{Spin}(1,8))$ in a similar way, using Jacobi functions.

The fact that the K-invariant eigenfunctions of Ω on G/H can be expressed as Jacobi functions holds for all semisimple symmetric pairs of rank one. More generally it holds for semisimple symmetric pairs (G,H) where the maximal abelian subspaces of $\mathfrak{p} \cap \mathfrak{q}$ (cf.§3.4) have dimension one. This follows from the explicit expression for the radial part of Ω with respect to the decomposition G = KAH (cf.[67], [72,Ch.II], [45,(4.12)],[46,p.307], [75,Ch.10,11]). For the cases $(G,H) = (O(p,q),O(1)\times O(p-1,q))$ this was already observed [42]. Unfortunately, K-finite eigenfunctions of nontrivial K-type cannot always be expressed in terms of Jacobi functions. For instance, on the space $SL(n,\mathbb{R})/GL(n-1,\mathbb{R})$ considered (for $n\geq 3$) by Kosters [89,Ch.4] and (for n=3) by Molčanov [107] the K-finite eigenfunctions do not factorize, in general, as $ka_tH \to Y(kM)F(t)$, but for the t-dependence we get vector-valued functions satisfying a system of second order o.d.e.'s. Thus many of the methods used in [37] fail here, because the theory of these vector-valued special functions is not yet developed.

It is still worthwhile to have knowledge about Jacobi functions as K-invariant eigenfunctions of Ω on G/H (so-called <u>intertwining functions</u>): in order to do harmonic analysis for K-invariant functions on G/H, in order to get partial information about full harmonic analysis on G/H and in order to get new group theoretic interpretations of formulas for Jacobi functions. For the first and third objective see Flensted-Jensen [42], Badertscher [6], for the second objective see, for instance, Kosters [89,Ch.4], Sekiguchi [122,§7]. Sekiguchi [122] characterized the eigenhyperfunctions of Ω on $G/H = U(p,q;\mathbb{F})/U(1,\mathbb{F})\times U(p-1,q;\mathbb{F})$ as Poisson transforms of the hyperfunctions on $K/K\cap M$ by using the full machinery of hyperfunction theory. W. Kosters (Leiden, to appear) did analogous work for $SL(n,\mathbb{R})/GL(n-1,\mathbb{R})$. It would be interesting to redo the results in [122] in the style of Helgason [68](cf.§4.2) by use of the explicit expressions (4.22) for the K-finite functions.

Badertscher [6] does harmonic analysis for K-invariant functions on G/H with $G = O(p,1)$, $K = O(p)\times O(1)$, $H = O(p-1,1)$ (so H is slightly smaller than in the case of [37]). The radial part of Ω with respect to the decomposition $G = KAH$ now becomes the Jacobi differential operator $L_{-\frac{1}{2},\frac{1}{2}p-1}$ on the full real axis. The eigenfunctions with eigenvalue $-\lambda^2-\rho^2$ are the linear combinations of the even function $\phi_\lambda^{(-\frac{1}{2},\frac{1}{2}p-1)}$ and the odd function $t \to \operatorname{sh} t\, \phi_\lambda^{(\frac{1}{2},\frac{1}{2}p-1)}(t)$. So, by decomposition into even and odd functions the spectral decomposition of $L_{-\frac{1}{2},\frac{1}{2}p-1}$ on $\mathbb{R}$ can be reduced to inversion of the Jacobi transform for $(\alpha,\beta) = (\pm\frac{1}{2},\frac{1}{2}p-1)$. The occurrence of multiplicity 2 for the K-invariant eigenfunctions of Ω already suggests that (G,H) is not a generalized Gelfand pair in this case.

Mizony [105], [106] and Faraut [38] point out that for $G = O(p,1)$, $H = O(p-1,1)$ and $A_+ := \{a_t \mid t > 0\}$ the subset HA_+H of G is a subsemigroup of G. Thus the continuous H-biinvariant functions on G with support included in HA_+H form a convolution algebra which turns out to be commutative. After restriction to a smaller algebra of C^∞-functions of at most exponential growth on A_+, the characters of this algebra can be determined. It turns out that they have the form

$$(4.25)\qquad f \mapsto \int_0^\infty f(a_t)\, \frac{2^{p-2}\Gamma(\frac{1}{2}p-\frac{1}{2})\Gamma(-\frac{1}{2}p+\frac{1}{2}-i\lambda)}{\Gamma(1-i\lambda)}\, \Phi_\lambda^{(\frac{1}{2}p-1,-\frac{1}{2})}(t)\cdot (\operatorname{sh} t)^{p-1}dt,$$

where Φ_λ is a Jacobi function of the second kind (cf.(2.15)) and $\operatorname{Im}\lambda$ is sufficiently large. Thus we have a group theoretic interpretation of the <u>Laplace-Jacobi transform</u> if $\beta = -\frac{1}{2}$,

$\alpha = 0, \frac{1}{2}, 1, \ldots$. Mizony [105] and Carroll [19] consider the Laplace-Jacobi transform also for more general α, β, without group theoretic interpretation, and they obtain inversion formulas. Mizony [106] points out that, in the case of group theoretic interpretation, the functions $Ha_tH \to \Phi_\lambda(t)$ can be considered as certain generalized matrix elements of "principal series" representations of the semigroup HA_+H. The interpretation as characters of a convolution algebra is interesting, since such an interpretation is not known for spherical distributions.

5. THE ABEL TRANSFORM

As we already observed in §2, the Jacobi transform $f \mapsto \hat{f}$ has a factorization

$$(5.1) \qquad \begin{array}{ccc} f & \longrightarrow & \hat{f} \\ & \searrow \quad \nearrow_{\mathcal{F}} & \\ & F_f & \end{array}$$

where $\mathcal{F}$ is the classical Fourier transform and $f \mapsto F_f$ is the <u>Abel transform</u>. This last transform can be defined both in a group theoretic (geometric) way and in a purely analytic way. Since fairly much is known about the properties of $\mathcal{F}$, a study of the Abel transform will teach us a lot about the Jacobi transform. Moreover, the Abel transform is an interesting object in its own right. Roughly the following aspects of the Abel transform will be discussed:

(a) the homomorphism property with respect to suitable convolution algebras;
(b) the transmutation property with respect to suitable differential operators;
(c) the bijection property with respect to suitable function spaces;
(d) the inversion of the Abel transform;
(e) the images of certain special functions;
(f) the dual Abel transform.

One can start reading this section either in §5.1, where the Abel transform is treated in the spherical rank one case, or in §5.3, where an analytic treatment of the Abel transform is presented. In §5.2 the transposition of formulas from group theoretic into analytic form is discussed. Finally, §5.4 contains a generalization of the Abel transform and §5.5 discusses results and references.

5.1. The spherical rank one case

The main reference for this subsection is [39]. Assume that G is a rank one group, use the notation of §3.1 and use the results and conventions of §4.1. The Haar measure dn on N can be normalized such that the Haar measure on G, normalized by (4.8) has the following expression with respect to the Iwasawa decomposition:

$$(5.2)\qquad \int_G f(x)\,dx = \int_K\int_{\mathbb{R}}\int_N f(ka_tn)e^{2\rho t}\,dk\,dt\,dn, f\in C_c(G).$$

For f in $C_c(G//K)$ define the Abel transform $f \mapsto F_f$ by

$$(5.3)\qquad F_f(t) := e^{\rho t}\int_N f(a_tn)\,dn,\quad t\in\mathbb{R}.$$

Combination of (4.7), (4.4), (5.2), (5.3) shows that, for f in $C_c(G//K)$,

$$(5.4)\qquad \hat{f}(\lambda) = \int_{\mathbb{R}} F_f(t)e^{i\lambda t}\,dt.$$

Thus the spherical Fourier transform is the composition of the Abel transform and the classical Fourier transform (cf. (5.1)). It can be shown that $f \mapsto F_f$ is an homomorphism of the convolution algebra $C_c(G//K)$ (or $\mathcal{D}(G//K)$) into the convolution algebra $C_{c,\mathrm{even}}(\mathbb{R})$ (or $\mathcal{D}_{\mathrm{even}}(\mathbb{R})$):

$$(5.4)\qquad F_{f*g} = F_f*F_g,\quad f,g\in C_c(G//K),$$

and that the mapping has the transmutation property (use(4.1)):

$$(5.5)\qquad F_{\Omega f}(t) = \left(\frac{d^2}{dt^2}-\rho^2\right)F_f(t),$$

where $f\in\mathcal{D}(G//K)$.

Let the dual Abel transform $g \mapsto E_g$ be the linear mapping of $C(\mathbb{R})$ into $C(G//K)$ which satisfies

$$(5.6)\qquad \int_G f(x)E_g(x)\,dx = \int_{\mathbb{R}} F_f(t)g(t)\,dt$$

for all f in $C_c(G//K)$. Then (use (5.2))

$$(5.7)\qquad E_g(x) = \int_K g(H(x^{-1}k))e^{-\rho H(x^{-1}k)}\,dk,\quad x\in G,\ g\in C(\mathbb{R}),$$

and there is the transmutation property

$$(5.8)\qquad \Omega E_g = E_{g''-\rho^2 g},\quad g\in\mathcal{E}(\mathbb{R})\ (=C^\infty(\mathbb{R})).$$

If $g(t) := e^{i\lambda t}$ then $E_g = \phi_\lambda$ (cf.(4.4)).

Now make the further assumption that $G = U(1,n;\mathbb{F})$ (cf. §3.1). Then we can rewrite (4.4) and (5.3) in a more concrete form. If x in $U(1,n;\mathbb{F})$ has matrix $(x_{ij})_{i,j=0,\ldots,n}$ then it can be shown that

$$(5.9) \qquad H(x) = \log(|x_{00}+x_{0n}|), \quad x \in G.$$

Let $k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ be an element of $K = U(1,\mathbb{F}) \times U(n,\mathbb{F})$. Then it follows from (5.9) that

$$H(a_{-t}k) = \log(|u \operatorname{ch} t - v_{nn} \operatorname{sh} t|).$$

Thus (4.4) can be rewritten as

$$(5.10) \qquad \phi_\lambda(a_t) = \int_{S(\mathbb{F}^n)} |\operatorname{ch} t + y_n \operatorname{sh} t|^{i\lambda-\rho} dy$$

where $S(\mathbb{F}^n)$ is the unit sphere in $\mathbb{F}^n$, $y = (y_1,\ldots,y_n) \in S(\mathbb{F}^n)$ and dy is the normalized $U(n,\mathbb{F})$-invariant measure on $S(\mathbb{F}^n)$. Note that $S(\mathbb{F}^n)$ is the homogeneous space K/M.

Next we rewrite (5.3). In terms of the elements $n_{z,w}$ (cf.(3.1)) the Haar measure on N equals $dn_{z,w} = c_0\, dz\, dw$, where dz and dw are Lebesgue measures on $\mathbb{F}^{n-1} = \mathbb{R}^{d(n-1)}$ and $\operatorname{Im}\mathbb{F} = \mathbb{R}^{d-1}$, respectively, and the positive constant c_0 has yet to be determined. For a K-biinvariant function f on G write

$$(5.11) \qquad f[\operatorname{ch} t] := f(a_t), \quad t \in \mathbb{R}.$$

Then

$$(5.12) \qquad f(x) = f[|x_{00}|], \quad x \in G.$$

For an even function g on $\mathbb{R}$ also write

$$(5.13) \qquad g[\operatorname{ch} t] := g(t), \quad t \in \mathbb{R}.$$

Now we can rewrite (5.3) (using (3.1) and (5.12)) as

$$F_f(t) = c_0 e^{\rho t} \int_{\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}} f[|\operatorname{ch} t + e^t(\tfrac{1}{2}|z|^2+w)|]\, dz\, dw,$$

hence

$$(5.14) \qquad F_f[x] = c_0 (2x)^\rho \int_{\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}} f[x|1+|z|^2+2w|]\, dz\, dw, \quad x \geq 1.$$

It was observed by Godement [57] that (5.14) reduces for $\mathbb{F} = \mathbb{R}$, $n = 2$ to

$$F_f[x] = c_0(2x)^{\frac{1}{2}} \int_{\mathbb{R}} f[x(1+z^2)]dz =$$

$$= c_0 2^{-\frac{1}{2}} \int_x^\infty f[y](y-x)^{-\frac{1}{2}}dy,$$

which is a version of the classical Abel transform (cf. Abel [1]). This explains the name of the transform $f \mapsto F_f$.

