Some studies on the deformation of the membrane in an RF MEMS switch

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Abstract

The upper membrane of an RF MEMS bends under a difference in voltage between the membrane and the dielectric beneath it. In this paper, the problem of the behaviour of the position of the upper membrane is posed. It is known from simulations by finite difference schemes that the solutions display all kinds of typical behaviour, one of which is that the graph of the capacity C against the voltage V displays hysteresis.

The authors derive several results, along two lines of attack. A number of model simulations is executed, by different approaches. We first discuss finite difference methods. A shooting method written in MATHEMATICA gives a graph of the solutions in (C, V)-space for a simplified version of the problem. We will argue that numerical path following approaches are useful in obtaining insight into the nature of solutions. We will exemplify this by a continuation of solutions running a program written in the continuation and bifurcation package AUTO.

In the second part of this paper, the focus lies on obtaining insight by analytical methods. It will be derived that solutions for which the membrane touches the dielectric, have no 'holes' in the surface where they touch. Some results concerning the stability of the solutions will be formulated. Finally, given some mild conditions on the energy, we prove that there exists a continuous path of equilibrium states between the open and the closed state.

1 Problem description

In this report we study an RF MEMS switch. The abbreviation RF MEMS stands for Radio Frequency MicroElectroMechanical Systems. An RF MEMS switch is a device that is designed to switch high-frequency band used in mobile telephones. This is achieved by drastically changing the capacity in a capacitor, as is explained as follows. A classic capacitor consists of two parallel plates. The capacitance depends inversely proportional on the distance between the two plates. In the RF MEMS switch this distance can be varied. The way it is varied depends on two forces: an elastic force which tries to keep the plates in an open state, which is the rest state of the device, and a

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force obtained by putting a constant voltage difference over the plates. This constant voltage does not lead to a current and does not affect the high-frequency signals.

Figure 1 shows a schematic cross-section of the RF MEMS switch. The thick black lines indicate the top electrode and bottom electrode. The dashed material is the dielectric with thickness d_{diel}. The presence of a charge Q on the top electrode results in an electrostatic force $F_{electrostatic}$, which is balanced by the mechanical spring forces F_{spring} on the anchors by which the top electrode membrane is suspended. If the top electrode touches the membrane, the contact force prevents it from moving into the dielectric. The thickness of the top electrode is h, the anchors are separated by a distance g from the bottom electrode. The shape of the top electrode is given by the function $u(x)$.

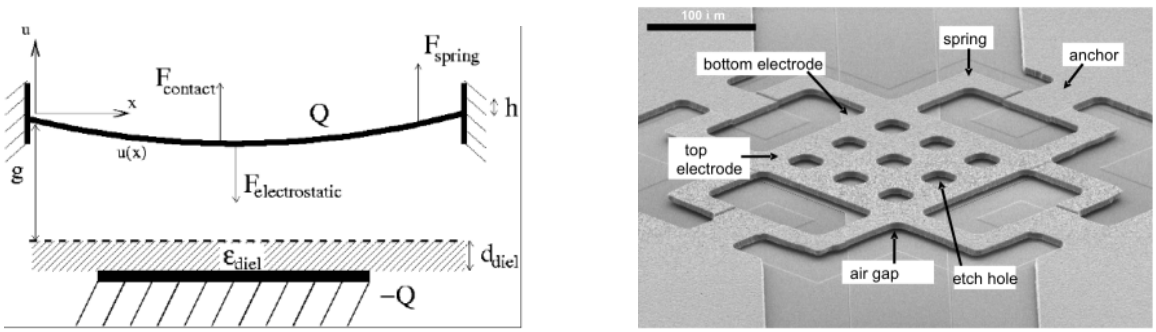

Figure 1: Left: schematic cross-section of a capacitive RF MEMS switch. Right: scanning electron microscope picture of a capacitive RF MEMS switch.

Problem formulation. The main problem description is the following: find all the displacement states i of the top electrode $u_{eq,Q,i}(x, y)$ for which the forces on the top electrode are in equilibrium at a fixed charge Q on the top electrode (or for a fixed voltage V between the electrodes).

Sub-problems:

- Is there always a continuous path of equilibrium states $u_{eq,Q,i}(x, y)$ between the open state $u_{eq,Q,i} = 0$ for all $x, y \in A_{top}$ and the closed state $u_{eq,\infty,N} = -g$ for all $x, y \in A_{bot}$.
- Is there a function $f(u_{eq,Q,i}(x, y), Q)$ that is monotonically increasing along this path?
- Can it be shown that along this continuous path $\frac{d}{dC}E_{mech} > 0$ is always valid? Here E_{mech} is the mechanical energy and C is its capacitance.
- Is there a simple way to determine whether a state is stable or unstable at a fixed voltage or charge?
- For which geometries and boundary conditions is the problem analytically solvable? Most interesting is the situation in which the top electrode springs are clamped (zero displacement and zero slope) at some points of its boundary.
- The dynamics of the structure under the presence of gas damping is a related interesting problem.

