The Radon-Nikodym theorem
for absolutely continuous probability measures

Suppose that one has a random variable $Z \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{E}\, Z = 1$. Define then for $F \in \mathcal{F}$

$$\mathbb{Q}(F) = \mathbb{E}\, \mathbf{1}_F Z.$$

One sees that $\mathbb{Q}$ is a measure on $(\Omega, \mathcal{F})$ and that $\mathbb{P}(\Omega) = 1$, hence $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$. If F is any event such that $\mathbb{P}(F) = 0$, then also $\mathbb{Q}(F) = 0$ as $\mathbf{1}_F = 0$ $\mathbb{P}$-a.s. One says that $\mathbb{Q}$ is *absolutely continuous w.r.t.* $\mathbb{P}$ and this is denoted $\mathbb{Q} \ll \mathbb{P}$. Theorem 2 below states that the above procedure is the only way to get absolutely continuous probability measures.

As a preparation for its proof we recall the Riesz-Frechét theorem. If $\mathcal{H}$ is a Hilbert space and T a continuous linear functional on it, then there exists a unique $y \in \mathcal{H}$ such that $Tx = \langle x, y \rangle$. We consider a special case of this theorem. The setting is that of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{H} = \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, where the inner product is given by $\langle X, Y \rangle = \mathbb{E}\, XY$. Strictly speaking, this $\mathcal{H}$ is not a normed space as $\mathbb{E}\, X^2 = 0$ only implies $X = 0$ a.s. Likewise we have completeness in the sense that a Cauchy sequence $(X_n) \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ has a limit which is unique almost surely only. The Riesz-Frechét theorem takes the following form

Lemma 1 *Let $T : \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a continuous linear functional. Then there exists an a.s. unique $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ such that $TX = \mathbb{E}\, XY$.*

Theorem 2 (Radon-Nikodym) *Consider a measurable space $(\Omega, \mathcal{F})$ on which are defined two probability measures $\mathbb{Q}$ and $\mathbb{P}$. Assume $\mathbb{Q} \ll \mathbb{P}$. Then there exists a $\mathbb{P}$-a.s. unique random variable Z with $\mathbb{P}(Z \geq 0) = 1$ and $\mathbb{E}\, Z = 1$ such that for all $F \in \mathcal{F}$ one has*

$$\mathbb{Q}(F) = \mathbb{E}\, \mathbf{1}_F Z. \tag{1}$$

Proof Let $\hat{\mathbb{P}} = \frac{1}{2}(\mathbb{P} + \mathbb{Q})$ and consider $T : \mathcal{L}^2(\Omega, \mathcal{F}, \hat{\mathbb{P}}) \to \mathbb{R}$ defined by $TX = \mathbb{E}_{\mathbb{Q}}\, X$. Then

$$\begin{aligned} |TX| &\leq \mathbb{E}_{\mathbb{Q}}\, |X| \\ &\leq 2\mathbb{E}_{\hat{\mathbb{P}}}\, |X| \\ &\leq 2(\mathbb{E}_{\hat{\mathbb{P}}}\, X^2)^{1/2}. \end{aligned}$$

Hence T is a continuous linear functional on $\mathcal{L}^2(\Omega, \mathcal{F}, \hat{\mathbb{P}})$, and by the *Riesz-Fréchet theorem* there exist $Y \in \mathcal{L}^2(\Omega, \mathcal{F}, \hat{\mathbb{P}})$ such that $\mathbb{E}_{\mathbb{Q}}\, X = 2\,\mathbb{E}_{\hat{\mathbb{P}}}\, XY = \mathbb{E}_{\mathbb{Q}}\, XY + \mathbb{E}\, XY$. Hence we have

$$\mathbb{E}_{\mathbb{Q}}\, X(1 - Y) = \mathbb{E}\, XY, \tag{2}$$

for all $X \in \mathcal{L}^2(\Omega, \mathcal{F}, \hat{\mathbb{P}})$. To find a convenient property of Y, we make two judicious choices for X in (2).

