

# Extending Time-consistent Convex Risk Measures from Discrete to Continuous Time: a Convergence Approach

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9th Winter School on Mathematical Finance

January 19th, 2010

- How to measure risk?
  - Convex risk measures in a one-period setting (static)
  - Dynamic convex risk measures
    - in a multiperiod setting
    - in a continuous time setting
- Convex risk measures in continuous time may be interpreted as limits of certain compound one-period convex risk measures.
- Extending discrete-time risk measures like AV@R, the semi-deviation and the Gini risk measure to continuous time

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# What is risk?

Let  $\Omega$  be a fixed set of scenarios. A financial position is described by a mapping  $X : \Omega \rightarrow \mathbb{R}$  where  $X(\omega)$  is the discounted net worth of the position at the end of the trading period if the scenario  $\omega \in \Omega$  is realized. What is the risk of  $X$ ?

One notion of risk used by banks and insurances is *Value at Risk* ( $V@R$ ). For the cdf  $F_X$  of the random variable  $X$  define

$$q_X^+(s) = \inf\{x \in \mathbb{R} | F_X(x) > s\}.$$

Then for fixed  $\alpha \in (0, 1]$  the Value at Risk of  $X$  to a level  $\alpha$  is defined by

$$\text{Risk}(X) = V@R_\alpha(X) = -q_X^+(\alpha) = \inf\{m \in \mathbb{R} | P(X+m < 0) \leq \alpha\}.$$

In other words  $V@R_\alpha(X)$  is the minimal amount of money I have to add to my position  $X$  such that with a probability greater than  $1 - \alpha$  I will not encounter any losses.

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# Axiomatic approach: static (one-period) convex risk measure (Artzner, Delbaen, Heath (1999), Föllmer, Schied/Frittelli, Rosazza Gianin (2002))

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and suppose that the set of all possible payoffs is given by  $L^\infty(\Omega, \mathcal{F}, P)$ . A mapping  $\rho : L^\infty(\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ , is a *convex risk measure* if it has the following properties:

- *Normalization*:  $\rho(0) = 0$
- *Translation Invariance*:  $\rho(X + m) = \rho(X) - m$  for all  $m \in \mathbb{R}$
- *Monotonicity*: If  $X \leq Y$  a.s., then  $\rho(X) \geq \rho(Y)$
- *Convexity*:  $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$  for  $0 \leq \lambda \leq 1$
- *Lower-Semicontinuity*: If  $(X_n)$  is a bounded sequence which converges to  $X$  a.s. then

$$\rho(X) \leq \liminf_n \rho(X_n).$$

# Examples

Typical examples for one-period risk measures are

- Average Value at Risk:

$$AV@R^\alpha(X) = \frac{1}{\alpha} \int_0^\alpha V@R^\lambda(X) d\lambda, \quad \alpha \in (0, 1].$$

If the distribution of  $X$  is continuous

$$AV@R^\alpha(X) = \mathbb{E}[-X | X \leq q_X^+(\alpha)].$$

- Semi-deviation risk measure:

$$S_{t_i}^{\lambda, p}(X) = \mathbb{E}[-X] + \lambda \| (X - \mathbb{E}[X])_- \|_p, \quad \lambda \in [0, 1], \quad p \in [1, \infty).$$

- Gini risk measure:

$$V^\theta(X) = \sup_{Q \ll P} \left\{ E_Q[-X] - \frac{1}{2\theta} C(Q|P) \right\}, \quad \theta > 0$$

where

$$C(Q|P) = \mathbb{E} \left[ \left( \frac{dQ}{dP} - 1 \right)^2 \right].$$

Given  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ . Let  $I$  be the set of time instances in which the agent is allowed to update his risk. We call a family of mappings  $\rho_{s,t} : L^\infty(\mathcal{F}_t) \rightarrow L^\infty(\mathcal{F}_s)$ ,  $s, t \in I$  and  $s \leq t$ , a *dynamic risk measure* if it has the following properties for  $X, Y \in L^\infty(\mathcal{F}_t)$ :

- *Normalization*:  $\rho_{s,t}(0) = 0$
- *Monotonicity*: If  $X \leq Y$ , then  $\rho_{s,t}(X) \geq \rho_{s,t}(Y)$  a.s.
- *$\mathcal{F}_s$ -Translation Invariance*:  $\rho_{s,t}(X + m) = \rho_{s,t}(X) - m$  for all  $m \in L^\infty(\mathcal{F}_s)$
- *$\mathcal{F}_s$ -Convexity*:  
 $\rho_{s,t}(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_{s,t}(X) + (1 - \lambda)\rho_{s,t}(Y)$  for all  $\lambda \in L^\infty(\mathcal{F}_s)$  such that  $0 \leq \lambda \leq 1$
- *$\mathcal{F}_s$ -Lower-Semicontinuity*: For any  $\mathcal{F}_t$ -adapted bounded sequences  $X_n$  converging a.s. to  $X$  we have  
 $\rho_{s,t}(X) \leq \liminf_n \rho_{s,t}(X_n)$  a.s.

