Watanabe's characterization of a Poisson process

Theorem *Let $N = \{N(t),\, t \geq 0\}$ be a pure jump process, whose jumps all have size $+1$; N is called a counting process for obvious reasons. Let $\{\mathcal{F}(t),\, t \geq 0\}$ be a filtration to which N is adapted. Let $\lambda > 0$ and assume that the process M, $M(t) = N(t) - \lambda t$, is a martingale w.r.t. this filtration. Then N is a Poisson process with intensity λ, relative to this filtration (see Definition 11.4.1).*

Exercise The idea of the proof is similar to what happens on page 169 for Brownian motion. Let $\phi_{N(t)}(u) = \mathbb{E}\exp(uN(t))$. We want to show that $\phi_{N(t)}(u)$ is given by (11.3.4). Therefore we define

$$X(t) = \exp(uN(t) - (e^u - 1)\lambda t).$$

1. Show, using the Itô formula, that $X(t)$ is a martingale. (In fact, it is much like the processes $S(t)$ of Example 11.5.2 and $Z(t)$ of (11.6.1)).

2. Conclude that $\mathbb{E}\, X(t) = 1$ for all $t \geq 0$ and that N has a Poisson distribution. With what parameter?

3. With hardly more effort we can show more. Since X is a strictly positive martingale it follows, and you prove it, that

$$\mathbb{E}\,[\frac{X(t)}{X(s)}|\mathcal{F}(s)] = 1.$$

4. Show that the above equation is equivalent to

$$\mathbb{E}\,[\exp(u(N(t) - N(s))|\mathcal{F}(s)] = \exp(\lambda(t-s)(e^u - 1)).$$

   Conclude that $N(t)-N(s)$, conditional on $\mathcal{F}(s)$, has the correct Poisson distribution. Argue that the ordinary, unconditional, distribution is the same.

5. Finally, conclude from the previous part that $N(t) - N(s)$ is independent of $\mathcal{F}(s)$.