# A Geometric View of Interest Rate Theory

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**Definitions:**

$\mathbf{p}(t,x)$ : Price, at $t$ of zero coupon bond maturing at $t+x$,

$\mathbf{r}(t,x)$ : Forward rate, contracted at $t$, maturing at $t+x$

$R(t)$ : Short rate.

$$\begin{aligned} r(t,x) &= -\frac{\partial \log p(t,x)}{\partial x} \\ p(t,x) &= e^{-\int_0^x r(t,s)ds} \\ R(t) &= r(t,0). \end{aligned}$$

## Heath-Jarrow-Morton-Musiela

**Idea:** Model the dynamics for the **entire forward rate curve**.

The yield curve itself (rather than the short rate $R$) is the explanatory variable.

Model forward rates. Use observed forward rate curve as initial condition.

$Q$-dynamics:

$$\begin{aligned} dr(t,x) &= \alpha(t,x)dt + \sigma(t,x)dW(t), \\ r(0,x) &= r_0^\star(x), \quad \forall x \end{aligned}$$

$W$: $d$-dimensional Wiener process

One SDE for every fixed $x$.

**Theorem:** (HJMM drift Condition) The following relations must hold, under a martingale measure $Q$.

$$\alpha(t,x) = \frac{\partial}{\partial x} r(t,x) + \sigma(t,x)\int_0^x \sigma(t,s)ds.$$

**Moral:** Volatility can be specified freely. The forward rate drift term is then uniquely determined.

# The Interest Rate Model

$$r_t = r_t(\cdot), \quad \sigma(t,x) = \sigma(r_t,x)$$

**Heath-Jarrow-Morton-Musiela equation:**

$$dr_t = \mu_0(r_t)dt + \sigma(r_t)dW_t$$

$$\begin{aligned}\mu_0 &= \frac{\partial}{\partial x} r_t(x) + \sigma(r_t,x)\int_0^x \sigma(r_t,s)ds \\ \sigma &= \sigma(r_t,x)\end{aligned}$$

The HJMM equation is an **infinite dimensional SDE** evolving in the space $\mathcal{H}$ of forward rate curves.

# A Hilbert Space

**Definition:**
For each $(\alpha,\beta) \in R^2$, the space $\mathcal{H}_{\alpha,\beta}$ is defined by

$$\mathcal{H}_{\alpha,\beta} = \{f \in C^\infty[0,\infty);\ \|f\| < \infty\}$$

where

$$\|f\|^2 = \sum_{n=0}^{\infty} \beta^{-n} \int_0^\infty \left[f^{(n)}(x)\right]^2 e^{-\alpha x} dx$$

where

$$f^{(n)}(x) = \frac{d^n f}{dt^n}(x).$$

We equip $\mathcal{H}$ with the inner product

$$(f,g) = \sum_{n=0}^{\infty} \beta^{-n} \int_0^\infty f^{(n)}(x) g^{(n)}(x) e^{-\alpha x} dx$$

# Properties of $\mathcal{H}$

**Proposition:**
The following hold.

- The linear operator
  $$\mathbf{F} = \frac{\partial}{\partial x}$$
  is bounded on $\mathcal{H}$
- $\mathcal{H}$ is complete, i.e. it is a Hilbert space.
- The elements in $\mathcal{H}$ are real analytic functions on $R$ (not only on $R_+$).

**NB:** Filipovic and Teichmann!

# Stratonovich Form of HJMM

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

where

$$\mu(r_t) = \mu_0(r_t) - \frac{1}{2}\frac{d\langle \sigma, W\rangle}{dt}$$

**Main Point:**
Using the Stratonovich differential we have no Itô second order term. Thus we can treat the SDE above as the ODE

$$\frac{dr_t}{dt} = \mu(r_t) + \sigma(r_t) \cdot v_t$$

where $v_t =$ “white noise”.

## Natural Questions

- What do the forward rate curves look like?
- What is the support set of the HJMM equation?
- When is a given model (e.g. Hull-White) consistent with a given family (e.g. Nelson-Siegel) of forward rate curves?
- When is the short rate Markov?
- When is a finite set of benchmark forward rates Markov?
- When does the interest rate model admit a realization in terms of a finite dimensional factor model?
- If there exists an FDR how can you construct a concrete realization?