If  $T$  is the time horizon of our model,  $\rho_{s,T}$  is often denoted by  $\rho_s$ .

- *Time-consistency*: for  $X, Y \in L^\infty(\mathcal{F}_t)$   $\rho_{s',t}(X) \leq \rho_{s',t}(Y)$  a.s. for some  $s'$  with  $t \geq s' \geq s$ , implies  $\rho_{s,t}(X) \leq \rho_{s,t}(Y)$  a.s.

see for instance Delbaen (2003) or Barrieu and El Karoui (2005).

Using time-consistency you can show that for every bounded  $\mathcal{F}_t$ -measurable  $X$

$$\rho_{s,t}(X) = \rho_{s,T}(X) = \rho_s(X),$$

i.e.,  $\rho_{s,t} = \rho_s|_{L^\infty(\mathcal{F}_t)}$  and thus the whole family  $(\rho_{s,t})_{s,t \in I}$  is uniquely determined by  $(\rho_s)_{s \in I}$ .

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# Duality in a discrete setting

Suppose that we are in a multiperiod discrete setting, i.e.,  $I = \{t_0, t_1, \dots, t_k\}$  where  $0 = t_0 < t_1 < \dots < t_k = T$ . For  $i = 0, \dots, k - 1$  define the set of one-step transition densities

$$D_{t_i} = \{\xi_{t_{i+1}} \in L^1_+(\mathcal{F}_{t_{i+1}}) \mid \mathbb{E}[\xi_{t_{i+1}} | \mathcal{F}_{t_i}] = 1 \text{ a.s.}\}.$$

We identify a probability measure  $Q$  with its density  $\xi_{t_{i+1}} \in D_{t_i}$ . Suppose that  $\rho_{t_i, t_{i+1}} : L^\infty(\mathcal{F}_{t_{i+1}}) \rightarrow L^\infty(\mathcal{F}_{t_i})$  is a **one-period risk measure**.

Define the *penalty function* on  $D_{t_i}$  of a one-period risk measure  $\rho_{t_i, t_{i+1}}$  as

$$\phi_{t_i}^{\rho_{t_i, t_{i+1}}}(Q) = \text{ess sup}_{X \in L^\infty(\mathcal{F}_{t_{i+1}})} \{\mathbb{E}_Q[-X | \mathcal{F}_{t_i}] - \rho_{t_i, t_{i+1}}(X)\}.$$

Every sequence  $\xi \in D_{t_0} \times D_{t_1} \times \dots \times D_{t_{k-1}}$  induces a  $P$ -martingale

$$M_{t_r}^\xi = \begin{cases} \prod_{j=i+1}^r \xi_{t_j} & \text{if } r \geq i+1 \\ 1 & \text{if } r \leq i \end{cases}$$

and a probability measure  $Q^\xi$  by  $\frac{dQ^\xi}{dP} = M_T^\xi$ . Set

$D = D_{t_0} \times D_{t_1} \times \dots \times D_{t_{k-1}}$ . For  $\xi \in D$  define

$\phi_{t_i}^{\rho_{t_i, t_{i+1}}}(Q^\xi) = \phi_{t_i}^{\rho_{t_i, t_{i+1}}}(\xi_{t_{i+1}})$ . Then from Cheridito and Kupper (2006), we obtain the following representation.

### Proposition

*Suppose that  $(\rho_s)_{s \in I}$  is a discrete-time risk measure. Then*

$$\rho_{t_i}(X) = \text{ess sup}_{Q \in D} \mathbb{E}_Q \left[ -X - \sum_{j=i}^{k-1} \phi_{t_j}^{\rho_{t_j, t_{j+1}}}(Q) \middle| \mathcal{F}_{t_i} \right].$$

# Dynamic risk measures in continuous time

Now suppose that  $I = [0, T]$ , i.e., the risk manager is allowed to update his information at *any* time. Risk Modelling can then be done using Backward Stochastic Differential Equations (Barrieu and El Karoui (2005)).