If we substitute

$$(5.15) \qquad f[x] := (2x)^{-i\lambda-\rho}, \quad \mathrm{Im}\lambda < 0,$$

in (5.14) then

$$(5.16) \qquad F_f[x] = c_0(2x)^{-i\lambda} \int_{\mathbb{F}^{n-1}\times \mathrm{Im}\,\mathbb{F}} |1+|z|^2+2w|^{-i\lambda-\rho} dz\, dw.$$

Here $f[\cdot]$ determines a function $f(\cdot)$ in $C_c(G//K)$ and F_f is well-defined although f does not have compact support. It is possible to determine c_0 from (5.16). Observe that, for K-biinvariant f,

$$\int_0^\infty f(a_t)\Delta(t)dt = \int_{\mathbb{R}} F_f(t)e^{\rho t}dt.$$

Substitute (5.15), (5.16) in this identity, put $\lambda := i\nu$ and let $\nu \uparrow -\rho$. Then we obtain

$$(5.17) \qquad c_0^{-1} = \int_{\mathbb{F}^{n-1}\times \mathrm{Im}\,\mathbb{F}} |1+|z|^2+2w|^{-2\rho} dz\, dw.$$

Remember the c-function, cf. (2.17), (2.18), (2.19). By application of (2.19) to (5.10) we can derive an integral representation for the c-function. Rewrite (5.10) as

$$\phi_\lambda(a_t) = e^{(i\lambda-\rho)t} \int_{S(\mathbb{F}^n)} \left|1+e^{-2t}\frac{1-y_n}{1+y_n}\right|^{i\lambda-\rho} |\tfrac{1}{2}(1+y_n)|^{i\lambda-\rho} dy.$$

Let $t > 0$. Then the integrand is dominated by $|\frac{1}{2}(1+y_n)|^{-\mathrm{Im}\lambda-\rho}$ if $-\rho \le \mathrm{Im}\lambda < 0$ and by 1 if $\mathrm{Im}\lambda \le -\rho$. Hence, an application of the dominated convergence theorem shows that $\phi_\lambda(a_t)$ satisfies (2.19), where

$$(5.18) \qquad c(\lambda) = \int_{S(\mathbb{F}^n)} |\tfrac{1}{2}(1+y_n)|^{i\lambda-\rho} dy, \quad \mathrm{Im}\lambda < 0.$$

Next we will express the constant factor in (5.16) in terms of the c-function. By the Bruhat and Iwasawa decompositions (cf. §3.1) the mapping $\bar{n} \mapsto u(\bar{n})M$ is a diffeomorphism of $\bar{N}$ onto an open dense subset of K/M. Here $u(\bar{n})$ is as in §3.1(c). The corresponding Jacobian occurs in the formula

$$\int_{K/M} h(kM)\,d(kM) = \int_{\bar{N}} h(u(\bar{n})M)\,e^{-2\rho H(\bar{n})}\,d\bar{n}, \quad h \in C(K/M). \tag{5.19}$$

This formula can be rewritten as

$$\int_{S(\mathbb{F}^n)} h((y_1,\ldots,y_n))\,dy = \tag{5.20}$$

$$= c_0 \int_{\mathbb{F}^{n-1}\times \mathrm{Im}\,\mathbb{F}} h\left(\frac{(-2z_1,\ldots,-2z_{n-1},1-|z|^2-2w)}{1+|z|^2+2w}\right) |1+|z|^2+2w|^{-2\rho}\,dz\,dw,$$

where $h \in C(S(\mathbb{F}^n))$. Formula (5.20) can also be derived by straightforward computation. Now put $h(y) := |\tfrac{1}{2}(1+y_n)|^{i\lambda-\rho}$ in (5.20) and combine with (5.18). Then we obtain

$$c(\lambda) = c_0 \int_{\mathbb{F}^{n-1}\times \mathrm{Im}\,\mathbb{F}} |1+|z|^2+2w|^{-i\lambda-\rho}\,dz\,dw, \quad \mathrm{Im}\lambda < 0, \tag{5.21}$$

and, in view of (5.15), (5.16):

$$F_f[x] = (2x)^{-i\lambda} \text{ if } f[x] := (c(\lambda))^{-1}(2x)^{-i\lambda-\rho}, \quad \mathrm{Im}\lambda<0. \tag{5.22}$$

Formula (5.22) has an analogue for Jacobi functions of the second kind defined by (2.15):

$$F_f(t) = e^{-i\lambda t}\,(t>0) \text{ if } f(k_1 a_t k_2) := (c(\lambda))^{-1}\Phi_{-\lambda}(t) \quad (t>0,\, k_1,k_2\in K),\ \mathrm{Im}\lambda < 0. \tag{5.23}$$

Here the function f is well-defined on $G\backslash\{e\}$ and so is its Abel transform defined by the right hand side of (5.14). It follows from (2.9) and (5.5) that $(d^2/dt^2+\lambda^2)F_f(t) = 0$ and (5.22), (5.14) and (2.15) show that $F_f(t) = e^{-i\lambda t}(1+o(1))$ as $t \to \infty$. Thus (5.23) is proved.

The results (5.22), (5.23) seem to be unobserved in literature until now. It would be of interest to find an higher rank analogue of (5.22).

5.2. Elimination of group variables from the integration formulas

In order to pass smoothly to the analytic treatment in §5.3, which does not use group theory, we will rewrite some of the previous integrals like (5.10), (5.14) in a form which does not involve group variables, thus allowing generalization to other values of α,β. The key observation is that, for $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$,

$$(5.24) \qquad \int_{S(\mathbb{F}^n)} f(\mathrm{Re}\,y_n + i\,|\mathrm{Im}\,y_n|)\,dy = \int_0^1\int_0^\pi f(r\,e^{i\psi})\,dm_{\alpha,\beta}(r,\psi),$$

where f is a function on the upper half unit disk and

$$(5.25) \qquad dm_{\alpha,\beta}(r,\psi) := = \frac{2\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(p+\frac{1}{2})}(1-r^2)^{\alpha-\beta-1}(r\sin\psi)^{2\beta}\,r\,dr\,d\psi,$$

and that

$$(5.26) \qquad c_0\int_{\mathbb{F}^{n-1}\times \mathrm{Im}\,\mathbb{F}} f(|1+|z|^2+2w|)\,dz\,dw = \int_0^\infty\int_0^\infty f(((1+s^2)^2+4t^2)^{\frac{1}{2}})\,dn_{\alpha,\beta}(s,t),$$

where f is a function on $(1,\infty)$ and

$$(5.27) \qquad dn_{\alpha,\beta}(s,t) := \frac{2^{2\rho+1}\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\,s^{2\alpha-2\beta-1}t^{2\beta}\,ds\,dt.$$

These formulas remain valid for $\mathbb{F} = \mathbb{R}$ $(\beta=-\frac{1}{2})$, but the measures $m_{\alpha,-\frac{1}{2}}$, $n_{\alpha,-\frac{1}{2}}$ degenerate to measures with one-dimensional support: the weak limits of $m_{\alpha,\beta}$, $n_{\alpha,\beta}$ as $\beta\downarrow -\frac{1}{2}$. Throughout the rest of the paper we will keep to this convention, so we will not give the formulas for $\beta = -\frac{1}{2}$ (or $\alpha=\beta$) separately.

Formulas (5.10), (5.14), (5.18), (5.21) now can be rewritten as

$$(5.28) \qquad \phi_\lambda(a_t) = \int_0^1\int_0^\pi |\mathrm{ch}\,t + r\,e^{i\lambda}\,\mathrm{sh}\,t|^{i\lambda-\rho}\,dm(r,\psi).$$

$$(5.29) \qquad F_f[x] = (2x)^\rho\int_0^\infty\int_0^\infty f[x((1+s^2)^2+4t^2)^{\frac{1}{2}}]\,dn(s,t),$$

$$(5.30)\qquad c(\lambda) = \int_0^1\int_0^\pi |\tfrac{1}{2}(1+r\ e^{i\psi})|^{i\lambda-\rho}\, dm(r,\psi) =$$

$$= \int_0^\infty\int_0^\infty ((1+s^2)^2+4t^2)^{-\frac{1}{2}(i\lambda+\rho)}\, dn(s,t),\quad \mathrm{Im}\lambda < 0.$$

The equality of the two integrals in (5.30) also follows by the transformation of integration variables $\frac{1}{2}(1+r\ e^{i\psi}) = (1+s^2-2it)^{-1}$. This is seen by straightforward computation or by use of (5.20). The explicit expression (2.18) of $c(\lambda)$ can also be obtained by evaluation of one of the integrals in (5.30).

5.3. The analytic case

A reference for this subsection is [81]. We will obtain a pair of dual integral transforms $f \mapsto F_f$, $g \mapsto E_g$ such that the transmutation properties (5.5), (5.8) hold for more general α,β. These transforms will be built up from two fractional integrals and a quadratic transformation. So let us first introduce these building blocks.

Let $L_{\alpha,\beta}$ be defined by (2.9). Then there is the quadratic transformation (QT)

$$(5.31)\qquad (L_{\alpha,\alpha}f)(t) = 4(L_{\alpha,-\frac{1}{2}}g)(2t)\ \text{if}\ f(t) = g(2t),$$

$$(5.32)\qquad \phi_{2\lambda}^{(\alpha,\alpha)}(t) = \phi_\lambda^{(\alpha,-\frac{1}{2})}(2t), \Phi_{2\lambda}^{(\alpha,\alpha)}(t) = \Phi_\lambda^{(\alpha,-\frac{1}{2})}(2t).$$

For $\mathrm{Re}\ \mu > 0$ define the fractional integral operators R_μ of Riemann-Liouville type and W_μ of Weyl type by

$$(5.33)\qquad (R_\mu f)(x) := \frac{1}{\Gamma(\mu)}\int_1^x f(y)(x-y)^{\mu-1}dy,$$

$$(5.34)\qquad (W_\mu f)(x) := \frac{1}{\Gamma(\mu)}\int_x^\infty f(y)(y-x)^{\mu-1}dy,$$

where $f \in L^1([1,\infty))$ and, in (5.34), $f \to 0$ sufficiently fast as $x \to \infty$ (cf. [35,Ch.13]). Some simple properties are:

$$(5.35)\qquad R_\mu \circ R_\nu = R_{\mu+\nu},\ DR_\mu = R_\mu D = R_{\mu-1},\ D^nR_n = \mathrm{id},$$

$$(5.36)\qquad W_\mu \circ W_\nu = W_{\mu+\nu},\ DW_\mu = W_\mu D = -W_{\mu-1},\ D^nW_n = (-1)^n\mathrm{id},$$

where $D := d/dx$ and $n \in \mathbb{N}$. Define the class $H_\sigma (\sigma>0)$ by

$$(5.37)\quad H_\sigma := \{f \in C^\infty((1,\infty)) \mid \forall n \in \mathbb{Z}_+\ f^{(n)}(x) = O(x^{-\sigma-n}),\ x \to \infty\}.$$

The following two mappings are bijections:

$$R_\mu : (x-1)^\alpha C^\infty([1,\infty)) \to (x-1)^{\alpha+\mu} C^\infty([1,\infty)),\ \operatorname{Re}\alpha > -1,$$

$$W_\mu : H_\sigma \to H_{\sigma-\operatorname{Re}\mu},\quad \operatorname{Re}\mu < \sigma.$$

A generalized integration-by-parts formula is given by

$$(5.38)\quad \int_1^\infty f(x)(W_\mu g)(x)(x-1)^\alpha (x+1)^\beta dx =$$

$$= \int_1^\infty (R_\mu^{(\alpha,\beta)} f)(x) g(x)(x-1)^{\alpha+\mu}(x+1)^{\beta+\mu} dx,$$

where $f \in C^\infty([1,\infty))$, $g \in C_c^\infty([1,\infty))$ and

$$(5.39)\quad (R_\mu^{(\alpha,\beta)} f)(x) := (x-1)^{-\alpha-\mu}(x+1)^{-\beta-\mu}\cdot$$

$$\cdot\ R_\mu (y \mapsto (y-1)^\alpha (y+1)^\beta f(y))(x).$$

$R_\mu^{(\alpha,\beta)}$ is a bijection of $C^\infty([1,\infty))$ onto itself and of $C^\infty([1,\infty)) \cap H_\sigma$ onto $C^\infty([1,\infty)) \cap H_{\sigma-\operatorname{Re}\mu}$.

Let $L_{\alpha,\beta}$ be the differential operator on $(1,\infty)$ obtained from $\mathcal{L}_{\alpha,\beta}$ by making the transformation $x = \operatorname{ch} 2t$:

$$(5.40)\quad (L_{\alpha,\beta} g)(x) := 4(x^2-1)g''(x) + 4((\alpha+\beta+2)x+\alpha-\beta)g'(x).$$

A straightforward computation yields the transmutation formula

$$(5.41)\quad (L_{\alpha,\beta} + (\alpha+\beta+1)^2) W_\mu f = W_\mu (L_{\alpha+\mu,\beta+\mu} + (\alpha+\beta+2\mu+1)^2) f,$$

where $f \in H_\sigma$, $\operatorname{Re}\mu < \sigma$. By using (5.41), (5.38) and the self-adjointness of $L_{\alpha,\beta}$ with respect to the weight function $(x-1)^\alpha (x+1)^\beta$ we obtain another transmutation formula

$$(5.42)\quad (L_{\alpha+\mu,\beta+\mu} + (\alpha+\beta+2\mu+1)^2) R_\mu^{(\alpha,\beta)} f = R_\mu^{(\alpha,\beta)} (L_{\alpha,\beta} + (\alpha+\beta+1)^2) f,$$

where $f \in C^\infty([1,\infty))$. Three applications of the beta integral yield:

(5.43) $$W_\mu((x-a)^{-\sigma}) = \frac{\Gamma(\sigma-\mu)}{\Gamma(\sigma)}(x-a)^{-\sigma+\mu},\quad 0<\mathrm{Re}\,\mu<\mathrm{Re}\,\sigma,$$

(5.44) $$(R_\mu^{(\alpha,\beta)}f)(1) = \frac{2^{-\mu}\Gamma(\alpha+1)}{\Gamma(\alpha+\mu+1)}f(1),\quad f\in C^\infty([1,\infty)),$$

(5.45) $$R_\mu^{(\alpha,\beta)}(x^\sigma) = \frac{\Gamma(\alpha+\beta+\sigma+1)}{\Gamma(\alpha+\beta+\sigma+\mu+1)}x^{\sigma-\mu}\,(\mathrm{mod}H_{\mu-\sigma-1}).$$

Now we consider maps $g\mapsto E_g^{(\alpha,\beta)}$ and $f\mapsto F_f^{(\alpha,\beta)}$ which are, schematically, compositions of the following maps:

(5.46) $$E^{(\alpha,\beta)}:(-\tfrac12,-\tfrac12)\xrightarrow{R_{\alpha-\beta}^{(-\frac12,-\frac12)}}(\alpha-\beta-\tfrac12,\alpha-\beta-\tfrac12)\xrightarrow{QT}$$
$$\longrightarrow(\alpha-\beta-\tfrac12,-\tfrac12)\xrightarrow{R_{\beta+\frac12}^{(\alpha-\beta-\frac12,-\frac12)}}(\alpha,\beta),$$

(5.47) $$F^{(\alpha,\beta)}:(\alpha,\beta)\xrightarrow{W_{\beta+\frac12}}(\alpha-\beta-\tfrac12,-\tfrac12)\xrightarrow{QT}$$
$$\longrightarrow(\alpha-\beta-\tfrac12,\alpha-\beta-\tfrac12)\xrightarrow{W_{\alpha-\beta}}(-\tfrac12,-\tfrac12).$$

We will work in the t-variable and we will normalize $E^{(\alpha,\beta)}$ and $F^{(\alpha,\beta)}$ such that $E_g^{(\alpha,\beta)}(0) = g(0)$ and $E^{(\alpha,\beta)}$ and $F^{(\alpha,\beta)}$ are adjoint to each other in a suitable sense. More concretely, we define

(5.48) $$E_g^{(\alpha,\beta)}(t) := (\Delta_{\alpha,\beta}(t))^{-1}\int_0^t g(s)A_{\alpha,\beta}(s,t)ds,$$

(5.49) $$F_f^{(\alpha,\beta)}(s) := \int_s^\infty f(t)A_{\alpha,\beta}(s,t)dt,$$

where

(5.50) $$A_{\alpha,\beta}(s,t) := \frac{2^{3\alpha+3/2}\Gamma(\alpha+1)\mathrm{sh}2t}{\Gamma(\frac12)\Gamma(\alpha-\beta)\Gamma(\beta+\frac12)}\int_s^t(\mathrm{ch}2t-\mathrm{ch}2w)^{\beta-\frac12}\cdot$$
$$\cdot(\mathrm{ch}\,w-\mathrm{ch}\,s)^{\alpha-\beta-1}\mathrm{sh}\,w\,dw,\quad 0<s<t,\ \alpha>\beta>-\tfrac12,$$

with degenerate cases

$$A_{\alpha,-\frac12}(s,t) = \tfrac12 A_{\alpha,\alpha}(\tfrac12 s,\tfrac12 t) =$$
$$= \frac{2^{3\alpha+\frac12}\Gamma(\alpha+1)}{\Gamma(\alpha+\frac12)\Gamma(\frac12)}\,\mathrm{sh}\,t(\mathrm{ch}\,t-\mathrm{ch}\,s)^{\alpha-\frac12},\quad \alpha>-\tfrac12.$$

Again we call the transform $f \mapsto F_f$ defined by (5.49) an <u>Abel transform</u>. It equals the classical Abel transform if $\alpha = \beta = 0$.