Finite element method. The equilibrium problem can be solved using finite element packages. Two examples are shown in figure 2. However it is not straightforward to find all solutions, especially since there can be multiple solutions for each value of V, Q and C. Some of the solutions are stable and others are unstable. In the left part of picture 2 a so-called VC-curve is displayed. We can see that the system may be in any of a finite number of states. This phenomenon is alled *hysteresis*.
Mathematically speaking, the equilibrium states are given by the critical points of the total energy function E_{tot}. The total energy is given by the sum of the electric energy E_{el} and the mechanical energy E_{mech}:

$$E_{tot} = E_{el} + E_{mech}.$$

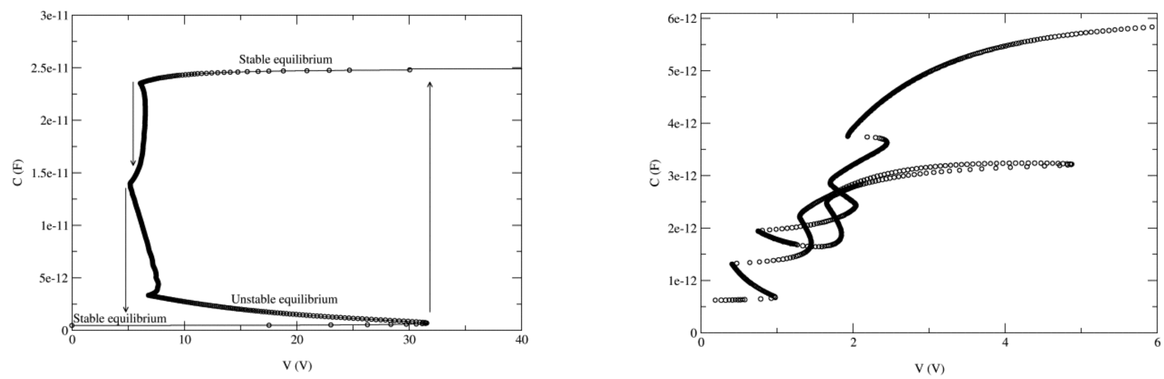

Figure 2: Calculated CV-curve (capicitance-voltage characteristic) of two different RF MEMS switches. On the left, a CV-curve of the switch of Figure 1 is depicted. On the right, the CV-curve for the so-called 'see-saw' RF MEMS switch is shown.

If we assume that the parallel plate approximation is valid, the capacitance C is given by the integral over the area of the bottom electrode A_{bot}:

$$
\begin{aligned}
C(u(x,y)) &= \int_{A_{bot}} \frac{\epsilon_0 d\,A}{g + u(x,y) + d_{diel}/\epsilon_{diel}} \\
E_{el} &= \frac{Q^2}{2C}
\end{aligned}
$$

If only the bending forces are taken into account, and if it is assumed that the thickness is very small and initial stress is zero, the mechanical energy is given by:

$$
E_{mech} = \int_{A_{top}} \frac{D}{2} |\Delta u|^2 \, dA \quad \text{where} \quad D = \frac{2h^3 Y}{3(1-\nu^2)}.
$$

Here Y is Young's modulus and ν is the Poisson ratio of the material.

Overview of results

The problem posed by the NXP-representatives is interesting, both from a practical as from a mathematical point of view. It should be stressed however that the problem as a whole is far too general and too difficult to tackle in only one week. However, we have obtained some interesting results that solve parts of the problem.

First of all, it is proven that under conditions that can easily interpreted and checked, the state is stable. Secondly, an inequality is formulated from which the stability can be derived. These results answer question 5 in the problem formulation above at least partially.

Then a result concerning the shape of solutions is stated. It is proven that the contact area has no 'holes', for any solution for which the electrode hits the membrane. Though this is not a precise answer to any of the questions in the problem formulation, it gives general insight in the type of solutions one would expect to have.

It can also be proven that given some mild conditions on the energy, there exists a continuous path of equilibrium states between the open and the closed state. This resolves the first question in the problem formulation.

Besides this all, it is argued that the continuation and bifurcation package AUTO helps a lot in acquiring insight into the nature of solutions to the problem. AUTO generates sets of solutions by *continuation*, i.e. it changes some parameter of the system (e.g. the voltage) and calculates one solution to this system for each value of the parameter. One advantage of this approach in state of finite element methods, is the fact that multiple solutions are easily found. For more on this, see the chapter on AUTO. We remark here that quickly after the Studygroup of Mathematics with Industry 2008, some people from NXP and the Studygroup Mathematics with Industry published an article on modelling the MEMS by continuation (or 'numerical path following') partly based on the discussions during the Studygroup (see [14]).

How to read this paper

This paper is organized as follows. It consists mainly of two parts. In the first part, we describe some numerical methods to gain insight into the problem. We describe the continuation by AUTO and a shooting method.

The second part consists of analytical approaches to the problem. We will first derive full solutions to the linearization of the problem. Linear problems with a suitable number of boundary conditions have a unique solution, so of course the hysteresis described above will not be found. After this, the results announced above will be presented and proven.

2 Methods for finding solutions

2.1 Finite difference scheme

A simple finite difference scheme for the 1D model which exhibits the most important qualitative aspects of the system can easily be implemented in MATLAB. Such a scheme can be compared to analytical approximations, and it can be used as a basis for a more advanced simulations in 2D.

The 1D system is of the form

$$\begin{aligned} \partial_x^4 w &= -\frac{\epsilon v^2}{1-\eta+w} + \phi(w), && x \in [0,1] \\ w = \partial_x w &= \partial_x^2 w = 0, && x = 0, x = 1 \end{aligned}$$

where v is a parameter corresponding to the voltage between the plates, and ϕ is a function modeling the contact force between the plate and the dielectric, essentially being zero when $w > -1$.

Assume a discrete n point numerical solution collected in a vector $\mathbf{w} = \{w_i\}_{i=1}^n$ on a grid $\mathbf{x} = \{(i+1)\Delta x\}_{i=1}^n$ with $\Delta x = 1/(n+1)$. Now $w_i \approx w(x_i)$, and we can write the biharmonic operator as a central difference

$$\begin{aligned} \partial_x^4 w|_{x_i} &= \frac{w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2}}{\Delta x^4} + \mathcal{O}(\Delta x^2), \quad i = 3, \ldots, n-2 \\ &= A\mathbf{w} + \mathcal{O}(\Delta x^2). \end{aligned}$$

Near the boundaries, we use $w_0 = w_n = 0$ and $w_1 - w_{-1} = 0$, $w_{n+1} - w_{n-1} = 0$, which is a second order central difference approximation to the boundary conditions, in order to get a consistent approximation. This discretized biharmonic operator can efficiently be inverted by using an iterative scheme like the conjugate gradient method(CG).

