Duisenberg School of Finance
Measure Theory and Stochastic Processes II
Old exam questions

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1. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T > 0$ be fixed. As a possible alternative to compute $\int_0^T W(s)\,\mathrm{d}W(s)$ we use the approximating sums $S(\Pi) = \sum_{i=1}^{n-1} W(t_{i-1})(W(t_{i+1}) - W(t_i))$, where $\Pi = \{t_0, \ldots, t_n\}$ is a partition of $[0, T]$ with $t_0 = 0$ and $t_n = T$. Split $S(\Pi) = C(\Pi) + I(\Pi)$, where $C(\Pi) = \sum_{i=1}^{n-1}(W(t_{i-1}) - W(t_i))(W(t_{i+1}) - W(t_i))$ and $I(\Pi) = \sum_{i=1}^{n-1} W(t_i)(W(t_{i+1}) - W(t_i))$.

   (a) What is the L^2-limit of $I(\Pi)$ for a sequence of partitions Π whose mesh tend to zero?

   (b) What is the expectation of $C(\Pi)$?

   (c) Write $D_i = W(t_{i-1}) - W(t_i)$ and

   $$\mathbb{E}\, C(\Pi)^2 = \sum_{i,j} \mathbb{E}\,(D_i D_{i+1} D_j D_{j+1}).$$

   Show that $\mathbb{E}\, C(\Pi)^2 = \sum_{i=1}^{n-1} \mathbb{E}\,(D_i^2 D_{i+1}^2)$.

   (d) Show that $\mathbb{E}\, C(\Pi)^2 = \sum_{i=1}^{n-1}(t_i - t_{i-1})(t_{i+1} - t_i)$.

   (e) Show that $\mathbb{E}\, C(\Pi)^2 \to 0$ as $||\Pi|| \to 0$.

   (f) What is limit of the $S(\Pi)$ as $||\Pi|| \to 0$?

2. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T > 0$ be fixed. We'd like to compute the quadratic variation of the process X, defined by $X(t) = W(t)^2$, over the interval $[0, T]$. Let $S(\Pi) = \sum_{i=0}^{n-1}(X(t_{i+1}) - X(t_i))^2 = \sum_{i=0}^{n-1}(W(t_{i+1})^2 - W(t_i)^2)^2$, where $\Pi = \{t_0, \ldots, t_n\}$ is a partition of $[0, T]$ with $t_0 = 0$ and $t_n = T$.

   (a) Show that

   $$\begin{aligned} S(\Pi) = {} & \sum_{i=0}^{n-1}(W(t_{i+1}) - W(t_i))^4 \\ & + 4\sum_{i=0}^{n-1}(W(t_{i+1}) - W(t_i))^3 W(t_i) \\ & + 4\sum_{i=0}^{n-1}(W(t_{i+1}) - W(t_i))^2 W(t_i)^2. \end{aligned}$$

   (b) Write, in order of appearance, $S(\Pi) = \mathrm{I} + \mathrm{II} + \mathrm{III}$. Show that the term I converges (almost surely) to zero for a sequence of partitions whose mesh tend to zero. *Hint:* $(W(t_{i+1}) - W(t_i))^2 \leq \max_i (W(t_{i+1}) - W(t_i))^2$.

(c) Show that the term II converges (almost surely) to zero (for the same sequence of partitions).

(d) Argue (relying on known results) that $S(\Pi)$ converges (in the L^2 sense) to $4\int_0^T X(s)\,\mathrm{d}s$ (for the same sequence of partitions).

(e) Use the Itô formula to write X as in Itô process and verify the expression for $[X,X](T)$.

(f) Apply the Itô formula to X^2 to get an alternative expression for $[X,X](T)$.

(g) Combine the Itô formula for W^4 and the previous item to obtain the expression for $[X,X](T)$ as in item (d).

3. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $W^a(t) = \frac{1}{\sqrt{a}} W(at)$ for $a > 0$.

(a) Show that $W^a(t)$ is also Brownian motion, i.e. show that all defining properties of a Brownian motion are satisfied.

(b) Let $X^a(t) = W(at)$, $t \geq 0$. What is $[X^a, X^a](t)$?

(c) Determine $c > 0$ such that $X^a(t)^2 - ct$ is a martingale relative to its own filtration.