It follows from (5.42), (5.31), (5.44) that

$$(5.51) \qquad g \mapsto E_g^{(\alpha,\beta)} : C^\infty_{\text{even}}(\mathbb{R}) \to C^\infty_{\text{even}}(\mathbb{R}),$$

$$(5.52) \qquad E_g^{(\alpha,\beta)}(0) = g(0),$$

$$(5.53) \qquad E_{g''}^{(\alpha,\beta)} = (L_{\alpha,\beta}+(\alpha+\beta+1)^2 E_g^{(\alpha,\beta)}.$$

In order to describe the mapping properties of $F^{(\alpha,\beta)}$ let us introduce, for σ in $\mathbb{R}$, the class

$$H_\sigma := \{f \in C^\infty((0,\infty)) \mid \forall n \in \mathbb{Z}_+ \; f^{(n)}(t) = O(e^{-\sigma t}), t \to \infty\}.$$

Then it follows from (5.41), (5.31) that

$$(5.54) \qquad f \mapsto F_f^{(\alpha,\beta)} : H_\sigma \to H_{\sigma-\alpha-\beta}, \quad \sigma > \alpha + \beta + 1,$$

$$(5.55) \qquad (F_f^{(\alpha,\beta)})'' = F_{(L_{\alpha,\beta}+(\alpha+\beta+1)^2)f}.$$

It follows from (5.43) that

$$(5.56) \qquad F_f^{(\alpha,\beta)}(s) = c_{\alpha,\beta}(\lambda)(2\cosh s)^{-i\lambda} \quad \text{if } f(t) = (2\cosh t)^{-i\lambda-\alpha-\beta-1},$$

where $c(\lambda)$ is given by (2.18) and $\operatorname{Im}\lambda < 0$.

By combination of the above results about $f \mapsto F_f$ with (2.11) and the characterization of ϕ_λ as special solution of (2.10) we obtain the integral representation

$$(5.57) \qquad \phi_\lambda(t) = 2(\Delta(t))^{-1} \int_0^t \cos\lambda s \; A(s,t)\,ds,$$

which, in the case $\alpha = \beta = 0$, goes back to Mehler [103]. Similarly, if we combine the above results about $f \mapsto F_f$ with the characterization of Φ_λ as solution of (2.10) satisfying

$$(5.58) \qquad \Phi_\lambda^{(\alpha,\beta)}(t) = e^{(i\lambda-\rho)t} (\operatorname{mod} H_{\operatorname{Im}\lambda+\rho+2}),$$

(in view of (2.15)) then we obtain

$$(5.59)\qquad e^{i\lambda s} = \int_s^\infty \frac{\Phi_\lambda(t)}{c(-\lambda)} A(s,t)dt, \quad \operatorname{Im}\lambda > 0.$$

Different proofs of (5.57), (5.59) were given in [81,(2.16), (2.17)] by the use of fractional integrals for hypergeometric functions.

Formula (5.57) together with (5.45) yields again (2.19) for $\alpha \geq \beta \geq -\frac{1}{2}$. We now give another proof of (2.17). For $\lambda \notin i\mathbb{Z}$ we have $\phi_\lambda = a(\lambda)\Phi_\lambda + b(\lambda)\Phi_{-\lambda}$ for certain coefficients $a(\lambda)$, $b(\lambda)$. From (2.19) and (2.16) we find $a(\lambda) = c(\lambda)$ if $\operatorname{Im}\lambda < 0$. By analyticity in λ and since $\phi_\lambda = \phi_{-\lambda}$ we have $b(\lambda) = a(-\lambda)$. This proves (2.17).

The kernel $A_{\alpha,\beta}(s,t)$ can be written as a hypergeometric function by making the substitution $\tau = (\operatorname{ch} t - \operatorname{ch} w)/(\operatorname{ch} t - \operatorname{ch} s)$ in (5.50) and by using Euler's integral [33,2.1(10)]:

$$(5.60)\qquad A_{\alpha,\beta}(s,t) = \frac{2^{3\alpha+2\beta+\frac{1}{2}}\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})} \operatorname{sh}2t(\operatorname{ch} t)^{\beta-\frac{1}{2}} \cdot$$

$$\cdot\ (\operatorname{ch} t - \operatorname{ch} s)^{\alpha-\frac{1}{2}}\ {}_2F_1\left(\tfrac{1}{2}+\beta, \tfrac{1}{2}-\beta; \alpha+\tfrac{1}{2}; \frac{\operatorname{ch} t - \operatorname{ch} s}{2\operatorname{ch} t}\right).$$

Instead of in kernel form we can write the formulas (5.57), (5.49) also in a more group-like way. In (5.57), (5.50) make the two successive transformations of variables $(s,w) \mapsto (s,\chi) \mapsto (r,\psi)$ given by $\operatorname{ch} w = \cos\chi \operatorname{ch} t$ and $\operatorname{ch} t + \operatorname{sh} t\, r\, e^{i\psi} = e^{s+i\chi}$. The resulting formula is (5.28) (with left hand side replaced by $\phi_\lambda(t)$), now proved for $\alpha \geq \beta \geq -\frac{1}{2}$. A different proof was given in [41,p.150]. In the case of Legendre polynomials ($\alpha=\beta=0$) this integral representation goes back to Laplace. Next consider (5.49), (5.50). With the convention (5.13) formula (5.49) can be written as

$$F_f[x] = \frac{2^{3\alpha+\beta+1}\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \cdot$$

$$\cdot \int_{z=x}^\infty \int_{y=z}^\infty f[y](y^2-z^2)^{\beta-\frac{1}{2}}(z-x)^{\alpha-\beta-1} y\, dy\, dz.$$

By making the transformation of variables $y = x((s^2+1)^2+t^2)^{\frac{1}{2}}$, $z = x(s^2+1)$ we obtain (5.29) for general $\alpha \geq \beta \geq -\frac{1}{2}$. Finally (5.30) follows from (5.28), (2.19), (5.29), (5.56).

In order to invert the Abel transform we introduce a version $\mathcal{W}_\mu^\tau$ ($\operatorname{Re}\mu>0, \tau>0$) of Weyl's fractional integral transform:

$$(5.61)\qquad (\mathcal{W}^{\tau}_{\mu}f)(s) := \frac{1}{\Gamma(\mu)}\int_s^{\infty} f(t)(\mathrm{ch}\tau t-\mathrm{ch}\tau s)^{\mu-1}d\,\mathrm{ch}\tau t,$$

where, for convenience, we assume that $f \in \mathcal{D}_{even}(\mathbb{R})$. Then it follows from (5.36) that $\mathcal{W}^{\tau}_{\mu}f$ has an analytic continuation to all complex μ: if $n = 0,1,2,\ldots$ and $\mathrm{Re}\,\mu > -n$ then

$$(5.62)\qquad (\mathcal{W}^{\tau}_{\mu}f)(s) = ((-1)^n/\Gamma(\mu+n))\cdot$$

$$\cdot\int_s^{\infty}\frac{d^n f(t)}{d(\mathrm{ch}\tau t)^n}(\mathrm{ch}\tau t-\mathrm{ch}\tau s)^{\mu+n-1}d\,\mathrm{ch}\tau t.$$

It follows (again using (5.36)) that $\mathcal{W}^{\tau}_{\mu}$ has inverse $\mathcal{W}^{\tau}_{-\mu}$ and that it is a bijection of $\mathcal{D}_{even}(\mathbb{R})$ onto itself.

For f in $\mathcal{D}_{even}(\mathbb{R})$ formula (5.49) can be rewritten as

$$(5.63)\qquad F_f^{(\alpha,\beta)} = 2^{3\alpha+\frac{1}{2}}\pi^{-\frac{1}{2}}\Gamma(\alpha+1)\mathcal{W}^1_{\alpha-\beta}\circ\mathcal{W}^2_{\beta+\frac{1}{2}}(f),$$

this formula has an analytic continuation to all complex α,β with $\alpha \neq -1,-2,\ldots$ and it can be inverted as

$$(5.64)\qquad f = 2^{-3\alpha-\frac{1}{2}}\pi^{\frac{1}{2}}(\Gamma(\alpha+1))^{-1}\mathcal{W}^2_{-\beta-\frac{1}{2}}\circ\mathcal{W}^1_{\beta-\alpha}(F_f^{(\alpha,\beta)}).$$

Let us summarize the various expressions for the Abel transform obtained until now:

(a) kernel form (5.49) with kernel given by integral representation (5.50) or by hypergeometric function (5.60);
(b) composition (5.63) of two fractional integrals;
(c) group-like form (5.29) with only essential integration variables being preserved;
(d) group form (5.14) using special structure of N;
(e) group form (5.3) which generalizes to arbitrary semisimple G.

We might add:

(f) geometric form (cf. [66]) involving integration over horospheres: the so-called Radon transform.

A similar list can be made for the integral representation for ϕ_λ.

5.4. A generalization of the Abel transform

Let

$$(5.65)\qquad (A_{\alpha,\beta;\gamma,\delta}f)(s) := \int_s^{\infty} f(t)A_{\alpha,\beta;\gamma,\delta}(s,t)dt,\ s > 0,$$

where

$$(5.66)\qquad A_{\alpha,\beta;\gamma,\delta}(s,t) := \frac{2^{2(\alpha+\beta-\gamma-\delta)}\Gamma(\alpha+1)}{\Gamma(\gamma+1)\Gamma(\alpha-\gamma)} \cdot$$

$$\cdot\ \mathrm{sh}2t(\mathrm{ch}\ t)^{\gamma-\delta-\alpha+\beta}(\mathrm{ch}^2 t-\mathrm{ch}^2 s)^{\alpha-\gamma-1} \cdot$$

$$\cdot\ {}_2F_1\left(\tfrac{1}{2}(\alpha+\beta-\gamma+\delta),\tfrac{1}{2}(\alpha-\beta-\gamma+\delta);\alpha-\gamma;1-\frac{\mathrm{ch}^2 s}{\mathrm{ch}^2 t}\right),$$

$\alpha > \gamma$, $f \in C^\infty((0,\infty))$ and sufficiently rapidly decreasing. Then it can be shown (unpublished work of the author) that

$$(5.67)\qquad (L_{\gamma,\delta}+(\gamma+\delta+1)^2)A_{\alpha,\beta;\gamma,\delta} = A_{\alpha,\beta;\gamma,\delta}(L_{\alpha,\beta}+(\alpha+\beta+1)^2),$$

$$(5.68)\qquad (A_{\alpha,\beta;\gamma,\delta}f)(s) = \frac{(2\mathrm{ch}s)^{i\lambda-\gamma-\delta-1}}{c_{\gamma,\delta}(-\lambda)} \text{ if}$$

$$f(t) := \frac{(2\mathrm{ch}t)^{i\lambda-\alpha-\beta-1}}{c_{\alpha,\beta}(-\lambda)},$$

$$(5.69)\qquad A_{\alpha,\beta;\gamma,\delta}\left(\frac{\phi_\lambda^{(\alpha,\beta)}}{c_{\alpha,\beta}(-\lambda)}\right) = \frac{\phi_\lambda^{(\gamma,\delta)}}{c_{\gamma,\delta}(-\lambda)},$$

$$(5.70)\qquad \phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t) = \int_0^t \phi_\lambda^{(\gamma,\delta)}(s)\Delta_{\gamma,\delta}(s)A_{\alpha,\beta;\gamma,\delta}(s,t)\,ds.$$

Formulas (5.68), (5.69) are valid for $\mathrm{Im}\lambda > -\gamma + |\delta|-1$. Formula (5.69) may be derived from (5.68) by using that

$$(5.71)\qquad \frac{\phi_\lambda^{(\alpha,\beta)}(t)}{c_{\alpha,\beta}(-\lambda)} = \sum_{k=0}^\infty \frac{(-i\lambda)_{2k}}{(-i\lambda+1)_k}\,\frac{(2\mathrm{ch}t)^{i\lambda-2k-\alpha-\beta-1}}{c_{\alpha,\beta}(-\lambda-2ki)}$$

(this follows from (2.15), (2.18)).

In view of [33,2.11(22)], (5.66) and (5.60) we have $A_{\alpha,\beta;-\frac{1}{2},-\frac{1}{2}} = A_{\alpha,\beta}$, hence $A_{\alpha,\beta;-\frac{1}{2},-\frac{1}{2}}f = F_f$ and formulas (5.67)-(5.70) generalize (5.55), (5.56), (5.59) and (5.57).