The right hand side of the equation is non-linear and has to be treated, typically with a sort of fixed-point iteration. The simplest possible scheme is to rewrite

$$\mathbf{w} = A^{-1}\left(-\frac{\epsilon v^2}{1-\eta+\mathbf{w}} + \phi(\mathbf{w})\right),$$

performing all operations in the parenthesis element-wise, and iterate. From a physical point of view it is clear that the equation is not uniquely solvable for a certain range of v. This will be reflected in the fact that the fixed-point iteration could converge to two different solutions. Which one of them is basically determined by the starting point for the iteration. This suggests how a CV-curve with the stable solutions of the system can be drawn: Starting at a low v, where the solution is unique, solve, and increment v. Using the last solution as starting point for the iterations means it will converge to a nearby solution, leading to a "continuous" branch of the CV-curve (this should be possible to prove without too much effort). Repeating the process for decreasing v starting at a large v will give the upper branch of the CV-curve.

2.2 Shooting method

In this section we use the shooting method to solve the nonlinear equation describing the shape of the membrane. The shooting method is in some sense the easiest 'method' to find solutions for an ODE. It relaxes the problem by ignoring one of the boundary conditions and replacing it by a 'free' initial condition instead. This initial condition can de adapted until the obtained solution satisfies the boundary condition that was omitted. For more on the shooting method, see [11].

We distinguish three cases

1. The plate touches the dielectric on some interval.
2. The plate touches the dielectric at one point.
3. The plate does not touch the dielectric.

Each of these cases describes a part of the CV-curve. First we treat in detail the shooting method for the situation when the plate touches the dielectric on some interval. Then we use standard *Mathematica 6* to solve the shooting problem and show the asymptotic behaviour of the CV-curve. The remaining two cases are solved using *Mathematica 6* as well; since this is done analogously we will not show the derivation. Finally we compute the CV-curve according to

$$C(v) = \frac{\Lambda \epsilon_0}{g} \int_{-1}^{1} \frac{dx}{1 + w(x; v) + \eta}. \tag{1}$$

Plate touches dielectric on some interval

Because of symmetry we consider the membrane on the interval $[0, 1]$ and assume that the membrane touches the dielectric on the interval $[0, a]$, where a is unknown in advance and $0 \leq a < 1$. The differential equation describing the shape of the membrane $w(x)$ between the support and the contact with the dielectric is

$$w^{(iv)} = -\frac{\epsilon v^2}{1 - \eta + w}, \tag{2}$$

with boundary conditions

$$w(1) = 0,\ w'(1) = 0,\ w(a) = -1,\ w'(a) = 0,\ w''(a) = 0. \tag{3}$$

It is convenient to change variable x according to $\tilde{x} = x - a$, $\tilde{w}(\tilde{x}) = w(x)$. Then the boundary conditions (3) are written as

$$\tilde{w}(1 - a) = 0,\ \tilde{w}'(1 - a) = 0,\ \tilde{w}(0) = -1,\ \tilde{w}'(0) = 0,\ \tilde{w}''(0) = 0, \tag{4}$$

and equation (2) remains the same

$$\tilde{w}^{(iv)} = -\frac{\epsilon v^2}{1 - \eta + \tilde{w}}. \tag{5}$$

In order to solve (2) and (3) we study the initial value problem for (5) with initial conditions

$$\tilde{w}(0) = -1,\ \tilde{w}'(0) = 0,\ \tilde{w}''(0) = 0,\ \tilde{w}'''(0) = P, \tag{6}$$

which has a solution $\tilde{w}(\tilde{x}; P)$.

Here the unknown parameter P_s has to be found from a condition that a solution $\tilde{w}(\tilde{x}; P_s)$ of (2) and (6) at some point $b > 0$ satisfies

$$\tilde{w}(b; P_s) = 0, \tilde{w}'(b; P_s) = 0. \tag{7}$$

Then by setting $a = 1 - b$ we conclude that $w(x) = \tilde{w}(\tilde{x}; P_s)$ solves (8) and (4). In case $b > 1$ a solution of (2) and (4) does not exist.

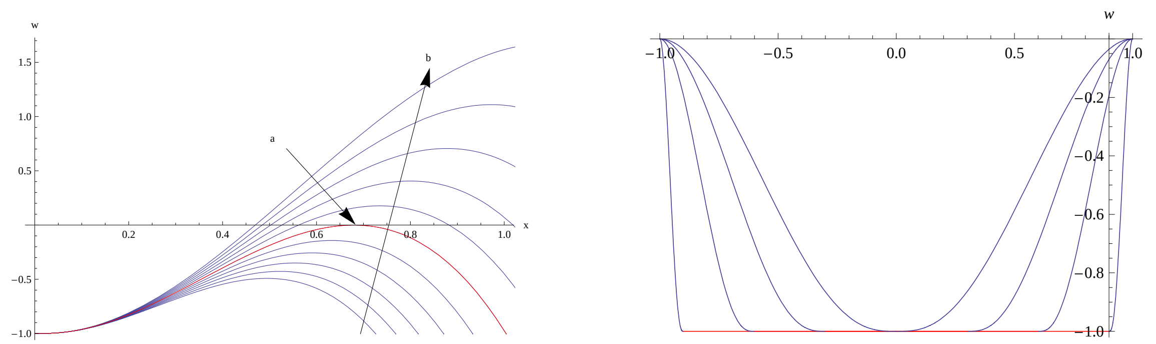

Figure 3: Left: The function $\tilde{w}(\tilde{x}; P)$ for different values of the shooting parameter P and $v = 890$. Here, $\tilde{w}(\tilde{x}; P)$ increases as P increases. The red curve corresponds to a solution $\tilde{w}(\tilde{x}; P_s)$ which satisfies (7) and solves (2) and (4). Right: The membrane shape for different v. The red line depicts a part of membrane in contact with dielectric. The blue curve is the shape of the membrane between the support and the dielectric.