(d) Suppose Y_0 is a random variable with $\mathbb{E}\,Y_0 = 0$ and $\mathbb{E}\,Y_0^2 = 1$ that is independent of the Brownian motion W. Then $Y(t) = Y_0 + W(t)$ does not define a martingale w.r.t. the filtration generated by W. Why not?

(e) Let $\mathcal{G}(t)$ be the smallest σ-algebra such that Y_0 and $W(s), s \leq t$ are measurable. Show that $\{Y(t),\, t \geq 0\}$ is a martingale w.r.t. the filtration $\{\mathcal{G}(t), t \geq 0\}$.

4. Let $a : [0,\infty) \to \mathbb{R}^+$ be a continuous function and $A(t) = \int_0^t a(s)\,\mathrm{d}s$, $t \geq 0$. Let N be a standard Poisson process, so with intensity $\lambda = 1$. Let $Z(t) = \exp(-A(t) + t + \int_0^t \log a(s)\,\mathrm{d}N(s))$.

(a) Show (use the Itô rule for jump processes) that Z is a solution to $\mathrm{d}Z(t) = Z(t-)(a(t)-1)(\mathrm{d}N(t) - \mathrm{d}t)$.

(b) Show that $\mathrm{d}[N,Z](t) = Z(t-)(a(t)-1)\mathrm{d}N(t)$.

(c) Show (use the product rule) that the product $(N-A)Z$ is a martingale.

(d) Define a new probability measure $\mathbb{P}'$ by $\mathbb{P}'(A) = \mathbb{E}\,\mathbf{1}_A Z(T)$. Show that $\{N(t) - A(t), 0 \leq t \leq T\}$ is a martingale under $\mathbb{P}'$.

(e) Show that $\mathbb{P}'(N(t) - N(s) = j) = \mathbb{E}\,[\mathbf{1}_{\{N(t)-N(s)=j\}} Z(t)]$ for $0 \leq s \leq t \leq T$.

In the remainder of the exercise we assume that a is a constant function, $a(t) = \lambda' > 0$ for all $t \geq 0$.

(f) Show directly from the definition of $\mathbb{P}'$ that $\mathbb{P}'(N(t)-N(s)=j) = \exp(-\lambda'(t-s))(\lambda'(t-s))^j/j!$ for $j = 0,1,\dots$ and $0 \le s \le t \le T$.

(g) Show that $N(t) - N(s)$ and $N(s)$ are independent random variables under $\mathbb{P}'$ for $0 \le s \le t \le T$.

(h) What kind of process is $N(t)$, $0 \le t \le T$ under $\mathbb{P}'$?

5. Let W be a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a some constant. Let $T > 0$ and put $X(t) = \exp(2W(t) - at)$, $t \ge 0$. Let $\{\mathcal{F}(t), t \ge 0\}$ be the filtration generated by W and $g(t,x) = xe^{(a-2)(t-T)}$.

   (a) Give a stochastic differential equation for the process X. Is X a martingale?

   (b) Put $Y(t) = g(t, X(t)), t \ge 0$. Show that $\mathrm{d}Y(t) = 2Y(t)\,\mathrm{d}W(t)$.

   (c) Show that $\{Y(t), t \ge 0\}$ is a martingale and that $Y(t) = \mathbb{E}\left[X(T)|\mathcal{F}(t)\right]$ for $t \le T$.

   (d) Give a partial differential equation that the function g satisfies, and explicitly the boundary condition on $g(T,x)$. Verify that g indeed solves this equation.

   (e) Let h be a function such that $h(t, X(t)) = \mathbb{E}\left[\log X(T)|\mathcal{F}(t)\right]$ for $0 \le t \le T$. Give also a partial differential equation that the function h satisfies. What is the boundary condition on h?

6. Consider a Brownian motion $\{W(t) : t \in [0,T]\}$ and let $\Pi = \{0 = t_0 < \cdots < t_n = T\}$ be a partition of $[0,T]$. Let $t_j^\alpha = \alpha t_j + (1-\alpha)t_{j+1}$ for any $\alpha \in [0,1]$, $j = 0, \dots, n-1\}$. Define

$$J^\alpha(\Pi) = \sum_{j=0}^{n-1} W(t_j^\alpha)(W(t_{j+1}) - W(t_j)).$$

   (a) Show that

$$J^\alpha(\Pi) = J^1(\Pi) + Q^\alpha(\Pi) + C^\alpha(\Pi),$$

   where

$$C^\alpha(\Pi) = \sum_{j=0}^{n-1}(W(t_j^\alpha) - W(t_j))(W(t_{j+1} - W(t_j^\alpha)),$$

$$Q^\alpha(\Pi) = \sum_{j=0}^{n-1}(W(t_j^\alpha) - W(t_j))^2.$$

   (b) Show that $\mathbb{E}\,Q^\alpha(\Pi) = (1-\alpha)T$ and $\mathbb{E}\,C^\alpha(\Pi) = 0$.