Sprinkhuizen-Kuyper [125,(3.1)] defines a generalized fractional integral operator $I_\nu^{\mu,\lambda}$ which operates on $C((0,1])$, but which can immediately be extended to an action on sufficiently rapidly decreasing continuous functions on $(0,\infty)$. Then

$$A_{\alpha,\beta;\gamma,\delta}(f\circ ch) = 2^{2(\beta-\delta)+3(\alpha-\gamma)}\frac{\Gamma(\alpha+1)}{\Gamma(\gamma+1)} \cdot$$

$$\cdot (I_{2\delta+1}^{\alpha-\beta-\gamma+\delta,\beta-\delta} f)\circ ch$$

and the composition property [125,(3.4)] can be translated as

(5.72) $$A_{\alpha_2,\beta_2;\alpha_3,\beta_3}\, A_{\alpha_1,\beta_1;\alpha_2,\beta_2} = A_{\alpha_1,\beta_1;\alpha_3,\beta_3}\ .$$

Like in [125,§3], analytic continuation of the operator (5.65) with respect to $\alpha,\beta,\gamma,\delta$ is possible by use of (5.72) and

(5.73) $$A_{\alpha,\beta;\alpha+1,\beta+1} = -2^{-5}(\alpha+1)^{-1}(\mathrm{ch}\ t)^{-1}d/d(\mathrm{ch}\ t),$$

(5.74) $$A_{\alpha,\beta;\alpha+2,\beta} = \frac{1}{2^6(\alpha+1)(\alpha+2)}\left(\left(\frac{d}{d(\mathrm{ch}t)}\right)^2 + \frac{2\beta+1}{\mathrm{ch}t}\,\frac{d}{d(\mathrm{ch}t)}\right).$$

The kernel (5.66) simplifies if $\alpha - \gamma = \beta + \delta$, $\beta - \delta$, $-\beta + \delta$ or $-\beta - \delta$. The kernel degenerates completely if $\alpha = \gamma$, $\beta = -\delta$ and then (5.69), (5.70) give rise to the symmetries

(5.75) $$\phi_\lambda^{(\alpha,\beta)}(t) = (\mathrm{ch}t)^{-2\beta}\phi_\lambda^{(\alpha,-\beta)}(t),$$

(5.76) $$\Phi_\lambda^{(\alpha,\beta)}(t) = (2\mathrm{ch}t)^{-2\beta}\ \Phi_\lambda^{(\alpha,-\beta)}(t),$$

which can also be derived from (2.4), (2.15).

It would be of interest to find integral transforms on groups which give interpretations of (5.65). One possibility would be to consider (5.3) with N replaced by some suitable subgroup of N. Another possibility is in Badertscher [6,§5], where formulas (3) and (14) give interpretations of our formulas (5.70) and (5.65) as a passage from K-biinvariant functions to left-K, right-H-invariant functions on $G(G = O(1,n),\ K = O(1)\times O(n),\ H = O(1,n-1)\times O(1))$.

5.5. Notes

The Abel transform (5.3) can also be considered in the case of higher rank. It is an interesting open problem to find analogues of (5.63) in those cases, maybe related to fractional integrals in several variables, and to find an explicit inversion formula. Partial answers to the inversion problem in higher rank are given by Gindikin & Karpelevic [56], Helgason [66, Theorem 2.6] and Aomoto [4]. Flensted-Jensen & Ragozin [51] were motivated by the structure of (5.57) to prove that also in the higher rank case spherical functions $\phi_\lambda(a)$, considered as function of λ, are Fourier transforms of L^1-functions.

Flensted-Jensen [45] considers integral transforms between function spaces on a complex semisimple Lie group and on its normal real form. The special case $SL(2,\mathbb{C})$ yields a pair of integral transforms connecting Jacobi functions of order $(\frac{1}{2},\frac{1}{2})$ and $(0,0)$ (cf.[45,(10.4),(10.5)]).

Flensted-Jensen [42,§3] gives an analogue of the Abel transform (5.3) for left-K, right-H invariant functions, where (G,K,H) are certain triples as in §3.4, 4.5. In particular, this gives an interpretation in the Jacobi cases of order (α,β), $\alpha \geq \beta \geq -\frac{1}{2}$, $\alpha,\beta \in \frac{1}{2}\mathbb{Z}$.

Lions [96,Ch.12] finds an analogue of the dual Abel transform (5.48) in the case of a differential operator $L+q$, $q \in \mathcal{E}_{even}(\mathbb{R})$, L given by (2.8), $t \mapsto t^{-2\alpha-1}\Delta(t)$ in $\mathcal{E}_{even}(\mathbb{R})$ but not necessarily positive, $\alpha \in \mathbb{C}$ but $-\alpha \notin \mathbb{N}$. He proves the existence of a unique continuous bijection E of $\mathcal{E}_{even}(\mathbb{R})$ such that $(Ef)(0) = f(0)$, $Ef'' = (L+q)(Ef)$. For the proof he considers the hyperbolic p.d.e.

$$(5.77) \qquad (L_t+q(t)-\partial^2/\partial s^2)v(s,t) = 0.$$

Chébli [21], [22], [24] and Trimèche [137] have analogues of the integral representation (5.57) under their conditions (2.35) and eventually (2.36). Chébli [21], [22] uses (5.77) for its derivation. Moreover, in view of (2.36), he can apply a maximum principle and he thus obtains the positivity of the kernel. Chébli [24] and Trimèche [137] obtain their integral representation from asymptotics of ϕ_λ in terms of Bessel functions and from properties of the Fourier transform. (Conversely, estimates for ϕ_λ can be derived from its integral representation, cf.§6.) Trimèche [137] obtains analogues of the Abel transform (5.49) and its dual (5.48) from the integral representation.

Carroll [18] uses the Jacobi function example as a model for a general theory of transmutation operators.

6. PROOF OF THE PALEY-WIENER AND PLANCHEREL THEOREM

This section contains proofs of the two above-mentioned Theorems 2.1 and 2.3 by use of the Abel transform. We start with some estimates which will be needed in the proofs and we end with some notes. The papers [41] and [81] can be used as a reference for this section

Assume that $\alpha \geq \beta \geq -\frac{1}{2}$. From (4.4), (5.10) or (5.28) it follows that $|\phi_{\mu+i\nu}(t)| \leq |\phi_{i\nu}(t)|\,(\mu,\nu\in\mathbb{R})$, $\phi_{-i\rho}(t) \equiv 1$, $\phi_{i\rho}(t) \equiv 1$ (since $\phi_\lambda = \phi_{-\lambda}$) and that $\nu \mapsto \phi_{i\nu}$ is a convex function on $\mathbb{R}$ for all real t. It follows that

$$(6.1)\qquad |\phi_\lambda(t) \leq 1 \text{ if } |\mathrm{Im}\lambda| \leq \rho, \quad t \in \mathbb{R}.$$

By combination of this result with (2.19) we obtain that ϕ_λ is bounded iff $|\mathrm{Im}\lambda| \leq \rho$. From (5.28) we see that

$$(6.2)\qquad |\phi_\lambda(t)| \leq e^{t|\mathrm{Im}\lambda|}\phi_0(t), \quad t \geq 0.$$

Now $\phi_0(t)$ can be written as a linear combination of two solutions of $(L+\rho^2)u = 0$ behaving like $e^{-\rho t}$ and $te^{-\rho t}$ as $t \to \infty$ (cf.[33,Ch.2]). Hence, for some $C > 0$:

$$(6.3)\qquad |\phi_\lambda(t)| \leq C(1+t)e^{t(|\mathrm{Im}\lambda|-\rho)} \text{for all } t \geq 0,\ \lambda \in \mathbb{C}.$$

See [41, Lemmas 14,15], [81, Lemma 2.3], [101,(2.8)] for estimates of derivatives of $\phi_\lambda(t)$.

Let now α,β be arbitrarily complex $(\alpha\neq-1,-2,\ldots)$. Observe that $(c(-\lambda))^{-1}$ has only finitely many poles for $\mathrm{Im}\lambda \geq 0$ (none if $\mathrm{Im}\lambda \geq 0$ and $\mathrm{Re}(\alpha\pm\beta+1)>0$). Then an application of Stirling's formula [33,1.18(2)] shows that for each $r > 0$ there is $C_r > 0$ such that

$$(6.4)\qquad |c(-\lambda)|^{-1} \leq C_r(1+|\lambda|)^{\mathrm{Re}\alpha+\frac{1}{2}} \text{ if } \mathrm{Im}\lambda \geq 0 \text{ and}$$
$$c(-\mu) \neq 0 \text{ for } |\mu-\lambda| \leq r.$$

Finally we need the following estimate for Φ_λ: For each $\delta > 0$ there is $C_\delta > 0$ such that

$$(6.5)\qquad |\Phi_\lambda(t)| \leq C_\delta e^{-(\mathrm{Im}\lambda+\mathrm{Re}\rho)t} \text{ if } t \geq \delta \geq 0,\ \mathrm{Im}\lambda \geq 0.$$

See Flensted-Jensen [41,pp.150-152] for a proof, which is analogous to Harish-Chandra's [61] proof in the group case (general rank). It proceeds by deriving a recurrence relation for the coefficients $\Gamma_m(\lambda)$ in the expansion

$$(6.6)\qquad \Phi_\lambda(t) = e^{(i\lambda-\rho)t}\sum_{m=0}^{\infty}\Gamma_m(\lambda)e^{-mt}$$

and proving that $\Gamma_m(\lambda)$ is of at most polynomial growth in m, uniformly in λ. (Stanton & Thomas [127] give more precise estimates for $\Gamma_m(\lambda)$.)

Let $\alpha \geq \beta \geq -\frac{1}{2}$. In view of (5.57), (5.49) the Jacobi transform $f \mapsto \hat{f}$ defined by (2.12) factorizes as in (5.1) if, for instance, $f \in C_c(\mathbb{R})$. In the spherical rank one case this was already observed in (5.4).

We now prove the Paley-Wiener Theorem 2.1. The operator $\mathcal{W}_\mu^\tau$ defined by (5.61) can be shown to be an isomorphism of topological vector spaces in the two following cases:

$$\mathcal{W}_\mu^\tau: \mathcal{D}_{\text{even}}(\mathbb{R}) \to \mathcal{D}_{\text{even}}(\mathbb{R}),$$

$$\mathcal{W}_\mu^\tau: (\text{ch } t)^{-\sigma}\mathcal{S}_{\text{even}}(\mathbb{R}) \to (\text{ch } t)^{-\sigma+\tau\mu}\mathcal{S}_{\text{even}}(\mathbb{R})$$

$$(\sigma \geq \tau\mu \geq 0).$$

Thus, in view of (5.63), we have isomorphisms

(6.7) $$f \mapsto F_f: \mathcal{D}_{\text{even}}(\mathbb{R}) \to \mathcal{D}_{\text{even}}(\mathbb{R}),$$

(6.8) $$f \mapsto F_f: (\text{ch}t)^{-\sigma}\mathcal{S}_{\text{even}}(\mathbb{R}) \to (\text{ch}t)^{-\sigma+\rho}\mathcal{S}_{\text{even}}(\mathbb{R})$$

($\sigma \geq \rho$ in (6.8)). It follows from (6.7), (6.8), (5.1) and standard mapping properties of $\mathcal{F}$ that the Paley-Wiener Theorem 2.1 holds for $\alpha \geq \beta \geq -\frac{1}{2}$ and also

Theorem 6.1. For $\alpha \geq \beta \geq -\frac{1}{2}$, $\sigma \geq \rho$ the Jacobi transform is a 1-1 map of $(\text{ch } t)^{-\sigma}\mathcal{S}_{\text{even}}(\mathbb{R})$ onto the space of even C^∞-functions g on $\{\lambda \in \mathbb{C} \mid |\text{Im}\lambda| \leq \sigma-\rho\}$, holomorphic in its interior and satisfying

$$\sup_{|\text{Im}\lambda| \leq \sigma-\rho} (1+|\lambda|)^n |g^{(m)}(\lambda)| < \infty, \quad n,m \in \mathbb{Z}_+ .$$

Flensted-Jensen [41, Theorem 4] shows that the bijections in Theorems 2.1, 6.1 are homeomorphisms and that, in Theorem 2.1, there is also a bijection between the dual spaces.

Theorem 2.1 can similarly be proved for general complex α,β ($\alpha \neq -1,-2,\ldots$) by use of the analytic continuation of (5.1) with respect to α,β (use (5.63), (5.62), (2.13)). Note that Theorem 2.1 both generalizes and is derived from the classical Paley-Wiener theorem, cf. for instance Rudin [118, Theor. 7.22].

Next let us prove the inversion formulas (2.21), (2.25) for the Jacobi transform in the case $\alpha \geq \beta \geq -\frac{1}{2}$. Let $g(t)$ denote the right hand side of (2.25). Because of Theorem 2.1, (6.3), (6.4) g is well-defined and continuous. Observe that $\overline{c(\lambda)} = c(-\lambda)$ if $\lambda \in \mathbb{R}$ and use (2.17). Then, for $t > 0$,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) \frac{\Phi_\lambda(t)}{c(-\lambda)} d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda+i\mu) \frac{\Phi_{\lambda+i\mu}(t)}{c(-\lambda-i\mu)} d\lambda, \quad \mu \geq 0,$$

because of the estimate

$$\left| \hat{f}(\lambda+i\mu) \frac{\Phi_{\lambda+i\mu}(t)}{c(-\lambda-i\mu)} \right| \leq CC_{\hat{f},n}\, C_\delta (1+|\lambda+i\mu|)^{-n+\alpha+\frac{1}{2}} \cdot$$

$$\cdot\, e^{(A_{\hat{f}}-t)\mu-\rho t}, \quad t \geq \delta > 0, \lambda \in \mathbb{R}, \mu \geq 0, n \in \mathbb{Z}_+$$

(use Theorem 2.1, (6.4), (6.5)). Hence $g(t) = 0$ if $t > A_{\hat{f}}$ so $g \in C_{c,\mathrm{even}}(\mathbb{R})$. Now, by injectivity, $f = g$ will follow from $F_f = F_g$. We have

$$F_g(s) = \frac{1}{2\pi} \int_s^{\infty} \left(\int_{-\infty}^{\infty} \hat{f}(\lambda+i\mu) \frac{\Phi_{\lambda+i\mu}(t)}{c(-\lambda-i\mu)} d\lambda \right) A(s,t)\,dt, \mu > 0, \ s > 0.$$

The above estimate together with an estimate for $A(s,t)$ following from (5.50) allows us to apply Fubini's theorem. Combination with (5.59) yields

$$F_g(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda+i\mu) e^{(i\lambda-\mu)s} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda s} d\lambda.$$

Thus, because of (5.1) and the inversion formula to the classical Fourier transform we obtain $F_f = F_g$ and we have proved (2.21) and (2.25).