First we take $X = \mathbf{1}_{\{Y \geq 1\}}$. We then get

$$0 \geq \mathbb{E}_{\mathbb{Q}} \mathbf{1}_{\{Y\geq 1\}}(1 - Y) = \mathbb{E}\,\mathbf{1}_{\{Y\geq 1\}}Y \geq \mathbb{P}(Y \geq 1),$$

and hence $\mathbb{P}(Y \geq 1) = 0$. By absolute continuity also $\mathbb{Q}(Y \geq 1) = 0$.

Second, we choose $X = \mathbf{1}_{\{Y<0\}}$ and get, note that $X(1 - Y) \geq 0$,

$$0 \leq \mathbb{E}_{\mathbb{Q}} \mathbf{1}_{\{Y<0\}}(1 - Y) = \mathbb{E}\,\mathbf{1}_{\{Y<0\}}Y \leq 0,$$

and hence $\mathbb{P}(Y < 0) = 0$ and $\mathbb{Q}(Y < 0) = 0$. We conclude that $\mathbb{Q}(Y \in [0, 1)) = \mathbb{P}(Y \in [0, 1)) = 1$.

We now claim that the random variable Z in the assertion of the theorem is given by

$$Z = \frac{Y}{1 - Y}\mathbf{1}_{[0,1)}(Y).$$

Note that Z takes its values in $[0, \infty)$. Write $Z = \lim_n Z_n$ with $Z_n = YS_n$, where $S_n = \mathbf{1}_{[0,1)}(Y)\sum_{k=0}^{n-1} Y^k$ and note that $0 \leq S_n, Z_n \leq n$. Let $F \in \mathcal{F}$. We apply (2) with $X = \mathbf{1}_F S_n$ to obtain, using $(1 - Y)S_n = \mathbf{1}_{[0,1)}(Y)(1 - Y^n)$,

$$\mathbb{E}_{\mathbb{Q}} \mathbf{1}_F\mathbf{1}_{[0,1)}(Y)(1 - Y^n) = \mathbb{E}\,\mathbf{1}_F Z_n.$$

Monotone convergence gives

$$\mathbb{E}_{\mathbb{Q}} \mathbf{1}_F\mathbf{1}_{[0,1)}(Y) = \mathbb{E}\,\mathbf{1}_F Z. \tag{3}$$

As $\mathbb{Q}(Y \in [0, 1)) = 1$ one has $\mathbb{Q}(F) = \mathbb{Q}(F \cap \{Y \in [0, 1)\}) = \mathbb{E}_{\mathbb{Q}} \mathbf{1}_F\mathbf{1}_{[0,1)}(Y)$, and (1) follows from (3). Moreover, as $\mathbb{Q}$ is a probability measure, we get with $F = \Omega$ that $\mathbb{E}\,Z = 1$.

The final issue to address is the $\mathbb{P}$-a.s. uniqueness of Z. Suppose Z' is another random variable satisfying the assertion of the theorem. Then (1) is also valid for any $F \in \mathcal{F}$ with Z replaced with Z'. Take $F = \{Z > Z'\}$. By subtraction one obtains $\mathbb{E}\,\mathbf{1}_{\{Z>Z'\}}(Z - Z') = 0$ and hence $\mathbb{P}(Z \leq Z') = 1$. By swapping the roles of Z and Z', one concludes $\mathbb{P}(Z = Z') = 1$, which finishes the proof. □

We conclude with some notation. The random variable Z in the theorem is often denoted $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ and called the *Radon-Nikodym derivative of* $\mathbb{Q}$ *w.r.t.* $\mathbb{P}$. With this notation Equation (1) gets an appealing form when using integrals,

$$\int \mathbf{1}_F \,\mathrm{d}\mathbb{Q} = \int \mathbf{1}_F \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \,\mathrm{d}\mathbb{P}.$$