The function $\tilde{w}(\tilde{x}; P)$ increases as function of P, see Figure 3(left). For small P, $w(\tilde{x}; P)$, as a function of $\tilde{x}$, increases, reaches a negative maximum and then decreases, see curves below the red one in Figure 3(left). For larger P, $\tilde{w}(\tilde{x}; P)$ increases and has positive first derivative where it crosses the line $\tilde{w} = 0$ for the first time, see curves above the red one in Figure 3(left). For $P = P_s$ the function $\tilde{w}(\tilde{x}; P)$ has a local maximum $\tilde{w} = 0$ (the red curve in Figure 3(left). This function satisfies the conditions (7) and b is the value of $\tilde{x}$ at which $\tilde{w}$ has the local maximum.

Next we present an alternative method for solving (2) and (4). This method is convenient for fast construction of a CV-curve because it requires solving a boundary value problem only once. Then using a scaling argument we get an easily calculable expression for C.

First we rescale $\tilde{x}$ according to $\hat{x} = \tilde{x}/(1-a)$ and $\hat{w}(\hat{x}) = \tilde{w}(\tilde{x})$. Then the boundary value problem (2) and (4) becomes

$$\hat{w}^{(iv)} = -\frac{\epsilon v^2 (1-a)^4}{1 - \eta + \hat{w}}, \tag{8}$$

$$\hat{w}(1) = 0,\ \hat{w}'(1) = 0,\ \hat{w}(0) = -1,\ \hat{w}'(0) = 0,\ \hat{w}''(0) = 0. \tag{9}$$

To solve (8) and (9) using the shooting method routine implemented in *Mathematica 6* we rewrite (8) as follows

$$\hat{w}^{(iv)}(\hat{x}) = -\frac{\epsilon V(\hat{x})^2}{1 - \eta + \hat{w}(\hat{x})},\ V'(\hat{x}) = 0. \tag{10}$$

Here the unknown $v^2(1-a)^4$ is described as an unknown constant function $V(\hat{x}) = V_s$. A solution $\hat{w}(\hat{x})$ and V_s of (10) describes the shape of the membrane $w(x) = \hat{w}(x)$ for $a = 0$ and V_s is the minimum value of v for which (2) and (3) has a solution. A solution $w(x)$ for arbitrary $v > V_s$ is written as

$$w(x) = \hat{w}((x-a)/(1-a)),\ a = 1 - \sqrt{\frac{V_s}{v}}.$$

The solution of $w(x)$ for $x \in [-1, 1]$ is given as

$$w(x) = \begin{cases} \hat{w}((|x|-a)/(1-a)), & \text{for } a < |x| \le 1, \\ 1, & \text{for } |x| \le 1, \end{cases}$$

see Figure 3(right).

With increasing v the contact with the dielectric increases and the membrane shape between the support and the dielectric becomes steeper.

The value of C is computed from (1) as

$$C(v) = \frac{2\Lambda\epsilon_0}{g}\left(\frac{1-\sqrt{V_s/v}}{\eta} + \sqrt{\frac{V_s}{v}} I_1\right), \text{ where } I_1 = \int_0^1 \frac{d\hat{x}}{1+\hat{w}(\hat{x})+\eta},$$

from which follows that $C(v)$ has a horizontal asymptotic

$$\lim_{v\to\infty} C(v) = \frac{2\Lambda\epsilon_0}{g\eta}, . \tag{11}$$

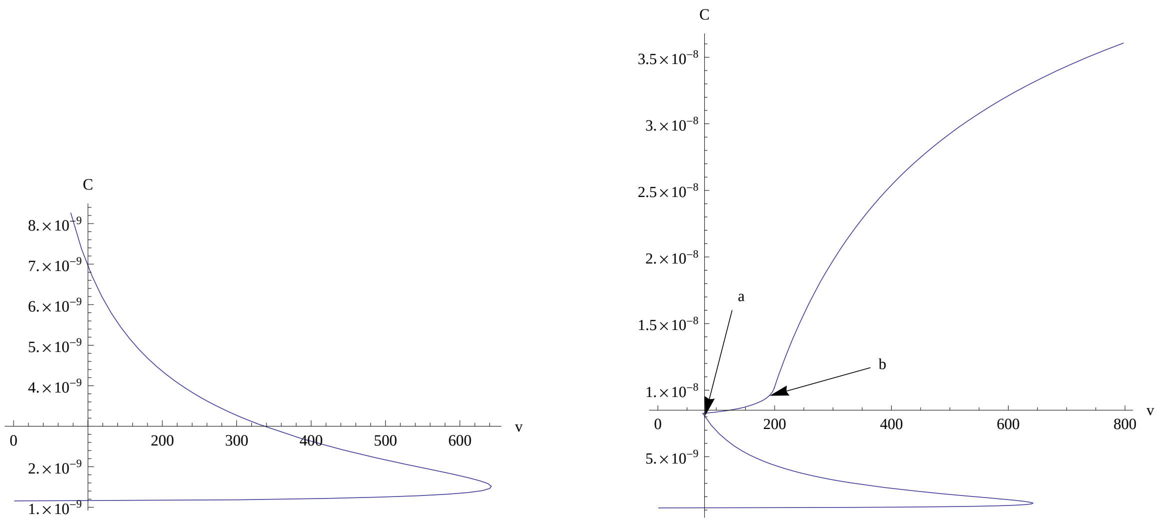

Figure 4: Left: The CV-curve, when the membrane does not touch the dielectric. The curve has a turning point at $v = 642$ because of the non-uniqueness of a solution to (2), (12) and (13). The curve starts at $w_0 = 0$ (this implies $v = 0$) and stops when the membrane touches the dielectric at $w_0 = -1$. Right: CV-curve for all three cases. The membrane first touches the dielectric at the point A. The change between the situations when the membrane touches the dielectric at one point, and on some interval is indicated by the point B.