By a slight adaptation of the above reasoning (cf. [82,§4]) Theorem 2.2 can be shown to hold for general α, β. Then Theorem 2.3 follows from Theorem 2.2 by moving μ to 0 in (2.21) and by taking account of the poles of the integrand which are passed. Formula (2.27) follows from (2.24) and (2.12), while the extension of (2.27) to an isometry of L^2-spaces is unique because of Theorem 2.1.

There is a close relationship between the Paley-Wiener and the Plancherel theorem, but there are many possibilities for the order of their proofs. For instance, Flensted-Jensen [41] first proves the Plancherel theorem and then uses it to prove the Paley-Wiener theorem, but in the proofs given in [81] and in the present section we have the converse order, while in Rosenberg's [117] proof for the spherical case of general rank (see also Helgason [72,Ch.4,§7], [71,§4.2]) ingredients of proofs of both theorems follow each other in logical order.

The Paley-Wiener theorem was first proved by Ehrenpreis & Mautner [31], [32] for SL(2,$\mathbb{R}$), respectively in the spherical case and in the case of arbitrary double K-type, by Helgason [65] in the spherical rank one case, by Gangolli [53] in the spherical case of general rank, by Helgason [69, Cor.10.2] for arbitrary K-types on rank one spaces G/K. Chébli [21], [22], [23] and Trimèche [137] obtain a Paley-Wiener theorem under their more general conditions (2.35), (2.36). Nussbaum [110] announces without proof that for positive Δ satisfying (2.36) but not necessarily (2.35) still a Paley-Wiener theorem can be proved.

7. CONVOLUTION

In this section the convolution structure associated with the Jacobi transform is discussed. A reference is [48]. We start in §7.1 with the hardware, i.e. explicit formulas for the product of two Jacobi functions and for the generalized translation, both in the spherical rank one case and in the analytic case. In §7.2 the corresponding harmonic analysis is treated: the Jacobi transform of L^p-functions, the convolution of two L^p-functions and the Kunze-Stein phenomenon. This subsection ends with some notes.

7.1. Product formulas and generalized translation

Let $G = U(1,n;\mathbb{F})$ as in §3.1. Remember that the spherical function ϕ_λ on G of argument a_t equals the Jacobi function $\phi_\lambda = \phi_\lambda^{(\alpha,\beta)}$ of argument t, for suitable order (α,β). Now apply the product formula (3.6) to the spherical function ϕ_λ with $x = a_s$, $y = a_t$ and use (5.12). Then for $k = \begin{pmatrix} u & 0 \\ 0 & V \end{pmatrix}$ in $K = U(1,\mathbb{F}) \times U(n,\mathbb{F})$ the integrand of (3.6) becomes

$$\phi_\lambda[|u \operatorname{ch} s \operatorname{ch} t + V_{nn} \operatorname{sh} s \operatorname{sh} t|].$$

Hence, just as we got (5.10) and (5.28) we can rewrite (3.6) in terms of Jacobi functions as

$$(7.1)\qquad \phi_\lambda(s)\phi_\lambda(t) = \int_{S(\mathbb{F}^n)} \phi_\lambda[|\mathrm{ch}\, s\, \mathrm{ch}\, t + y_n\, \mathrm{sh}\, s\, \mathrm{sh}\, t|]dy$$

$$(7.2)\qquad = \int_0^1\int_0^\pi \phi_\lambda[|\mathrm{ch}\, s\, \mathrm{ch}\, t + r\, e^{i\psi}\mathrm{sh}\, s\, \mathrm{sh}\, t|]dm(r,\psi).$$

If $f_1, f_2 \in \mathcal{D}(G//K)$ then

$$(7.3)\qquad (f_1 * f_2)(a_t) = \int_G f_1(y) f_2(y^{-1}a_t)dy$$

$$= \int_G f_1(y)\left(\int_K f_2(y^{-1}ka_t)\,dk\right)dy$$

$$= \int_0^\infty f_1(a_s)(T_s f_2)(a_t)\Delta(s)ds,$$

where, for $f \in C^\infty_=(G//K)$,

$$(7.4)\qquad (T_s f)(a_t) := \int_K f(a_s k a_t)dk$$

$$= \int_{S(\mathbb{F}^n)} f[|\mathrm{ch}\, s\, \mathrm{ch}\, t + y_n \mathrm{sh}\, s\, \mathrm{sh}\, t|]dy$$

$$= \int_0^1\int_0^\pi f[|\mathrm{ch}\, s\, \mathrm{ch}\, t + r\, e^{i\psi}\mathrm{sh}\, s\, \mathrm{sh}\, t|]dm(r,\psi)$$

$T_s f$ is called the <u>generalized translate</u> of f. Easy group theoretical arguments show that $(T_s f)(a_t)$ is symmetric in s and t and that

$$(7.5)\qquad \begin{cases} (L_{(s)} - L_{(t)})(T_s f)(a_t) = 0, \\ T_0 f = f, \quad \frac{\partial}{\partial s}(T_s f)(a_t)|_{s=0} = 0. \end{cases}$$

Furthermore, from the definition of spherical functions (§3.2) together with (4.7) we obtain

$$(7.6)\qquad (f_1 * f_2)^\wedge(\lambda) = \hat{f}_1(\lambda)\hat{f}_2(\lambda).$$

Also remember the homomorphism property (5.4) of the Abel transform.

Let us now extend the above results to other values of

α,β $(\alpha\geq\beta\geq-\frac{1}{2})$, without using group theory. First we sketch a proof of (7.2) (cf.[48,§4]). Observe that (5.28) implies that

$$(7.7)\qquad (\tfrac{1}{2}(\mathrm{ch}2s+\mathrm{ch}2t))^{i\lambda-\rho}\phi_\lambda\left[\frac{2^{\frac{1}{2}}\mathrm{ch}\,s\,\mathrm{ch}\,t}{(\mathrm{ch}2s+\mathrm{ch}2t)^{\frac{1}{2}}}\right] =$$

$$= \int_0^1\int_0^\pi |\mathrm{ch}\,s\,\mathrm{ch}\,t + r\,e^{i\psi}\mathrm{sh}\,s\,\mathrm{sh}\,t|^{i\lambda-\rho}\,dm\,(r,\psi).$$

Next it can be shown (cf.[48, Theorem 4.2]) that

$$(7.8)\qquad \Phi_\lambda(s)\phi_\lambda(t) = \sum_{n=0}^\infty A_n(\tfrac{1}{2}(\mathrm{ch}2s+\mathrm{ch}2t))^{i\lambda-\rho-n}\cdot$$

$$\cdot\phi_{\lambda+in}\left[\frac{2^{\frac{1}{2}}\mathrm{ch}\,s\,\mathrm{ch}\,t}{(\mathrm{ch}2s+\mathrm{ch}2t)^{\frac{1}{2}}}\right],\qquad s>t\geq 0,$$

where the coefficients A_n are the ones occurring in

$$(7.9)\qquad \Phi_\lambda(s) = \sum_{n=0}^\infty A_n(\tfrac{1}{2}(\mathrm{ch}2s+1))^{i\lambda-\rho-n},\quad s>0.$$

A similar expansion was independently obtained by Bellandi Fo & Capelus de Oliveira [9]. Combination of (7.7), (7.8), (7.9) yields

$$(7.10)\quad \Phi_\lambda(s)\phi_\lambda(t) = \int_0^1\int_0^\pi \Phi_\lambda[|\mathrm{ch}\,s\,\mathrm{ch}\,t + r\,e^{i\psi}\mathrm{sh}\,s\,\mathrm{sh}\,t|]\,dm(r,\psi).$$

Finally, (7.10) together with (2.17) yields (7.2).

In §5.2 we described how to pass from the kernel form (5.57) of the integral reprsentation for ϕ_λ to the group-like form (5.28). We can use a similar change of integration variables in converse direction, namely $e^{i\chi}\mathrm{ch}\,u = \mathrm{ch}\,s\,\mathrm{ch}\,t + r\,e^{i\psi}\mathrm{sh}\,s\,\mathrm{sh}\,t$, in order to derive from the product formula (7.2) the kernel form

$$(7.11)\qquad \phi_\lambda(s)\phi_\lambda(t) = \int_0^\infty \phi_\lambda(u)K(s,t,u)\Delta(u)\,du,$$

where

$$(7.12)\qquad K(s,t,u) := \frac{2^{1-2\rho}\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(\mathrm{sh}\,s\,\mathrm{sh}\,t\,\mathrm{sh}\,u)^{-2\alpha}\cdot$$

$$\cdot\int_0^\pi (1-\mathrm{ch}^2 s-\mathrm{ch}^2 t-\mathrm{ch}^2 u+2\mathrm{ch}\,s\,\mathrm{ch}\,t\,\mathrm{ch}u\cos\chi)_+^{\alpha-\beta-1}\cdot$$

$$\cdot(\sin\chi)^{2\beta}d\chi$$

if $|s-t| < u < s+t$ and $K(s,t,u) := 0$ otherwise, and $x_+^\gamma := x^\gamma$ if $x > 0$ and 0 otherwise. By the use of Euler's integral representation 2.1(10), formula (7.12) can be rewritten as

$$K(s,t,u) = \frac{2^{-2\rho}\Gamma(\alpha+1)(\text{ch } s \text{ ch } t \text{ ch } u)^{\alpha-\beta-1}}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})(\text{sh } s \text{ sh } t \text{ sh } u)^{2\alpha}} \cdot \tag{7.13}$$

$$\cdot (1-B^2)^{\alpha-\frac{1}{2}}\ {}_2F_1(\alpha+\beta,\alpha-\beta;\alpha+\tfrac{1}{2};\tfrac{1}{2}(1-B)),\ |s-t|<u<s+t,$$

where

$$B := \frac{\text{ch}^2 s+\text{ch}^2 t+\text{ch}^2 u-1}{2\text{ch } s \text{ ch } t \text{ ch } u}\ . \tag{7.14}$$

Observe that K is nonnegative and symmetric in its three variables. From (7.11) with $\lambda = i\rho$ we obtain that

$$\int_0^\infty K(s,t,u)\Delta(u)du = 1. \tag{7.15}$$

<u>The generalized translate $T_s f$</u> of a function f(in $\mathcal{D}_{\text{even}}(\mathbb{R})$, for convenience), is defined by

$$(T_s f)(t) := \int_0^\infty f(u)K(s,t,u)\Delta(u)du \tag{7.16}$$

$$= \int_0^1\int_0^\pi f[|\text{ch } s \text{ ch } t + r\, e^{i\psi}\text{sh } s \text{ sh } t|]dm(r,\psi).$$

Then, obviously, $T_s f \in \mathcal{D}_{\text{even}}(\mathbb{R})$, $T_s f \geq 0$ if $f \geq 0$ and, by use of (7.11),

$$(T_s f)^\wedge(\lambda) = \phi_\lambda(s)\hat{f}(\lambda). \tag{7.17}$$

Also $(T_s f)(t) = (T_t f)(s)$ is C^∞ in (s,t), $T_0 f = f$, $T_s f = T_{-s} f$ and, by (7.17),

$$(L_{(s)}-L_{(t)})(T_s f)(t) = 0. \tag{7.18}$$

For $f,g \in \mathcal{D}_{\text{even}}(\mathbb{R})$ define the convolution product $f*g$ by

$$(f*g)(t) := \int_0^\infty (T_t f)(s)g(s)\Delta(s)ds = \tag{7.19}$$

$$= \int_0^\infty\int_0^\infty f(r)g(s)K(r,s,t)\Delta(r)\Delta(s)drds.$$

Then $f*g = g*f \in \mathcal{D}_{even}(\mathbb{R})$, $f*g \geq 0$ if $f,g \geq 0$ and, by (7.17),

$$(7.20) \qquad (f*g)^\wedge(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).$$

From (7.20) we conclude that the convolution product is associative and that

$$(7.21) \qquad F_{f*g} = F_f * F_g.$$

7.2. Harmonic analysis

Let $\alpha \geq \beta \geq -\frac{1}{2}$, $(\alpha,\beta) \neq (-\frac{1}{2},-\frac{1}{2})$. Write L^p for $L^p(\mathbb{R}_+, \Delta(t)dt)$ and $\|f\|_p$ for the L^p-norm of f. We will first discuss the mapping properties of the Jacobi transform on L^p, cf. [48, §3]. Let $1 \leq p < 2$ and take q such that $p^{-1} + q^{-1} = 1$. Let

$$(7.22) \qquad D_p := \{\lambda \in \mathbb{C} \mid |\mathrm{Im}\lambda| < (2p^{-1}-1)\rho\}.$$

It follows from (6.3) and (6.1) that

$$(7.23) \qquad \|\phi_\lambda\|_q < \infty \quad \text{if} \quad \lambda \in D_p, \quad 1 < p \leq 2$$

$$(7.24) \qquad \|\phi_\lambda\|_\infty < \infty \quad \text{if} \quad \lambda \in \bar{D}_1.$$

Thus, by Hölder's inequality we obtain from (2.12) that

$$(7.25) \qquad |\hat{f}(\lambda)| \leq \|f\|_p \|\phi_\lambda\|_q \quad \text{if } f \in L^p, \ \lambda \in D_p, \ 1 < p \leq 2,$$

$$(7.26) \qquad |\hat{f}(\lambda)| \leq \|f\|_1, \quad f \in L^1, \ \lambda \in \bar{D}_1.$$

By the analyticity of $\phi_\lambda(t)$ in λ we conclude that f is holomorphic in the strip D_p if $f \in L^p$. Moreover, if f is in L^1 then f is continuous on $\bar{D}_1$ and a similar argument as for the classical Fourier transform shows that $f(\lambda) \to 0$ as $\mathrm{Re}\lambda \to \pm\infty$, uniformly on $\bar{D}_1$.