BSDE = backward stochastic differential equation

## Definition of a BSDE

Assume that we have a  $d$ -dimensional Brownian Motion  $(W_t^1, \dots, W_t^d)$  on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$  where  $(\mathcal{F}_t)$  is the standard filtration. Let  $g : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a function such that

- $z \mapsto g(t, \omega, z)$  is convex for every fixed  $(t, \omega) \in [0, T] \times \Omega$
- for every fixed  $z \in \mathbb{R}^d$ ,  $(t, \omega) \rightarrow g(t, \omega, z)$  is progressively measurable
- there exists a  $K > 0$  with  $|g(t, \omega, z)| \leq K(1 + |z|^2)$  a.s.
- ...

The solution of a BSDE with driver  $g(t, \omega, z)$  and terminal condition  $X \in L^\infty(\mathcal{F}_T)$  is a pair of (suitably integrable) progressively measurable processes  $(Y_t, Z_t)$  with values in  $\mathbb{R} \times \mathbb{R}^d$ , which satisfy

$$Y_t = X + \int_t^T g(s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

Let  $Y^g(-X)$  be the solution  $(Y_t)$  of the BSDE with driver  $g$  and terminal condition  $-X$ . Define

$$\rho_t^g(X) = Y_t^g(-X).$$

Then  $\rho^g$  is a dynamic risk measure!

# Framework: Risk measures in discrete time

**Setting:** For fixed  $N$  let  $B_{N,l}^j$  be independent Bernoulli random variable with

$$P[B_{N,l}^j = 1] = P[B_{N,l}^j = -1] = \frac{1}{2}; \quad j = 1, \dots, d, \quad N \in \mathbb{N}, \\ l = 1, \dots, N. \quad \text{Let}$$

$$R^{N,j}(t_i) = \sqrt{\frac{T}{N}} \sum_{l=1}^i B_{N,l}^j, \quad t_i = iT/N, \quad i = 1, \dots, N, \quad j = 1, \dots, d$$

and constant on the intervals  $[t_i, t_{i+1})$ . Let

$$R^N(t_i) = (R^{N,1}(t_i), \dots, R^{N,d}(t_i)).$$

Denote by  $\mathcal{F}^N = (\mathcal{F}_t^N)_{0 \leq t \leq T}$  the filtration generated by the random walk.

Assume that there exists a standard Brownian motion  $W_t$  such that

$$\sup_{0 \leq s \leq T} |R^N(s) - W_s| \rightarrow 0 \text{ in } L^2.$$

# A robust way of measuring risk in discrete time

## Definition

For a collection of one-period risk measures  $(F_{t_i}^N)_{i=0,\dots,k-1}$  with penalty functions  $(\phi_{t_i}^{F_{t_i}^N})_{i=0,\dots,k-1}$  we define its (tilted) robust extension as

$$\rho_{t_i}^N(X) = \sup_{\mu^N} \hat{\mathbb{E}}^{\mu^N} \left[ -X - \sum_{j=i}^{N-1} \phi_{t_j}^{F_{t_j}^N} \left( 1 + \mu_{t_i}^N B_{i+1}^N \right) \Delta t_{j+1} \mid \mathcal{F}_{t_i} \right]$$

where for every bounded,  $\mathcal{F}^N$ -adapted process  $\mu^N$ ,  $\hat{\mathbb{P}}^{\mu^N}$  is the measure under which  $R_t^N - \sum_{t_j \leq t} \mu_{t_j}^N \Delta t_{j+1}$  is a martingale.

**Goal:** Start with one-period risk measures  $F_{t_i}^N$  like AV@R, semi-deviation etc.

→ Define the robust extension in discrete time.

→ Extend it to continuous time by convergence.

## Proposition

Suppose we are given a collection of one-period risk measures  $(F_{t_i}^N)_{i=0, \dots, k-1}$ . For  $z \in \mathbb{R}^d$ ,  $M \in \mathbb{R}$  let

$$g^N(t_i, z, M) = F_{t_i}^N(-zB_{t_{i+1}}^N - M).$$

Then for every  $X^N \in L^\infty(\mathcal{F}_T^N)$  there exists a process  $Z^N$  and a martingale  $M^N$  orthogonal to  $R^N$  such that

$$\begin{aligned} \rho_{t_i}^N(X^N) &= -X^N + \sum_{t_i \leq t_j < T} g^N(t_j, Z_{t_j}^N, M_{t_{j+1}}^N)(t_{j+1} - t_j) \\ &\quad - \sum_{t_i \leq t_j < T} Z_{t_j}^N(R_{t_{j+1}}^N - R_{t_j}^N) - (M_T^N - M_{t_i}^N). \end{aligned}$$