CV-curve and influence of model parameters

Summarizing the results of the CV-curves for all three situations, see Figure 4, we construct the CV-curve for all v, see Figure 4(right). The complete CV-curve has discontinuous derivative at the transition point when the membrane touches the dielectric at the first time (point A in Figure 4(right)). At the transition between the situations when the membrane touches the dielectric at one point and on some interval (point B in Figure 4(right)), the CV-curve is C^1. For some interval of v three values of C are possible (see Figure 4(right)). This is a consequence of the non-uniqueness of the solution to the original problem for w. We call this interval a non-uniqueness interval for v.

3 The continuation problem

Auto is a software package that is used for finding and displaying solutions and tracking up bifurcations of solutions of ordinary differential equations (ODEs) by continuation of some system parameter.[1] A bifurcation is, loosely formulated, a sudden change in the qualitative behaviour of ODEs when some system parameter (or *bifurcation parameter*) crosses a certain threshold (the *critical value*). For example, an equilibrium solution may lose stability when the bifurcation parameter crosses a critical value. For more on the notion of bifurcation, see [6].

By continuation we mean the process of changing this system parameter and calculating the deformation of a solution when this parameter is changed. A typical continuation starts out with some (acquired) solution for the system with a certain value for the system parameter. Then the parameter is changed, and the solution is calculated for each value of the parameter. Auto also detects bifurcations when they take place. So, in order to do a continuation, one has to find one solution for a specific value of the bifurcation parameter (often zero is a smart choice). By changing a parameter (i.e. by a *continuation* in one of the parameters) the solution generically changes as well. This solution can be found by Auto, for each value of the bifurcation parameter.

Most continuation software, and especially Auto, allows for continuation in two or more parameters as well. Auto is able not only to perform continuation of equilibria to ODEs, but also the continuation of periodic solutions of ODEs, fixed points of discrete dynamical systems, and even solutions to partial differential equations (PDEs) that can in some sense be transformed to ODEs, like spatially uniform solutions (i.e. solutions that do not depend on any spatial variable) of a system of parabolic[2] partial differential equations (parabolic PDEs), travelling wave solutions to a system of parabolic PDEs, and even more.

It is not of our interest *how* Auto finds this solution. For convenience, we only note here that all continuation methods basically rely upon some version of Newton's method (and therefore the Implicit Function Theorem).

We want to stress that continuation always leads to a (discretized) *continuum* of solutions. This is an advantage with respect to the other numerical methods we described so far. Moreover, a continuation and bifurcation package such as Auto is able to detect bifurcations of the system as well. This is a second main advantage.

We show the method of continuation applied to our problem. Nonlinear differential equations usually do not have known explicit solutions, but linear differential equations do. The differential equation for which the voltage V and capacitance C are calculated reads

$$\begin{aligned} u^{(4)} &= -\frac{V^2}{2}\frac{\epsilon_0}{Au+\frac{d}{\epsilon}} + Ak_1 e^{-k_2 u}; \\ u''(0) &= u''(1) = 0, \quad u(0) = u(1) = 0, \end{aligned} \tag{12}$$

for $A = 1$. A is a dummy parameter, that only serves for the transition from a linear problem which is easily solvable to the nonlinear problem that is not. The associated linear problem is obtained by setting $A = 0$:

$$\begin{aligned} u^{(4)} &= -\frac{V^2}{2}\frac{\epsilon_0\epsilon}{d}; \\ u''(0) &= u''(1) = 0, \quad u(0) = u(1) = 0. \end{aligned} \tag{13}$$

By solving the linear equation above, Auto knows a solution of the 'nonlinear' problem for $A = 0$. By continuation in A it subsequently finds solutions for the nonlinear problem with $A \neq 0$. For each of these solutions the capacitance C and voltage V can be calculated. We have displayed a CV-curve in Figure 5. On the left of the picture one sees a hysteresis.

[1] The package has been developed initially by E. Doedel and subsequently expanded by a range of authors, see [3].

[2] We do not explain the notion of a *parabolic* PDE here; it is of no importance to us. But see any introductory text on partial differential equations.

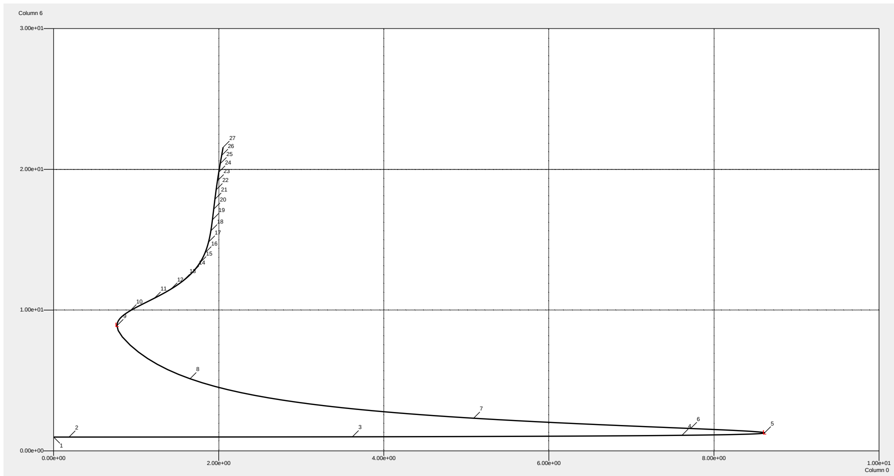

Figure 5: CV-curve generated by Auto.