Finally we have that the Jacobi transform is injective on L^p $(1 \leq p \leq 2)$, cf. [48, Theorem 3.2] for the proof. It uses (2.27) for $f \in L^p \cap L^2$, $g \in \mathcal{D}_{even}$.

Let us now discuss the convolution product of L^p-functions. Definition (7.16) of $T_s f$ can still be used if $f \in L^p$ $(1 \leq p \leq \infty)$. An application of Hölder's inequality together with (7.15) then shows that $T_s f \in L^p$ and

$$(7.27) \qquad \|T_s f\|_p \leq \|f\|_p.$$

Next, definition (7.19) of $f*g$ remains valid if $f \in L^p$, $g \in L^q$ such that $1 \le p,q,r \le \infty$ and $p^{-1} + q^{-1} - 1 = r^{-1}$. Then, by standard techniques, $f*g \in L^r$ and

(7.28) $$\|f*g\|_r \le \|f\|_p \|g\|_q .$$

However, because of (7.25) we can do better than in (7.28). Observe that for $f,g \in \mathcal{D}_{\mathrm{even}}(\mathbb{R})$, $1 \le p < 2$, $p^{-1}+q^{-1} = 1$ we have

$$(2\pi)^{\frac{1}{2}}\|f*g\|_2 = \|\hat{f}.\hat{g}\|_2 \le \|\hat{g}\|_\infty \|\hat{f}\|_2 \le$$

$$\le \|f\|_2 \|g\|_p \sup_{\lambda\in\mathbb{R}} \|\phi_\lambda\|_q = \|f\|_2 \|g\|_p \|\phi_0\|_q .$$

Thus, for some $A_p > 0$ we have

(7.29) $$\|f*g\|_2 \le A_p \|f\|_2 \|g\|_p, \quad f \in L^2,\ g \in L^p,\ 1 \le p < 2.$$

This phenomenon was first discovered by Kunze & Stein [90] on $SL(2,\mathbb{R})$. From (6.24) it can be derived (cf. [48, Theor. 5.5]) that

(7.30) $$\|f*g\|_q \le A_q \|f\|_2 \|g\|_2, \quad f,g \in L^2, \quad 2 < q \le \infty.$$

For more general Δ satisfying (2.35) and eventually (2.36) product formula, generalized translation and convolution were treated by Chébli [20], [21], Trimèche [137]. Chébli obtains a positive convolution kernel, by applying a maximum principle to (7.18). In [21] he also gets a Kunze-Stein phenomenon. Trimèche [137] gets his convolution structure by transplantation from the case $\Delta(t) \equiv 1$, by using the Abel transform. Braaksma & de Snoo [17] and Markett [98] discuss generalized translation with respect to operators $L = d^2/dt^2 + (2\alpha+1)t^{-1}\, d/dt + q(t)$ for certain potentials q. They get estimates for the generalized translation operator by using the Riemann function for the p.d.e. $(L_s - L_t)v(s,t) = 0$.

Flensted-Jensen [42] gives a group theoretic interpretation of the convolution structure for the Jacobi transform if $\alpha \ge \beta \ge -\frac{1}{2}$, $\alpha,\beta \in \frac{1}{2}\mathbb{Z}$. This is done in terms of K-H-invariant functions on $G = O(p,q)$. In very interesting work Badertscher [6,§5] transplants convolution for K-biinvariant functions on $G = O(p,1)$ to convolution for K-H-invariant functions by using his Abel-type transform.

8. ADDITION FORMULA, POSITIVE DEFINITE SPHERICAL FUNCTIONS AND DUAL CONVOLUTION STRUCTURE

In §3.3 we already derived a group theoretic version (3.25) of the addition formula for Jacobi functions. Here an analytic version of the addition formula will be obtained and it will be shown how this formula follows from (3.25) for special α,β. Next follow two applications of the addition formula: the examination of positive definiteness of spherical functions and the occurrence of a positive dual convolution structure. Both applications are connected with the positivity of the expansion coefficients in the addition formula of ϕ_λ for certain λ. The reference for the group theoretic derivation of the addition formula and for the positive definite spherical functions will be [50], while the analytic derivation of the addition formula and the treatment of the dual convolution structure is based on [49].

8.1. The addition formula, analytic form

In order to obtain analytic versions of (3.24), (3.25) for general α,β $(\alpha>\beta>-\frac{1}{2})$ we have to expand

$$|\mathrm{ch}\ t - r\,e^{i\psi}\mathrm{sh}\ t|^{i\lambda-\rho} \tag{8.1}$$

and

$$\phi_\lambda[|\mathrm{ch}\ s\ \mathrm{ch}\ t - r\,e^{i\psi}\mathrm{sh}\ s\ \mathrm{sh}\ t|], \tag{8.2}$$

respectively, as functions of $r\,e^{i\psi}$ on the upper half unit disk. This expansion should be in terms of some complete system of functions which are orthogonal with respect to the measure $dm(r,\psi)$ and which generalize the spherical functions for (K,M) (cf. the group theoretic derivation of (5.28) from (4.4) and of (7.2) from (3.6)). The most appropriate choice of the orthogonal system is given by the functions $\chi_{k,\ell}^{(\alpha,\beta)}$ $(\alpha>\beta>-\frac{1}{2}, k,\ell\in\mathbb{Z}, k\geq\ell\geq 0)$ defined by

$$\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi) := \tag{8.3}$$

$$:= R_\ell^{(\alpha-\beta-1,\beta+k-\ell)}(2r^2-1)\,r^{k-\ell}\,R_{k-\ell}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\psi),$$

where $R_n^{(\alpha,\beta)}$ is the renormalized Jacobi polynomial (2.3). They are polynomials in the two variables $r^2, r\cos\psi$, orthogonal with respect to the measure $dm_{\alpha,\beta}$. It can be computed that

$$(8.4)\qquad \pi_{k,\ell}^{(\alpha,\beta)} := \left(\int_0^1\int_0^\pi (\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi))^2 dm_{\alpha,\beta}(r,\psi)\right)^{-1} =$$

$$= \frac{(2k-2\ell+\beta)(k+\ell+\alpha)(\alpha-\beta)_\ell(2\beta+1)_{k-\ell}(\alpha+1)_k}{(k-\ell+2\beta)(k+\alpha)\ell!(k-\ell)!(\beta+1)_k}.$$

The associated Jacobi functions $\phi_{\lambda,k,\ell}^{(\alpha,\beta)}$ were defined by (4.15). Now we can state the analytic versions of (3.24), (3.25):

Theorem 8.1. Let $\alpha > \beta > -\frac{1}{2}$, then

$$(8.5)\qquad |\mathrm{ch}\, t - r\, e^{i\psi}\mathrm{sh}\, t|^{i\lambda-\rho} =$$

$$= \sum_{k=0}^\infty \sum_{\ell=0}^k \phi_{\lambda,k,\ell}^{(\alpha,\beta)}(t)\,\pi_{k,\ell}^{(\alpha,\beta)}\,\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi),$$

$$(8.6)\qquad \phi_\lambda^{(\alpha,\beta)}[|\mathrm{ch}\, s\, \mathrm{ch}\, t - r\, e^{i\psi}\mathrm{sh}\, s\, \mathrm{sh}\, t|] =$$

$$= \sum_{k=0}^\infty \sum_{\ell=0}^k \phi_{\lambda,k,\ell}^{(\alpha,\beta)}(s)\,\phi_{-\lambda,k,\ell}^{(\alpha,\beta)}(t)\,\pi_{k,\ell}^{(\alpha,\beta)}\,\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi).$$

The double series in (8.6), (8.7) converge absolutely, uniformly for (s,t,r,ψ) in compact subsets of their domain.

For $\alpha = \beta$ or $\beta = -\frac{1}{2}$ the theorem remains valid if one puts $r = 1$ or $\phi = 0,\pi$, respectively in (8,5), (8.6). Then both expansions degenerate to a single series ($\ell=0$ if $\alpha=\beta$, $k-\ell=0$ or 1 if $\beta=-\frac{1}{2}$). The case $\alpha = \beta = 0$ was proved in [145,§15.71] and the case $\alpha = \beta$ by Henrici [73,(80)].

In order to prove Theorem 8.1 we imitate the proof in [82] of the addition formula for Jacobi polynomials. First we show that the expansions formally hold, i.e. that

$$(8.7)\qquad \int_0^1\int_0^\pi |\mathrm{ch}\, t - r\, e^{i\psi}\mathrm{sh}\, t|^{i\lambda-\rho}\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)\,dm_{\alpha,\beta}(r,\psi) =$$

$$= \phi_{\lambda,k,\ell}^{(\alpha,\beta)}(t),$$

$$(8.8)\qquad \int_0^1\int_0^\pi \phi_\lambda^{(\alpha,\beta)}[|\mathrm{ch}\, s\, \mathrm{ch}\, t - r\, e^{i\psi}\mathrm{sh}\, s\, \mathrm{sh}\, t|]\cdot$$

$$\cdot\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)\,dm_{\alpha,\beta}(r,\psi) = \phi_{\lambda,k,\ell}^{(\alpha,\beta)}(s)\,\phi_{-\lambda,k,\ell}^{(\alpha,\beta)}(t).$$

For $k = \ell = 0$ these formulas coincide with (5.28) and (7.2), respectively. In fact, we will derive (8.7), (8.8) from (5.28), (7.2) by using the following lemma:

Lemma 8.2. Let $f \in C^\infty((0,\infty))$, $\alpha > \beta > -\frac{1}{2}$, $k,\ell \in \mathbb{Z}$, $k \geq \ell \geq 0$,

$$(8.9) \qquad (D_{k,\ell}^{(\alpha,\beta)} f)(x) := \left(\frac{d^2}{dx^2} + \frac{2(\beta+k-\ell)+1}{x}\frac{d}{dx}\right)^{\ell}\left(\frac{d}{d(x^2)}\right)^{k-\ell} f(x).$$

Then, for a,b in $\mathbb{R}$, $a > |b|$:

$$(8.10) \qquad \int_0^1\int_0^\pi f(|a+bre^{i\psi}|)\,\chi_{k,\ell}^{(\alpha,\beta)}(r,\psi)\,dm_{\alpha,\beta}(r,\psi) =$$

$$= \frac{a^{k+\ell}b^{k-\ell}}{2^{2\ell}(\alpha+1)_{k+\ell}}\int_0^1\int_0^\pi (D_{k,\ell}^{(\alpha,\beta)} f)(|a+bre^{i\psi}|)\cdot$$

$$\cdot\, dm_{\alpha+k+\ell,\beta+k-\ell}(r,\psi).$$

Lemma 8.2 is proved by using a Rodrigues type formula for $\chi_{k,\ell}^{(\alpha,\beta)}$ and by integration by parts (cf. [82, Lemma 4.1]). Formula (8.7) follows from (5.28) and (8.10) by using that

$$(8.11) \qquad D_{k,\ell}^{(\alpha,\beta)}\left(\frac{\Gamma(\alpha+1)x^{i\lambda-\alpha-\beta-1}}{c_{\alpha,\beta}(-\lambda)}\right)$$

$$= \frac{(-1)^{k+\ell}2^{2k+2\ell}\Gamma(\alpha+k+\ell+1)x^{i\lambda-\alpha-\beta-2k-1}}{c_{\alpha+k+\ell,\beta+k-\ell}(-\lambda)}.$$

Formula (8.11) is derived by straightforward differentiation and by substitution of (2.18). Formula (8.8) follows from (7.2) and (8.10) by using that

$$(8.12) \qquad D_{k,\ell}^{(\alpha,\beta)}\left(\frac{\Gamma(\alpha+1)\Phi_\lambda^{(\alpha,\beta)}[x]}{c_{\alpha,\beta}(-\lambda)}\right)$$

$$= \frac{2^{4k+2\ell}\Gamma(\alpha+k+\ell+1)\Phi_\lambda^{(\alpha+k+\ell,\beta+k-\ell)}[x]}{c_{\alpha+k+\ell,\beta+k-\ell}(-\lambda)}$$

(by termwise differentiation in (5.71) or by (5.69), (5.73), (5.74)).

The proof of Theorem 8.1 is completed by showing that the right hand sides of (8.5), (8.6) actually converge absolutely and uniformly to the left hand sides. This follows from the following lemma (cf. [49,Theor.3.6]):

Lemma 8.3. Let f be a C^∞-function on the closed upper half unit disk. Then, for each $\kappa > 0$:

$$\hat f(k,\ell) := \int_0^1\int_0^\pi f(re^{i\psi})\chi_{k,\ell}(r,\psi)\,dm(r,\psi) = O(k^{-\kappa}) \text{ as } k\to\infty,$$

uniformly in ℓ, and

$$f(re^{i\psi}) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \hat f(k,\ell)\,\pi_{k,\ell}\chi_{k,\ell}(r,\psi)$$

with absolute and uniform convergence. Moreover, if f depends on an additional parameter s running over a set S such that all partial derivatives of f with respect to $r\cos\psi$, $r\sin\psi$ are uniformly bounded in $(re^{i\psi},s)$ then the absolute convergence of the above series is also uniform on S.