# Convergence Theorem for discrete-time risk measures to continuous-time risk measures

## Theorem

Let  $g$  be a driver function such that for every  $z \in \mathbb{R}^d$  :

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |g^N(t, z, 0) - g(t, z)|^2 \right] \xrightarrow{N \rightarrow \infty} 0.$$

Then there exists a continuous-time dynamic risk measure  $(\rho_s)_{s \in [0, T]}$  such that for **every** sequence of discrete payoffs  $X^N$  which converge in  $L^2$  to a continuous-time payoff  $X$  we have

$$\sup_t |\rho_t^N(X^N) - \rho_t(X)| \xrightarrow{N \rightarrow \infty} 0 \quad \text{in } L^2$$

Moreover,  $\rho$  is the solution of a continuous-time BSDE with terminal condition  $-X$  and driver  $g$ .

# Examples of discrete-time risk measures extended to continuous time: semi-deviation

Suppose that the one-period risk measures are given by the **semi-deviation**:

The robust extension of semi-deviation converges to  $\rho_t$ , where for any terminal condition  $X$ ,  $(\rho_t(X), Z_t)$  is the solution of

$$\rho_t(X) = -X + \int_t^T g(Z_s) ds - \int_t^T Z_s dW_s$$

with  $g(z)$  defined by

$$g(z) = \lambda \left( \frac{1}{2^d} \left( \sum_{l=1, \dots, d, k_l \in \{1, 2\}} (-1)^{k_l} z^l \right)_-^p \right)^{1/p}, \quad z = (z^1, \dots, z^d) \in \mathbb{R}^d.$$

# Examples of discrete-time risk measures extended to continuous time: Average Value at Risk

Suppose that the one-period risk measures are given by  
**Average Value at Risk:**

Let

$x_i(z)$  =  $i$ -th largest element of the set

$$\{(-1)^{k_1} z^1 + \dots + (-1)^{k_d} z^d \mid k_l \in \{1, 2\}, l = 1, \dots, d\}.$$

The robust extension of AV@R converges to  $\rho_t$ , where  $(\rho_t(X), Z_t)$  is the solution of the BSDE with terminal condition  $-X$  and driver

$$g(z) = -\frac{1}{\alpha} \left( x_{2^d - \lceil 2^d \alpha \rceil + 1} \left( \alpha - \frac{\lceil 2^d \alpha \rceil - 1}{2^d} \right) + \frac{1}{2^d} \sum_{j=1}^{\lceil 2^d \alpha \rceil - 1} x_{2^d - j + 1}(z) \right).$$

In particular, if  $\alpha < 1/2^d$  we have

$$g(z) = |z_1| + |z_2| + \dots + |z_d|.$$

# Examples of discrete risk measures extended to continuous time

Suppose that the one-period risk measures are given by the **Gini risk measure**: Define

$$I(z) = \sup \left\{ I \in \{2^d, \dots, 1\} \mid \text{for all } j \in \{2^d, \dots, I\} : \frac{1}{\theta(2^d + 1 - j)} > -\frac{\sum_{i=I}^{2^d} x_i(z)}{2^d + 1 - I} + x_j(z) \text{ and } \frac{1}{\theta(2^d + 1 - I)} \leq -\frac{\sum_{i=I}^{2^d} x_i(z)}{2^d + 1 - I} + x_{I-1}(z) \right\}.$$

The driver for the BSDE of the extension of the Gini risk measure to continuous time is given by

$$g(z) = -\frac{1}{2\theta(2^d + 1 - l(z))} + \frac{1}{2\theta} - \frac{\sum_{j=l(z)}^{2^d} x_j(z)}{2^d + 1 - l(z)} - \frac{\theta \left( \sum_{j=l(z)}^{2^d} x_j(z) \right)^2}{2(2^d + 1 - l(z))} + \frac{\theta}{2} \sum_{j=l(z)}^{2^d} x_j^2(z)$$

for  $z = (z^1, \dots, z^d) \in \mathbb{R}^d$ . In the special case that  $d = 1$  we get

$$g(z) = \begin{cases} |z| - \frac{1}{2\theta}, & \text{if } |z| \geq 1/\theta \\ \frac{\theta}{2} z^2, & \text{if } |z| < 1/\theta. \end{cases}$$