4 Analytical results

4.1 The linearized problem

It is possible to fully solve the linearized problem for three different cases: (i) the case in which the membrane does not touch the dielectric at all (ii) the case in which the membrane touches the dielectric in one point only and (iii) the case in which the membrane touches the dielectric on an interval. Since linear problems have unique solutions, it is clear from the outset that the typical hysteresical behaviour does not show up in the linearized model. It has therefore no use for the problem in that sense. Since it might be useful to show some techniques however, we have summarized how to tackle the linearized problem in the appendix.

4.2 Collected analytical results

We prove some results for a functional E that may be interpreted as the total energy. The functional can be written as integral over some domain Ω in $\mathbb{R}^2$. To read this section, it might be necessary to consult a text on variational methods, see for example [4] or [5].

First, it is proved that the solution for the membrane cannot touch the dielectric 'with holes', i.e. in one dimension, the membrane is stuck to the dielectric between every two points where the membrane touches it. Secondly, it is derived that every critical point u for which $u = 0$ on some open set $\Omega_1 \subset \Omega$, has $\Delta u = 0$ on $\partial\Omega_1$. Thirdly, we prove that stationary solutions for the energy E for which it holds that $\frac{\partial C}{\partial V} < 0$ are necessarily unstable. The final result is argued but still not completely proved. It states that if for both large V and small V a unique critical point exists, then under some conditions on the energy functional, a continuous family of solutions connects the two solutions.

Just for notation sake: the main functional we consider is

$$E = \frac{D}{2}\int_\Omega u''^2 - \frac{V^2}{2}\int_\Omega \frac{\epsilon_0}{(u + d/\epsilon)} + \int_\Omega k_1 e^{-k_2 u}, \tag{14}$$

where Ω is a domain (e.g. a rectangle, or a circle) in $\mathbb{R}^2$ or an interval in $\mathbb{R}$, depending on the question considered. The second integral is the *capacity*

$$C = \int_\Omega \frac{\epsilon_0}{(u + d/\epsilon)}.$$

The boundary conditions are $u = g$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Unless stated otherwise, all integrals are over Ω.

Short list of results

1. For any minimiser (or general critical point) u of the *infinitely-hard bottom* problem

$$\min\Big\{ \frac{D}{2}\int \Delta^2 u - \frac{V^2}{2} C \;\Big|\; u \geq 0 \Big\} \tag{15}$$

   there exists *no* nonempty open sets $\Omega_1 \subset \Omega$ satisfying $u|_{\Omega_1} > 0$ and $u|_{\partial\Omega_1} = 0$. In particular, in dimension $n = 1$, the contact set $\{x \in \Omega \,|\, u(x) = 0\}$ is a (possibly empty) interval; in two-dimensions it means that the contact set has only simply connected components (no rings).

2. If u is minimizer of (15), or more generally a critical point, then if $u = 0$ on an open set $\Omega_1 \subset \Omega$, then $\Delta u = 0$ on $\partial\Omega_1$ (also of course on the interior of Ω_1).

3. Stationary points of E lying on a branch for which $\frac{dC}{dV} < 0$, are necessarily *unstable*. That is, there exists a perturbation w such that

$$E''(u) \cdot w \cdot w < 0.$$

   More generally, consider energies of the form

$$F(u, C, V) = \int f(x, u, \nabla u, \Delta u, \dots)dx + G(V, C),$$

   where $C = \int c(u(x))dx$ and V is a parameter (the Voltage for example). Then if $\frac{\partial^2 G}{\partial C \partial V} < 0$, any solution lying on a branch for which $\frac{dC}{dV} < 0$, is unstable.

4. The last result is more tentative; it should be true, but requires additional work to prove: if there exists a unique critical point of E both for small V and for large V, and provided that some type of coercivity holds for the energy functional (14), then there exists a continuous family of solutions connecting these two.

Sketches of the proofs

1 In 1D: let u be a stationary point, satisfying, where $u > 0$,

$$-Du'''' = \frac{\epsilon_0}{(u + d/\epsilon)^2}. \tag{16}$$

Note that the right hand side of (16) is strictly positive. Suppose u has two contact points $x_1 < x_2$. Since $u(x_i) = u'(x_i) = 0$ and $u \geq 0$, we must have $u''(x_i) \geq 0$. Furthermore, $-(u'')'' > 0$, hence it follows from the maximum principle that $u''(x) \geq \min\{u''(x_1), u''(x_2)\} \geq 0$ for $x \in (x_1, x_2)$. This implies, again by the maximum principle, that $u \leq 0$ on (x_1, x_2). We thus conclude that $u \equiv 0$ on $[x_1, x_2]$.

In more dimensions exactly the same (pair of maximum principle) arguments proves that the contact region can have no holes, as asserted.

2 We do not give a full proof, but illustrate the main idea. On a one-dimensional domain $\Omega = [-L, L]$, let $u_R(x)$ be a smooth family of symmetric solutions with “forced” contact region $[-R, R]$, with $R < L$. By symmetry, we only need to consider the left half of the solution:

$$\begin{cases} -Du_R'''' = \frac{\epsilon_0}{(u_R+d/\epsilon)^2} & \text{for } -L < x < -R, \\ u_R(-L) = g,\ u_R'(-L) = 0, \\ u_R(-R) = 0,\ u_R'(-R) = 0. \end{cases}$$

Now, u_R is a critical point of E if and only if $\frac{dE(u_R)}{dR} = 0$.