Now we will sketch how (8.5), (8.6) follow from (3.24), (3.25) for special values of α,β. By abuse of notation we consider the spherical function ψ_δ for (K,M) as a function on $K/M = S(\mathbb{F}^n)$. Then (3.22) takes the form

$$(8.13)\qquad \phi_{\lambda,\delta}(t) = \int_{S(\mathbb{F})^n} |\mathrm{ch}\,t - y_n\,\mathrm{sh}\,t|^{i\lambda-\rho}\,\psi_\delta(y)\,dy.$$

For ψ_δ we have in the three cases $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$:

$\mathbb{F} = \mathbb{R}$ (see [34,Ch.11]): $\alpha = \frac12 n-1$, $\beta = -\frac12$,

$$\psi_\delta(y) := R_\ell^{(\frac12 n-3/2,-\frac12+k-\ell)}(2y_n^2-1)\,y_n^{k-\ell} \quad (\ell = 0,1,\ldots; k=\ell \text{ or } \ell-1.$$

$\mathbb{F} = \mathbb{C}$ (see [140],[80]): $\alpha = n-1$, $\beta = 0$,

$$\psi_\delta(y) := R_{k\wedge\ell}^{(n-2,|k-\ell|)}(2|y_n|^2-1)\,|y_n|^{k-\ell}\cdot$$
$$\cdot e^{i(k-\ell)(\arg y_n)} \quad (k,\ell = 0,1,\ldots).$$

Hence

$$\mathrm{Re}\,\psi_\delta(y) = \mathrm{Re}\,\psi_\delta^\vee(y) =$$
$$= R_\ell^{(n-2,k-\ell)}(2\,y_n{}^2-1)\,|y_n|^{k-\ell} R_{k-\ell}^{(-\frac12,-\frac12)}\left(\frac{\mathrm{Re}\,y_n}{|y_n|}\right),$$

where $k,\ell \in \mathbb{Z}$, $k \ge \ell \ge 0$.

$\mathbb{F} = \mathbb{H}$ (see [123], [77]) : $\alpha = 2n-1$, $\beta = 1$,

$$\psi_\delta(y) = R_\ell^{(2n-3,k-\ell+1)}(2|y_n|^2-1)\,|y_n|^{k-\ell} R_{k-\ell}^{(\frac12,\frac12)}\left(\frac{\mathrm{Re}\,y_n}{|y_n|}\right),$$

where $k, \ell \in \mathbb{Z}$, $k \geq \ell \geq 0$.

By inserting these expressions for ψ_δ into (8.14) and by taking the "radial" part of the normalized invariant measure on $S(\mathbb{F}^n)$ we obtain that $\phi_{\lambda,\delta}(t) = \phi_{\lambda,\check{\delta}}(t)$ is given by the left hand side of (8.7) for suitable values of α,β. By applying (8.7) we obtain (8.5), (8.6) from (3.24), (3.25).

A group theoretic derivation of (8.6) in the case $\alpha = \beta = 0, \frac{1}{2}, 1, \ldots$ ($\mathbb{F} = \mathbb{R}$) was given in [139, Chap.10, §3.5]. Durand [29] obtained an addition formula for Jacobi functions of the second kind.

8.2. A criterium for positive definite spherical functions

Remember the definition of positive definite spherical functions on G and their relationship with unitary representations as given in §3.2. For $G = U(1,n;\mathbb{F})$ let $\pi_{\lambda,0}$ ($\lambda \in \mathbb{C}$) be the (unique) irreducible subquotient representation of the representation π_λ (cf.(3.21)) which contains the representation 1 of K. Then precisely those δ from $(K/M)^\wedge$ occur in $\pi_{\lambda,0}$ for which both $\phi_{\lambda,\delta}$ and $\phi_{\bar{\lambda},\delta}$ are nonzero (cf.[84,§3]). Furthermore, the spherical function associated with $\pi_{\lambda,0}$ equals ϕ_λ. The spherical function ϕ_λ is positive definite iff $\pi_{\lambda,0}$ is unitarizable, i.e., if there exists a possible new G-invariant inner product on some G-invariant dense subspace of $H(\pi_{\lambda,0})$. Thus it is important to know which spherical functions ϕ_λ are positive definite. We derive a criterium by using the addition formula (3.25). For f in $C_c(G)$ write

$$f_\delta(s) := \int_K \int_K f(k^{-1} a_s \ell)\, \psi_\delta(k)\, dk\, d\ell, \quad \delta \in (K/M)^\wedge.$$

Then it follows from (3.25) that

$$\int_G \phi_\lambda(x^{-1}y) f(x) \overline{f(y)}\, dx\, dy = \sum_{\delta \in (K/M)^\wedge} d_\delta \cdot$$

$$\cdot \int_0^\infty \phi_{\lambda,\delta}(a_s)\, \overline{\phi_{\bar{\lambda},\delta}(a_t)}\, f_\delta(s)\, \overline{f_\delta(t)}\, \Delta(s) \Delta(t)\, ds\, dt.$$

We conclude:

Lemma 8.4. <u>ϕ_λ is positive definite on G iff for all δ in $(K/M)^\wedge$ with $\phi_{\lambda,\delta} \neq 0 \neq \phi_{\bar{\lambda},\delta}$ there is $c_\delta > 0$ such that $\phi_{\bar{\lambda},\delta} = c_\delta \phi_{\lambda,\delta}$.</u>

For real λ we already know that π_λ is unitary and if λ is not real or imaginary then $\overline{\phi_\lambda(t)} = \phi_{\bar\lambda}(t) \neq \phi_\lambda(t)$, so then ϕ_λ cannot be positive definite because of (3.8). So we will use Lemma 8.4 in the case of imaginary λ. In §8.1 we pointed out that $t \mapsto \phi_{\lambda,\delta}(a_t)$ equals the associated Jacobi functions $\phi_{\lambda,k,\ell}$ (cf.(4.15)) for certain k,ℓ. By use of (2.18) it follows from Lemma 8.4 that:

Theorem 8.5. ϕ_λ is a positive definite spherical function on $G = U(1,n;\mathbb{F})$ iff

$$\lambda \in \mathbb{R}\,,\ \lambda = \pm i\rho \text{ or } i\lambda \in \begin{cases} (-\rho,\rho) & (\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}), \\ (-2n+1,2n-1) & (\mathbb{F} = \mathbb{H}). \end{cases}$$

On comparing Theorem 8.5 with [48, Theorem 6.3] we see that, in the cases $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, a bounded spherical function ϕ satisfies (3.7) for all f in $C_c(G)$ iff ϕ satisfies (3.7) for all f in $C_c(G//K)$ but that this equivalence no longer holds if $\mathbb{F} = \mathbb{H}$.

Theorem 8.5 (for all rank one cases) was earlier proved by Kostant [87], and, for $\mathbb{F} = \mathbb{R}$, by Takahashi [129]. Takahashi [132] deals with the exceptional rank one case.

8.3. Dual convolution structure

Let $G = U(1,n;\mathbb{F})$ and let $f \in L^p(G//K)$ $(1 \le p < 2)$. Then $\hat f$ is even, bounded and analytic on $\mathbb{R}$ (cf.§7.2). Hence it follows by a slight variation on the Bochner-Godement and Plancherel-Godement theorem (cf. [39,Ch.1]) that f is positive definite iff $\hat f(\lambda) \ge 0$ for $\lambda \ge 0$ and that for such positive definite f we have $\hat f$ in $L^1(\mathbb{R}_+;\nu)$ and

$$f(x) = \int_0^\infty \hat f(\lambda)\phi_\lambda(x)\,d\nu(\lambda), \qquad x \in G \tag{8.14}$$

(ν given by (2.26)). Now let $f(x) := \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)$ $(x \in G, \lambda_1,\lambda_2 \in \mathbb{R})$. Then $f \in L^p(G//K)$ for each $p > 1$ (cf.(6.3), (4.8)) and f is positive definite because ϕ_{λ_1} and ϕ_{λ_2} are so. We conclude that

$$a(\lambda_1,\lambda_2,\lambda_3) := (\phi_{\lambda_1}\phi_{\lambda_2})^\wedge(\lambda_3) = \int_G \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)\phi_{\lambda_3}(x)\,dx = \tag{8.15}$$

$$= \int_0^\infty \phi_{\lambda_1}(t)\phi_{\lambda_2}(t)\phi_{\lambda_3}(t)\Delta(t)\,dt$$

is well-defined and even, bounded and analytic in λ_3, that $a(\lambda_1,\lambda_2,\lambda_3) \geq 0$ $(\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R})$ and that

$$(8.16)\qquad \phi_{\lambda_1}(x)\phi_{\lambda_2}(x) = \int_0^\infty \phi_{\lambda_3}(x)a(\lambda_1,\lambda_2,\lambda_3)d\nu(\lambda_3).$$

Now we can define the dual convolution product $f\circ g$ of f,g in $L^1(\mathbb{R}_+;\nu)$ by

$$(8.17)\quad (f\circ g)(\lambda_1) = \int_0^\infty\int_0^\infty f(\lambda_2)g(\lambda_3)a(\lambda_1,\lambda_2,\lambda_3)d\nu(\lambda_2)d\nu(\lambda_3).$$

Then $f\circ g \in L^1(\mathbb{R}_+;\nu)$ and

$$(8.18)\qquad \|f\circ g\|_1 \leq \|f\|_1\|g\|_1.$$

Let $f \mapsto \check{f}$ denote the inverse spherical Fourier transform

$$(8.19)\qquad \check{f}(x) := \int_0^\infty f(\lambda)\phi_\lambda(x)d\nu(\lambda),\quad f \in L^1(\mathbb{R}_+;\nu).$$

Then

$$(8.20)\qquad (f\circ g)^\vee(x) = \check{f}(x)\check{g}(x).$$

In order to develop a dual convolution structure with nonnegative kernel for the Jacobi transform in the case of more general α,β we need a substitute without use of representations for the usual positive definiteness proof of $\phi_\mu\phi_\nu$ $(\mu,\nu\in\mathbb{R})$. We observed that, in the group case, it is sufficient to know that $(\phi_\mu\phi_\nu)^\wedge$ is nonnegative on $\mathbb{R}$. This last property is equivalent to

$$(8.21)\qquad \int_G\int_G \phi_\mu(x^{-1}y)\phi_\nu(x^{-1}y)f(x)\overline{f(y)}dxdy \geq 0$$

for all f in $\mathcal{D}(G//K)$, so it is not necessary to prove (8.21) for all f in $C_c(G)$. The left hand side of (8.21) can be rewritten as

$$\int_0^\infty\int_0^\infty \left(\int_K \phi_\mu(a_{-s}ka_t)\phi_\nu(a_{-s}ka_t)dk\right)f(a_s)\overline{f(a_t)}\cdot$$
$$\cdot\Delta(s)\Delta(t)dsdt.$$

Combination with (3.25) yields

$$\sum_{\delta\in(K/M)^\wedge} d_\delta \left|\int_0^\infty \phi_{\mu,\delta}(a_s)\phi_{\nu,\check{\delta}}(a_s)f(a_s)\Delta(s)ds\right|^2,$$

which is nonnegative. ($\check{\delta}$ is contragredient representation to δ.)

We will now imitate this method in cases without group theoretic interpretation. We need the following lemma (cf. [49, Lemmas 4.1, 4.2, 4.3]).

Lemma 8.6. *Let $\alpha \geq \beta \geq -\frac{1}{2}$, $1 \leq p < 2$ and $f \in L^p(\mathbb{R}_+;\Delta(t)dt)$. Then $\hat{f}(\lambda) \geq 0$ for $\lambda \geq 0$ iff $\int_0^\infty (f*g)(t)\overline{g(t)}\Delta(t)dt \geq 0$ for all g in $\mathcal{D}_{even}(\mathbb{R})$. If f is moreover continuous on $[0,\infty)$ and $\hat{f}(\lambda) \geq 0$ for $\lambda \geq 0$ then $\hat{f} \in L^1(\mathbb{R}_+;\nu)$ and (2.25) holds.*

If $\lambda_1,\lambda_2 \in \mathbb{R}$ then $\phi_{\lambda_1}\phi_{\lambda_2} \in L^p(\mathbb{R}_+,\Delta(t)dt)$ for all $p > 1$ (cf. §7.2) and for all g in $\mathcal{D}_{even}(\mathbb{R})$ we have

(8.22)
$$\int_0^\infty ((\phi_{\lambda_1}\phi_{\lambda_2})*g)(t)\overline{g(t)}\Delta(t)dt =$$
$$= \int_0^\infty\int_0^\infty\left(\int_0^1\int_0^\pi \phi_{\lambda_1}[|\mathrm{ch}\, s\, \mathrm{ch}\, t - r\, e^{i\psi}\mathrm{sh}\, s\, \mathrm{sh}\, t|]\cdot\right.$$
$$\left.\cdot\phi_{\lambda_2}[|\mathrm{ch}\, s\, \mathrm{ch}\, t - r\, e^{i\psi}\mathrm{sh}\, s\, \mathrm{sh}\, t|]dm(r,\psi)\right)\cdot$$
$$\cdot g(s)\overline{g(t)}\Delta(s)\Delta(t)ds\, dt = \sum_{k=0}^\infty \sum_{\ell=0}^k \pi_{k,\ell}\cdot$$
$$\cdot\left|\int_0^\infty \phi_{\lambda_1,k,\ell}(s)\phi_{\lambda_2,k,\ell}(s)g(s)\Delta(s)ds\right|^2 \geq 0,$$

where we used the addition formula (8.6) and the property that $\overline{\phi_{\lambda,k,\ell}(t)} = \phi_{-\lambda,k,\ell}(t)$ if λ is real. Now the properties of the positive dual convolution structure for $L^1(\mathbb{R}_+;\nu)$ follow in the same way as in the group case.

The method of applying the addition formula in order to prove the positivity of the dual convolution structure was earlier used in [83] in connection with Jacobi polynomials. The dual convolution kernel (8.15) was explicitly computed by Mizony [104] for $\alpha = \beta = 0$ or $\frac{1}{2}$. It is an open problem to find it for other α,β. Mayer-Lindenberg [100, §3] discusses dual convolution in the spherical rank one case. Nussbaum [109], also working in the rank one case, considers functions ϕ in $C(G//K)$ which satisfy (3.7) for all f in $C_c(G//K)$ (a weaker form of positive definiteness). Such ϕ are inverse Jacobi transforms of certain positive measures on $\mathbb{R}_+ \cup i\mathbb{R}_+$. He shows that radial functions on a ball around 0 in G/K which are positive definite in this sense have an extension to a similar function on G/K. Trimèche [137] and Chébli [21], working with a more general Δ satisfying (2.35) and (in [21]) (2.36), show that any distribution T satisfying $\langle T, f*\tilde{f}\rangle \geq 0$ for all f in $\mathcal{D}_{even}(\mathbb{R})$ (convolution in generalized sense) is the inverse ϕ_λ-transform of appropriate positive measures on $\mathbb{R}_+$ and $i\mathbb{R}_+$. By use of the Abel transform they reduce this property of T to the case $\Delta = 1$, for which the result can be found in [54,Ch.2,§6.3,Theor.5]. In [137] the Plancherel formula (2.37) is deduced from this result.