## Finite Dimensional Realizations

**Main Problem:**
When does a given interest rate model possess a finite dimensional realisation, i.e. when can we write $r$ as

$$\begin{aligned} z_t &= \eta(z_t)dt + \delta(z_t) \circ dW(t), \\ r(t,x) &= G(z_t,x), \end{aligned}$$

where $z$ is a **finite-dimensional** diffusion, and

$$G : R^d \times R_+ \to R$$

or alternatively

$$G : R^d \to \mathcal{H}$$

$\mathcal{H} =$ the space of forward rate curves

**Examples:**

$$\begin{aligned} \sigma(r,x) &= e^{-ax}, \\ \sigma(r,x) &= xe^{-ax}, \\ \sigma(r,x) &= e^{-x^2}, \\ \sigma(r,x) &= \log\left(\frac{1}{1+x^2}\right), \\ \sigma(r,x) &= \int_0^\infty e^{-s} r(s)ds \cdot x^2 e^{-ax}. \end{aligned}$$

Which of these admit a finite dimensional realisation?

## Invariant Manifolds

**Def:**
Consider an interest rate model

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t) \circ dW_t$$

on the space $\mathcal{H}$ of forward rate curves. A manifold (surface) $\mathcal{G} \subseteq \mathcal{H}$ is an **invariant manifold** if

$$r_0 \in \mathcal{G} \Rightarrow r_t \in \mathcal{G}$$

$P$-a.s. for all $t > 0$

# Main Insight

There exists a finite dimensional realization.

**iff**

There exists a finite dimensional invariant manifold.

# Characterizing Invariant Manifolds

**Proposition:** (Björk-Christensen)
Consider an interest rate model on Stratonovich form

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

A manifold $\mathcal{G}$ is invariant under $r$ if and only if

$$\begin{aligned} \mu(r) &\in T_{\mathcal{G}}(r), \\ \sigma(r) &\in T_{\mathcal{G}}(r), \end{aligned}$$

at all points of $\mathcal{G}$. Here $T_{\mathcal{G}}(r)$ is the tangent space of $\mathcal{G}$ at the point $r \in \mathcal{G}$.

# Main Problem

**Given:**

- An interest rate model on Stratonovich form

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t) \circ dW_t$$

- An inital forward rate curve $r_0$:

$$x \longmapsto r_0(x)$$

**Question:**
When does there exist a finite dimensional manifold $\mathcal{G}$, such that

$$r_0 \in \mathcal{G}$$

and

$$\begin{aligned} \mu(r) &\in T_{\mathcal{G}}(r), \\ \sigma(r) &\in T_{\mathcal{G}}(r), \end{aligned}$$

A manifold satisfying these conditions is called a **tangential manifold**.

## Lie Brackets

**Definition:**
Given two vector fields $f_1(r)$ and $f_2(r)$, their **Lie bracket** $[f_1, f_2]$ is a vector field defined by

$$[f_1, f_2] = (Df_2)f_1 - (Df_1)f_2$$

where $D$ is the Frechet derivative.

**Fact:**
If $\mathcal{G}$ is tangential to $f_1$ and $f_2$, then it is also tangential to $[f_1, f_2]$.

**Definition:**
Given vector fields $f_1(r), \ldots, f_n(r)$, the **Lie algebra**

$$\{f_1(r), \ldots, f_n(r)\}_{LA}$$

is the smallest linear space of vector fields, containing $f_1(r), \ldots, f_n(r)$, which is closed under the Lie bracket.

## Main result

- Given any fixed initial forward rate curve $r_0$, there exists a finite dimensional invariant manifold $\mathcal{G}$ with $r_0 \in \mathcal{G}$ if and only if the Lie-algebra

  $$\mathcal{L} = \{\mu, \sigma\}_{LA}$$

  is finite dimensional.

- Given any fixed initial forward rate curve $r_0$, there exists a finite dimensional realization if and only if the Lie-algebra

  $$\mathcal{L} = \{\mu, \sigma\}_{LA}$$

  is finite dimensional. The dimension of the realization equals $dim\,\{\mu, \sigma\}_{LA}$.