Writing $E(u) = \int_\Omega \frac{D}{2}u''^2 + g(u)dx$, where $g(u) = -\frac{V^2}{2}\frac{\epsilon_0}{(u+d/\epsilon)} + k_1 e^{-k_2 u}$, we obtain

$$E(u_R) = 2\int_{-L}^{-R} \frac{D}{2}u_R''^2 dx + 2\int_{-L}^{-R} g(u_R)dx + 2Rg(0).$$

Calculating this derivative with respect to R we infer that

$$\frac{dE(u_R)}{dR} = E_u(u_R)\frac{\partial u_R}{\partial R} - Du_R''^2(-R) - 2g(u_R(-R)) + 2g(0) = -Du_R''^2(-R),$$

since $u_R(-R) = 0$ and $E_u(u_R) = 0$, because u_R is a critical point when keeping R *fixed*. It follows that $u''(-R) = 0$ if u is a critical point of E.

This argument can be extended to higher dimensions quite easily, under the *assumption* that the solution for fixed contact region varies smoothly with the geometry of the contact region. Without this assumption, more complicated arguments are needed.

Remark 1. *We note that if the contact set is a single point, then the second derivative in this point need* not *be zero. Indeed, in one dimension for example, there is a branch of solutions with contact only in the midpoint of the domain (interval) and with varying second derivative.*

3 Let us first give the argument for the specific energy E in the one-dimensional case: consider

$$\frac{D}{2}\int u''^2 - \frac{V^2}{2}\int \frac{\epsilon_0}{(u+d/\epsilon)} + k_1 e^{-k_2 u}.$$

Let us look at stationary points, i.e., solutions of

$$Du'''' = -\frac{V^2}{2}\frac{\epsilon_0}{(u+d/\epsilon)^2} + k_1 k_2 e^{-k_2 u}, \tag{17}$$

which are, on the branch under consideration, parametrized by V. Let us write $\overline{u}$ for the derivative of the solutions u with respect to V along the branch. Taking the derivative of (17) along the branch, we obtain

$$D\overline{u}'''' = V^2\frac{\epsilon_0\overline{u}}{(u+d/\epsilon)^3} - V\frac{\epsilon_0}{(u+d/\epsilon)^2} - k_1 k_2^2 e^{-k_2 u}\overline{u}. \tag{18}$$

The second variation of the energy in the direction $\overline{u}$ gives

$$E''(u)\cdot\overline{u}\cdot\overline{u} = D\int \overline{u}''^2 - V^2\int \frac{\epsilon_0\overline{u}}{(u+d/\epsilon)^3} + k_1 k_2^2\int e^{-k_2 u}\overline{u}^2.$$

After performing partial integration twice on the first term, we can substitute (18) and, with most terms cancelling, we obtain

$$E''(u)\cdot\overline{u}\cdot\overline{u} = -V\int \frac{\epsilon_0\overline{u}}{(u+d/\epsilon)^2}.$$

This simplifies as

$$E''(u)\cdot\overline{u}\cdot\overline{u} = -V\int \frac{\epsilon_0\overline{u}}{(u+d/\epsilon)^2} = VC'(u)\overline{u} = V\frac{dC}{dV}.$$

Hence $\frac{dC}{dV} < 0$ implies that u is unstable.

For the general case, critical points $u = u(V)$ satisfy, subscripts denoting partial derivatives,

$$F_u(u(V)) \cdot w + G_C(V, C(u(V)))\, C_u(u(V)) \cdot w = 0 \qquad \text{for any } w.$$

Taking the derivative with respect to V gives (always evaluating at $u = u(V)$),

$$F_{uu} \cdot w \cdot u_V + G_{CV}\, C_u \cdot w + G_{CC}\, C_u \cdot w\, C_u \cdot u_V + G_C\, C_{uu} \cdot w \cdot u_V = 0 \qquad \text{for any } w. \tag{19}$$

On the other hand, the second variation (for fixed V) gives

$$F''(u) \cdot w \cdot w' = F_{uu} \cdot w \cdot w' + G_{CC}\, C_u \cdot w\, C_u \cdot w' + G_C\, C_{uu} \cdot w \cdot w',$$

hence, using (19), we obtain for $w = w' = u_V$

$$F''(u) \cdot u_V \cdot u_V = -G_{CV}\, C_u \cdot u_V.$$

When we write $\overline{C}(V) = C(u(V))$, this reduces to

$$F''(u) \cdot u_V \cdot u_V = -G_{CV} \frac{d\overline{C}}{dV}.$$

Hence, if $\frac{\partial^2 G}{\partial C \partial V} < 0$ then solutions on branches where $\frac{d\overline{C}}{dV} < 0$ are always unstable.

Remark 2. *One can also consider the problem where we put a charge $Q = VC$ on the switch. In that case the physically relevant energy does* not *include the energy stored in the battery, which is given by $-V^2C$. The energy E_Q thus becomes*

$$E_Q = \frac{D}{2} \int u''^2 + \frac{Q^2}{2C} + \int k_1 e^{-k_2 u},$$

and the arguments above show that solutions on curves with $\frac{dC}{dQ} < 0$ are always unstable.

4 Such a result follows from degree theory, see e.g. [8]. However, it still needs to be checked rigorously that there indeed does exist a unique critical point for very large and very small V. For $V = 0$ this is obvious, the energy being convex in that case, but the situation for large V is less straightforward, since the energy contains both convex and concave parts, although in numerical experiments uniqueness is observed.

