Algebraic modal logic

Summer 2013

Homework 1

(due Friday, 14 June at the beginning of the lecture)

1. Draw the Hasse diagrams of (a) $\mathbb{N} \times \mathbb{N}^{\partial}$ and (b) $\mathbf{2} \times \mathbf{M}_2$, for both product and lexicographic orders. [8 pts]

2. (a) Embed $\mathbf{M}_n (2 \leq n < \infty)$ into a direct product of two chains. [5 pts]
   (b) Express the order on $\mathbf{M}_n$ as the intersection of two totally ordered extensions [1]. [5 pts]

3. (Exercise 1.14, BD & HP) Let P be a finite ordered set.
   (a) Show that $Q = \downarrow \text{Max } Q$, for all $Q \in \mathcal{O}(P)$, where $\downarrow \text{Max } Q$ is the set of maximal elements of Q [5 pts]
   (b) Establish a one-to-one correspondence between elements of $\mathcal{O}(P)$ and antichains in P. [5 pts]

4. Show that if a poset P is finite, then each join-prime element of $\mathcal{O}(P)$ has the form $\downarrow p$ for some $p \in P$. [8 pts]

5. Prove that the inverse of a lattice isomorphism is a lattice isomorphism. [8 pts]

6. The finite-cofinite algebra of a set X is defined to be

$$FC(X) := \{A \subseteq X \mid A \text{ is finite or } X \setminus A \text{ is finite } \}$$

   Show that (i) $FC(X)$ is a boolean algebra (ii) $FC(\mathbb{N})$ is not complete. [6 pts]

7. Recall the definition of a Heyting algebra.

   Definition 0.1. A distributive lattice $(A, \wedge, \vee, \bot, \top)$ is said to be a *Heyting algebra* if for every $a, b \in A$ there exists an element $a \to b$ such that for every $c \in A$ we have:

$$c \leq a \to b \text{ iff } a \wedge c \leq b$$

   Prove that a complete distributive lattice L is a Heyting algebra if and only if it satisfies the infinite distributive law (Hint : Use the defintion of $a \to b$ in a Heyting algebra) [10 pts]

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i)$$

8. (a) Give an example of a Heyting algebra which is not a Boolean algebra. [4 pts]
   (b) Give a counterexample to the Birkhoff's representation theorem ($L \cong \mathcal{O}(J(L))$) if the distributive lattice L is infinite. [4 pts]

9. Exercise 5.1.4 from Modal logic by Blackburn, de Rijke and Venema [12 pts]

[1] A totally ordered extension of a partial order R is defined as an order S which is (i) a total order (ii) if aRb then aSb.