## Deterministic Volatility

$$\sigma(t, r, x) = \sigma(x)$$

Consider a **deterministic** volatility function $\sigma(x)$. Then the Ito and Stratonovich formulations are the same:

$$dr = \{\mathbf{F}r + S\}\,dt + \sigma dW$$

where

$$\mathbf{F} = \frac{\partial}{\partial x}, \quad S(x) = \sigma(x)\int_0^x \sigma(s)ds.$$

The Lie algebra $\mathcal{L}$ is generated by the two vector fields

$$\mu(r) = \mathbf{F}r + S, \quad \sigma(r) = \sigma$$

**Proposition:**
For an FDR to exist $\sigma$ has to be "quasi exponential", i.e. of the form

$$\sigma(x) = \sum_{i=1}^{n} p_i(x)e^{\alpha_i x}$$

where $p_i$ is a polynomial.

## Constant Direction Volatility

$$\sigma(t, r, x) = \varphi(r)\lambda(x)$$

**Theorem**
The model admits a finite dimensional realization if and only if $\lambda$ is quasi-exponential. The scalar field $\varphi(r)$ can be arbitrary.

## Short Rate Realizations

**Question:**
When is a given forward rate model realized by a short rate model?

$$\begin{aligned} r(t,x) &= G(t,R_t,x) \\ dR_t &= a(t,R_t)dt + b(t,R_t) \circ dW \end{aligned}$$

**Answer:**
There must exist a 2-dimensional realization. (With the short rate $R$ and running time $t$ as states).

**Proposition:** The model is a short rate model only if

$$dim\,\{\mu,\sigma\}_{LA} \leq 2$$

**Theorem:** The model is a generic short rate model if and only if

$$[\mu,\sigma]\,//\sigma$$

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## “All short rate models are affine“

**Theorem:** (Jeffrey) Assume that the forward rate volatitliy is of the form

$$\sigma(R_t,x)$$

Then the model is a generic short rate model if and only if $\sigma$ is of the form

$$\begin{aligned} \sigma(R,x) &= c && \text{(Ho-Lee)} \\ \sigma(R,x) &= ce^{-ax} && \text{(Hull-White)} \\ \sigma(R,x) &= \lambda(x)\sqrt{aR+b} && \text{(CIR)} \end{aligned}$$

($\lambda$ solves a certain Ricatti equation)

**Slogan:**
Ho-Lee, Hull-White and CIR are the **only generic** short rate models.

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## Constructing an FDR

**Problem:**
Suppose that there actually **exists** an FDR, i.e. that

$$dim\,\{\mu,\sigma\}_{LA} < \infty.$$

How do you **construct** a realization?

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## Constructing the invariant manifold

**Proposition:**
Suppoose that the Lie algebra $\{\mu,\sigma\}_{LA}$ is spanned by the vector fields $f_1,\dots,f_n$. Fix a point $r_0 \in X$. Then the induced invariant manifold is parametrized by

$$G : R^n \rightarrow \mathcal{G}$$

where

$$G(t_1,\dots,t_n) = e^{f_n t_n} \dots e^{f_2 t_2} e^{f_1 t_1} r_0$$

Here the operator $e^{f_i t}$ is defined as the flow mapping of the ODE

$$\frac{dr_t}{dt} = f_i(r_t)$$

## Constructing a Realization:

- Choose a finite number of vector fields $f_1, \ldots, f_d$ which span $\{\mu, \sigma\}_{LA}$.
- Compute the invariant manifold $G(z_1, \ldots, z_d)$ using
$$G(t_1, \ldots, t_n) = e^{f_n t_n} \ldots e^{f_2 t_2} e^{f_1 t_1} r_0$$
- Make the *Ansatz*
$$dZ_t = a(Z_t)dt + b(Z_t) \circ dW_t.$$
- From $r_t = G(Z_t)$ it follows that
$$G_\star a = \mu, \qquad G_\star b = \sigma.$$
- Use these (linear!) equations to solve for the vector fields $a$ and $b$.

## Example: Deterministic Direction Volatility

**Model:**

$$\sigma_i(r, x) = \varphi(r)\lambda(x).$$

**Minimal Realization:**

$$\begin{cases} dZ_0 &= dt, \\ dZ_0^1 &= [c_0 Z_n^1 + \gamma\varphi^2(G(Z))]dt + \varphi(G(Z))dW_t, \\ dZ_i^1 &= (c_i Z_n^1 + Z_{i-1}^1)dt, \qquad i = 1, \ldots, n, \\ dZ_0^2 &= [d_0 Z_q^2 + \varphi^2(G(Z))]dt, \\ dZ_j^2 &= (d_j Z_q^2 + Z_{j-1}^2)dt, \qquad j = 1, \ldots, q. \end{cases}$$

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