4.3 Functional estimates

In this section, two estimates for the first and second variation of the total energy are derived.

The energy functional modeling the deformation of a clamped plate Ω under influence of an electrical field due to a potential difference with a fixed plate reads

$$\begin{aligned} E[u] &= E_{\text{mech}} + E_{\text{el}} \\ &= \int_\Omega - \left[\frac{1}{2} D (\Delta u)^2 - \frac{1}{2} V^2 \frac{\varepsilon_0}{u + g + \frac{d}{\varepsilon_0}} \right] \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{20}$$

where $u \in H_0^2(\Omega)$, implying $u = \frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Equilibria for the system are the zeroes of the first variation,

$$\delta E[\tilde{u}, h] = 0, \quad \forall h \in H_0^2(\Omega).$$

These can be stable and unstable equilibria (e.g. saddle points). A stable equilibrium is a minimum for the functional, such states are characterized by the fact that the second variation at the equilibrium state is strictly positive,

$$\delta^2 E[\tilde{u}, h] > 0, \quad \forall h \in H_0^2(\Omega)$$

We here wish to give a sufficient condition for an equilibrium to be stable.

The first variation is found by putting $u = \tilde{u} + \varepsilon h$, where $h \in H_0^2(\Omega)$ is a test function, and taking the derivative wrt. ε at $\varepsilon = 0$. We then obtain

$$\partial E[\tilde{u}, h] = \int_\Omega -\left[D\Delta^2 \tilde{u} - \frac{1}{2}V^2 \frac{\varepsilon_0}{(u + g + \frac{d}{\varepsilon_0})^2}\right] h \,\mathrm{d}x\mathrm{d}y.$$

The variation lemma yields the boundary value problem for the system from this functional. Let's assume we have a solution for the system, now the question is whether the solution is stable or not. The second variation in the direction $h \in H_0^2(\Omega)$ is found to be

$$\delta^2 E[u, h] = \int_\Omega \left[D(\Delta h)^2 - \frac{1}{2}V^2 \frac{h^2 \varepsilon_0}{(u + g + \frac{d}{\varepsilon_0})^3}\right] \mathrm{d}x\mathrm{d}y. \tag{21}$$

In this form it is difficult to check positivity. However, we can prove a Cauchy-type inequality for the test functions in the space $H_0^2(\Omega)$ when Ω has a simple shape. For the case of a rectangle with sides L_1 and L_2 we have

$$\int_\Omega (\Delta h)^2 \mathrm{d}x\mathrm{d}y \geq \frac{4}{\max[L_1^4, 2L_1^2 L_2^2, L_2^4]} \int_\Omega h^2 \mathrm{d}x\mathrm{d}y$$

Using this inequality together with equation (21) we have the following estimate for the second variation,

$$\delta^2 E[u, h] \geq \int_\Omega \left[\frac{4D}{\max[L_1^4, 2L_1^2 L_2^2, L_2^4]} - V^2 \frac{\varepsilon_0}{(u + g + \frac{d}{\varepsilon_0})^3}\right] h^2 \mathrm{d}x\mathrm{d}y$$

Necessary conditions for the stability of the functional can be obtained from this estimate. For example, take $u^* = \min(u)$, then

$$\delta^2 E[u, h] \geq \left[\frac{4D}{\max[L_1^4, 2L_1^2 L_2^2, L_2^4]} - V^2 \frac{\varepsilon_0}{(u^* + g + \frac{d}{\varepsilon_0})^3}\right] \int_\Omega h^2 \mathrm{d}x\mathrm{d}y,$$

and it is sufficient to check the positivity of the constant.

Appendix

As has been said, the linearized problem can be fully elaborated for three different cases: (i) the case in which the membrane does not touch the dielectric at all (ii) the case in which the membrane touches the dielectric at $x = 0$ only and (iii) the case in which the membrane touches the dielectric on an disc $\bar{B}_a$, for $0 < a < 1$. We will make a few remarks on how to do this, in the case of radially symmetric MEMS switch. We consider case (iii). It will be clear from the result that the typical hysteresical behaviour does not show up in the linearized model.

In order not to bother with too many parameters, one rescales the problem. For example, in the 2-D radially symmetric version of the problem one obtains for the capacitance:

$$C(w) = \frac{2\pi\epsilon_0 \Lambda^2}{g} \int_0^1 \frac{r}{1 + \eta + w(r)} \,dr.$$

and, by computing the Euler-Lagrange equation corresponding to this energy we find

$$\Delta_r^2 w = -\frac{\delta v^2}{(1 + \eta + w)^2}. \tag{22}$$

where $\Delta_r = \frac{1}{r}\frac{d}{dr}$, δ some algebraic expression in terms of the other parameters, v a non-dimensionalized voltage and w a scaled version of the distance u. This problem can subsequently be linearized around $w = 0$:

$$\Delta^2 w = \omega^4 \left(-\frac{1+\eta}{2} + w \right),$$

where $\omega = ^4\sqrt{\frac{2\epsilon v^2}{(1+\eta)^3}}$ is just a scaling. Regarding w as a radially symmetric function depending on r only we get

$$w(r) = AJ_0(\omega r) + BY_0(\omega r) + CI_0(\omega r) + DK_0(\omega r) + \frac{1+\eta}{2}, \tag{23}$$

where J_0 and Y_0 are Bessel functions of the first and second kind respectively and I_0 and K_0 are modified Bessel functions of first and second kind. We add the following boundary conditions:

$$w(1) = w'(1) = w'(a) = w''(a) = 0, \quad w(a) = -1.$$

By rewriting this system as a four-dimensional first-order system, one obtains the constants A, B, C and D.

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