   (c) Show that $\mathbb{V}\mathrm{ar}\,C^\alpha(\Pi) = \alpha(1-\alpha)\sum_{j=0}^{n-1}(t_{j+1} - t_j)^2$.

(d) Show that $\mathbb{V}\mathrm{ar}\, Q^\alpha(\Pi) = 2(1-\alpha)^2 \sum_{j=0}^{n-1}(t_{j+1}-t_j)^2$.

(e) Let Q^α be the L^2-limit of the $Q^\alpha(\Pi)$ as we take a sequence of partitions with $||\Pi|| \to 0$, i.e. Q^α is such that $\mathbb{E}\,(Q^\alpha(\Pi) - Q^\alpha)^2 \to 0$. Identify Q^α and show that it is indeed the L^2-limit.

(f) What is the L^2-limit C^α of the $C^\alpha(\Pi)$ for partitions with $||\Pi|| \to 0$? Show that indeed $\mathbb{E}\,(C^\alpha(\Pi) - C^\alpha)^2 \to 0$.

(g) Show (use the definition of the Itô-integral) that $J^\alpha(\Pi)$ converges (in L^2) to $\int_0^T W(s)\mathrm{d}W(s) + (1-\alpha)T$ and that this is equal to $\frac{1}{2}W(T)^2 + (\frac{1}{2} - \alpha)T$.

7. Consider a Brownian motion $W = \{W(t) : t \geq 0\}$ and $\{\mathcal{F}(t) : t \geq 0\}$ a filtration for W. Fix some $T > 0$ and put $M(t) = \mathbb{E}\,[W(T)^3|\mathcal{F}(t)]$, $t \geq 0$.

   (a) Show that M is a martingale. What are $M(0)$ and $M(T)$?

   (b) Show by direct computation, using properties of Brownian motion, that $M(t) = 3(T-t)W(t) + W(t)^3$ for $t \in [0,T]$.

   (c) What is $M(t)$ for $t \geq T$.

   (d) Give an expression for $\mathrm{d}M(t)$ for $t \leq T$. The result should again reveal that M is a martingale.

   (e) Find $\Theta(s)$ such that $W(T)^3 = \int_0^T \Theta(s)\mathrm{d}W(s)$.

   (f) What is $\Theta(s)$ for $s > T$ such that $M(t) = \int_0^t \Theta(s)\mathrm{d}W(s)$ for $t > T$?

8. Consider the process X given by $X(t) = X(0) + W(t)$, with $W = \{W(t) : t \geq 0\}$ a Brownian motion and $X(0)$ a random variable independent of W. Let $\mathcal{F}(t)$ be the smallest σ-algebra such that $W(s)$ is $\mathcal{F}(t)$ -measurable for $s \leq t$ and such that $X(0)$ is $\mathcal{F}(t)$-measurable. Let $h(x) = x^2$, $T > 0$ and $g(t,x) = \mathbb{E}^{t,x}h(X(T))$ for $t \leq T$.

   (a) Show that $\mathbb{E}\,[X(T)^2|\mathcal{F}(t)] = T - t + X(t)^2$ for $t \leq T$.

   (b) What is $g(t,x)$?

   (c) Give a partial differential equation to which g is a solution. What is the terminal condition $g(T,x)$?

9. Let $N = \{N(t), t \geq 0\}$ be a Poisson process with intensity λ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mathcal{F}(t), t \geq 0\}$ be the filtration generated by N. Let $Z(t) = (\frac{\tilde\lambda}{\lambda})^{N(t)} \exp(-(\tilde\lambda - \lambda)t)$, $t \geq 0$. Let $T > 0$ and define a new probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_T$ by $\tilde{\mathbb{P}}(A) = \mathbb{E}\,(\mathbf{1}_A Z(T))$. It is known that the process $Z = \{Z(t), t \geq T\}$ is a martingale.