9. TWO SPECIAL ORTHOGONAL SYSTEMS MAPPED ONTO EACH OTHER BY THE JACOBI TRANSFORM

It is well-known that the functions $x \mapsto e^{-\frac{1}{2}x^2} H_n(x)$ (H_n Hermite polynomial) form a complete orthogonal system of eigenfunctions with respect to the Fourier transform and, similarly, the functions $x \mapsto 2^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^\alpha(x^2)$ (L_n^α Laguerre polynomial) with respect to the Hankel transform. For the Jacobi transform we cannot expect such a system because $\phi_\lambda(t)$ is not symmetric in λ, t. Still it would be pleasant to have two explicit orthogonal systems in $L^2(\mathbb{R}_+;\Delta(t)dt)$ and $L^2(\mathbb{R}_+; |c(\lambda)|^{-2}d\lambda)$ which are mapped onto each other by the Jacobi transform. In this section I will present such systems (author's result, unpublished until now). The proofs will only be sketched.

First observe that, for Re $\mu > 0$:

$$(9.1) \qquad \int_0^\infty (2\operatorname{ch} t)^{-\mu-\rho} \phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt =$$

$$= c_{\alpha,\beta}(-i\mu)\int_{-\infty}^\infty e^{i\lambda s}(2\operatorname{ch} s)^{-\mu} ds =$$

$$= \frac{2^{\alpha+\beta-\mu}\Gamma(\alpha+1)\Gamma(\frac{1}{2}(\mu-i\lambda))\Gamma(\frac{1}{2}(\mu+i\lambda))}{\Gamma(\frac{1}{2}(\alpha+\beta+1+\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+\mu))}$$

in view of (5.4), (5.56), (2.18) and [33,1.5(26)]. This formula is quite useful for evaluating Jacobi transforms of functions which are given as series in inverse powers of cht. For instance, by use of (9.1) and (2.15), the integral

$$\int_0^\infty \phi_\mu^{(\gamma,\beta)}(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt$$

can be evaluated as a quotient of products of gamma functions. A special case of this evaluation can be found in Sprinkhuizen [126,§8.3], where it is obtained by use of fractional integrals in two variables.

Now fix α,β,δ such that $\alpha \geq \beta \geq -\frac{1}{2}$, $\delta > -1$. For $n = 0,1,2,\ldots$ let

$$(9.2) \qquad r_n(t) := (\operatorname{ch} t)^{-\alpha-\beta-\delta-2} R_n^{(\alpha,\delta)}(1-2\operatorname{th}^2 t),$$

where $R_n^{(\alpha,\delta)}$ is a Jacobi polynomial, cf. (2.3). It follows from the orthogonality properties of Jacobi polynomials (cf. [34,§10.8]) that

$$(9.3) \qquad \int_0^\infty r_n(t) r_m(t) \Delta_{\alpha,\beta}(t)\,dt = \\ = \frac{2^{2\alpha+2\beta+1} (\Gamma(\alpha+1))^2 \Gamma(n+\delta+1)\, n!}{(2n+\alpha+\delta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\delta+1)} \delta_{n,m}$$

and that the system $\{r_n\}$ is complete in $L^2(\mathbb{R}_+;\Delta(t)dt)$. By [34, 10.8(13)] and (2.3) $r_n(t)$ can be expanded as

$$(-1)^n \frac{(\delta+1)_n}{(\alpha+1)_n} \sum_{k=0}^n \frac{(-n)_k (n+\alpha+\delta+1)_k}{(\delta+1)_k\, k!} (\operatorname{ch} t)^{-\alpha-\beta-\delta-2-2k}$$

On combining this with (9.1) we obtain that

$$(9.4) \qquad \int_0^\infty r_n(t) \phi_\lambda^{(\alpha,\beta)}(t) \Delta_{\alpha,\beta}(t)\,dt = s_n(t) := \\ = \frac{(-1)^n 2^{2\alpha+2\beta+1} \Gamma(\alpha+1)(\delta+1)_n \Gamma(\frac{1}{2}(\delta+1-i\lambda))\Gamma(\frac{1}{2}(\delta+1+i\lambda))}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+2))\Gamma(\frac{1}{2}(\alpha-\beta+\delta+2))(\alpha+1)_n} \cdot \\ \cdot {}_4F_3\left(\begin{matrix} -n, n+\alpha+\delta+1, \frac{1}{2}(\delta+1-i\lambda), \frac{1}{2}(\delta+1+i\lambda) \\ \delta+1, \frac{1}{2}(\alpha+\beta+\delta+2), \frac{1}{2}(\alpha-\beta+\delta+2) \end{matrix} \middle| 1 \right).$$

In view of (9.3) and (2.27) we must have

$$(9.5) \qquad (2\pi)^{-1} \int_0^\infty s_n(\lambda) s_m(\lambda) |c_{\alpha,\beta}(\lambda)|^{-2} d\lambda = \text{RHS of (9.3)}$$

On the other hand, the right hand side of (9.4) can be expressed in terms of <u>Wilson polynomials</u>

$$(9.6) \qquad p_n(t^2;a,b,c,d) := (a+b)_n (a+c)_n (a+d)_n \cdot \\ \cdot {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a-t, a+t \\ a+b, a+c, a+d \end{matrix} \middle| 1 \right),$$

cf. Wilson [146]. For positive a,b,c,d these are orthogonal polynomials in t^2 on $\mathbb{R}_+$ satisfying the orthogonality relations [146,(3.1)]. They are symmetric in a,b,c,d. By (9.4) and (9.6) we have

$$(9.7)\qquad s_n(\lambda) = \frac{(-1)^n 2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(\delta+1)_n |\Gamma(\frac{1}{2}(\delta+1-i\lambda)|^2}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+2))\Gamma(\frac{1}{2}(\alpha-\beta+\delta+2))(\alpha+1)_n}\cdot$$
$$\cdot p_n(-\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+1),\tfrac{1}{2}(\delta+1),\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha-\beta+1))/p_n(0).$$

By substitution of (9.7) into (9.5) we obtain the orthogonality relations [146,(3.1)] for the Wilson polynomials. Thus the Plancherel Theorem 2.4 and the orthogonality relations for the Wilson polynomials imply each other. If, more generally, $\alpha > -1$ and β is arbitrarily real then , by Theorem 2.4, the functions s_n are orthogonal with respect to the measure ν (cf.(2.26)) which may include a discrete part, see [146,(3.3)].

The Hankel transform acting on Laguerre polynomials can be obtained as a limit case of (9.4). Indeed, use of Stirling's formula [33,1.18(2)] gives

$$2^{2(\alpha+\beta+1)}\int_0^\infty {}_1F_1(-n;\alpha+1;t^2)\,{}_0F_1(\alpha+1;-\tfrac{1}{4}\lambda^2t^2)\cdot$$
$$\cdot t^{2\alpha+1}e^{-\frac{1}{2}t^2}dt =$$
$$= \lim_{\delta\to\infty}\delta^{\alpha+1}\int_0^\infty (\operatorname{ch} t)^{-\alpha-\beta-\delta-2}R_n^{(\alpha,\delta)}(1-2\operatorname{th}^2 t)\cdot$$
$$\cdot\phi^{(\alpha,\beta)}_{\lambda\delta^{1/2}}(t)\Delta(t)dt =$$
$$= (-1)^n 2^{2\alpha+2\beta+1}\Gamma(\alpha+1)\lim_{\delta\to\infty}\cdot$$
$$\cdot\frac{\delta^{\alpha+1}(\frac{1}{2}(\alpha+\beta+\delta+2))_n\Gamma(\frac{1}{2}(\delta+1-i\lambda\delta^{\frac{1}{2}}))\Gamma(\frac{1}{2}(\delta+1+i\lambda\delta^{\frac{1}{2}}))}{\Gamma(\frac{1}{2}(\alpha+\beta+\delta+2))\Gamma(\frac{1}{2}(\alpha-\beta+\delta+2))\;(\frac{1}{2}(\alpha-\beta+\delta+2))_n}\cdot$$
$$\cdot{}_4F_3\left(\begin{matrix}-n,n+\alpha+\delta+1,\frac{1}{2}(\alpha+\beta+1+i\lambda\delta^{\frac{1}{2}}),\frac{1}{2}(\alpha+\beta+1-i\lambda\delta^{\frac{1}{2}})\\ \alpha+1,\frac{1}{2}(\alpha+\beta+\delta+2),\frac{1}{2}(\alpha+\beta+\delta+2)\end{matrix}\,\middle|\,1\right)$$
$$= 2^{3\alpha+2\beta+2}(-1)^n\Gamma(\alpha+1)e^{-\frac{1}{2}\lambda^2}\,{}_1F_1(-n;\alpha+1;\lambda^2).$$

Thus we have obtained the well-known formula (cf.[35,8.9(3)])

$$(9.8)\qquad \int_0^\infty t^{\alpha+\frac{1}{2}}e^{-\frac{1}{2}t^2}L_n^\alpha(t^2)J_\alpha(\lambda t)(\lambda t)^{\frac{1}{2}}dt =$$
$$= (-1)^n e^{-\frac{1}{2}\lambda^2}\lambda^{\alpha+\frac{1}{2}}L_n^\alpha(\lambda^2).$$

A special case of (9.4) was obtained by Boyer & Ardalan [15] in a group theoretic context. The paper gives the following motivation for the choice of a special orthogonal basis like (8.2). A spherical principal series representation of $O(1,n)$ can be realized both on $H_1 := L^2(O(1,n-1)/O(n-1))$ and on $H_2 := L^2(O(n)/O(n-1))$. A canonical (continuous) basis for H_1 with respect to $O(n-1)$ involves Jacobi functions, while a canonical basis for H_2 with respect to $O(n-1)$, transferred to H_1 by the intertwining operator, involves functions of the form (9.2). Then the overlap functions between these two bases involve functions of the form (9.4).

It is an open problem to extend the results of this section such that Wilson polynomials (9.6) occur without the restriction $a = b$ in the parameters.

10. FURTHER RESULTS

First we list some further references on the harmonic analysis of the Jacobi transform. Hasegawa [62] gives sufficient conditions on f in L^2 in order that $\hat{f} \in L^1$. Schindler [120] derives transplantation theorems in the case $\alpha = \beta$. Stanton & Tomas [127], Vretare [141] and Achour & Trimèche [2], [3] obtain multiplier results by using various techniques (in [2], [3] an analogue of the Littlewood-Paley g-function). Van de Wetering [143] obtains necessary and sufficient conditions for $\hat{f}$ in order that f is variation diminishing. Meaney [101] considers differentiability properties of the inverse Jacobi transform of a L^1-function and applies this to the study of sets of synthesis. Chébli [24] gives a theory of almost periodic functions with respect to differential operators of the form (2.8).

Of course, special cases of the above results allow immediate translation to the context of noncompact symmetric spaces of rank one. The following papers exclusively deal with harmonic analysis on rank one spaces G/K but make essential use of properties of Jacobi functions. Berenstein & Zalcman [12] give criteria for pairs of double cosets KxK, KyK in order that they satisfy the Pompeiu property on a noncompact rank one symmetric space. Kawazoe [79] studies the radial maximal function on G/K. Lohoué & Rychener [97] obtain properties of the resolvent of the Laplacian on G/K by using inversion of the Abel transform.

Samii [119] obtains curious inequalities for the spherical functions on $SO_0(1,n)/SO(n)$ by using an explicit expression for the operator which intertwines the Euclidean and the non-Euclidean Poisson transform for the unit ball in $\mathbb{R}^n$.

Mizony [104] and Grünbaum [59] raise the question whether $\phi_\lambda^{(\alpha,\beta)}(t)$ is eigenfunction of a differential operator in λ, as is the case when $\alpha = \beta = \frac{1}{2}$ (cf.2.11)). Grünbaum (see also [60]) would like to apply such a property to obtain a p.d.o. commuting with the operation of both time and band limiting on a rank one space. However, the answer is probably negative.

Van den Ban [7] considers the integral representation (4.4) as an integral over a contour in $(K/M)_c$, (complexification) and he next deforms the contour. In the rank one case a sume comes out of two integrals which represent $c(\lambda)\Phi_\lambda$ and $c(-\lambda)\Phi_{-\lambda}$, respectively

Roehner & Valent [116] use Jacobi functions in connection with birth and death processes. They prove the Plancherel theorem for the Jacobi transform by writing it as a Hankel transform followed by a Kontorovich-Lebedev transform. This factorization can be seen from the integral representation

$$\phi_\lambda^{(\alpha,\beta)}(t) = c(\operatorname{sh}t)^{-\alpha}\int_0^\infty K_{i\lambda}(x)J_\alpha(x \operatorname{sh} t)x^\beta dx,$$

cf. [34,7.7(31)]. See Faraut [40] and Terras [134] for the occurrence of the Kontorovich-Lebedev transform on rank one spaces.

Berezin & Karpelevič [13] state without proof that the spherical functions for $(G,K) = (SU(n,n+k),S(U(n)xU(n+k))$ restricted to $A \simeq \mathbb{R}^n$ equal

$$(t_1,\ldots,t_n) \to \frac{c.\det(\phi_{\lambda_i}^{(k,0)}(t_j))_{i,j=1,\ldots,n}}{\prod_{1\le i<j\le n}(\operatorname{ch}2t_i-\operatorname{ch}2t_j)}.$$

A rigorous proof was later given by Hoogenboom [74].

Historical remarks about the Mehler-Fock transform in connection with separation of variables problems can be found in Robin [115,Ch.IX]. Sneddon [124] gives applications of this transform.

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