(a) Let $k_2 \geq k_1$ be nonnegative integers and $t_2 \geq t_1 \geq 0$. Show that

$$\mathbb{P}(N(t_2) = k_2, N(t_1) = k_1) = e^{-\lambda t_2} \lambda^{k_2} \frac{(t_2 - t_1)^{k_2 - k_1} t_1^{k_1}}{(k_2 - k_1)! k_1!}.$$

(b) Show that for $t_1 \leq t_2 \leq T$

$$\tilde{\mathbb{P}}(N(t_2) = k_2, N(t_1) = k_1) = \mathbb{E}\left(\mathbf{1}_{\{N(t_2)=k_2, N(t_1)=k_1\}} Z(t_2)\right).$$

(c) Compute explicitly for $0 \leq t_1 \leq t_2 \leq T$ and integers $k_2 \geq k_1 \geq 0$ the probability $\tilde{\mathbb{P}}(N(t_2) = k_2, N(t_1) = k_1)$.

(d) Guess a formula for $\tilde{\mathbb{P}}(N(t_m) = k_m, \ldots, N(t_1) = k_1)$, for $T \geq t_m \geq \cdots \geq t_1 \geq 0$ and integers $k_m \geq \cdots \geq k_1 \geq 0$.

(e) What kind of process should $\{N(t) : t \leq T\}$ be under the measure $\tilde{\mathbb{P}}$?

Let N_1 and N_2 be independent Poisson processes with intensities λ_1 and λ_2 respectively. It is known that N_1 and N_2 have no common jumps. Let $\{\hat{\mathcal{F}}_t : t \geq 0\}$ be the smallest filtration to which both N_1 and N_2 are adapted. Put $\hat{Z}(t) = Z_1(t) Z_2(t)$, with $Z_i(t) = (\frac{\tilde{\lambda}}{\lambda})^{N_i(t)} \exp(-(\tilde{\lambda}_i - \lambda_i)t)$, for $i = 1, 2$ and $t \geq 0$. Let $T > 0$ and define a new probability measure $\hat{\mathbb{P}}$ on $\hat{\mathcal{F}}_T$ by $\hat{\mathbb{P}}(A) = \mathbb{E}(\mathbf{1}_A \hat{Z}(T))$. It is known that both processes $\hat{Z}_1$ and $\hat{Z}_2$ are martingales w.r.t. $\{\hat{\mathcal{F}}_t : t \geq 0\}$.

(f) Show that $\hat{Z}$ is a martingale.

(g) Let $t \leq T$. Show that $N_1(t)$ and $N_2(t)$ are independent random variables under the probability measure $\hat{\mathbb{P}}$.

10. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the process $X = \{X(t) : t \geq 0\}$ that solves the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = (6 - 3X(t))\,\mathrm{d}t + 6\,\mathrm{d}W(t),$$

and assume that $X(0)$ is some given random variable, independent of W. Let $Y(t) = e^{3t} X(t)$.

(a) Give an SDE that $Y(t)$ satisfies and write $Y(t)$ as the sum of an Itô integral and 'smooth' terms.

(b) Compute $X(t)$ from $Y(t)$ explicitly in an expression that contains an Itô integral, recognize that $X(t)$ is of the form $X(t) = f(t, Z(t))$ for some function $f(t, z)$ and use this to verify that $X(t)$ is indeed the solution to the given SDE.

(c) Let $\mu(t) = \mathbb{E}\,X(t)$ and $\tilde{X}(t) = X(t) - \mu(t)$. Show that $\frac{\mathrm{d}}{\mathrm{d}t}\mu(t) = 6 - 3\mu(t)$ and $\mathrm{d}\tilde{X}(t) = -3\tilde{X}(t)\,\mathrm{d}t + 6\,\mathrm{d}W(t)$.

(d) Let $v(t) = \mathbb{V}\text{ar}\, X(t)$. Show that $\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -6v(t) + 36$. What is $v(t)$ if $v(0) = 6$?

11. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. You know that $[W, W](T) = T$ and in this exercise you are going to (re)prove this. We need partitions $\Pi = \{0 = t_0, \ldots, t_n = T\}$ and $V(\Pi) = \sum_{i=1}^{n}(W(t_i) - W(t_{i-1}))^2$.

   (a) Show that for any such Π one has $\mathbb{E}\, V(\Pi) = T$.

   (b) Show that for any such Π one has $\mathbb{V}\text{ar}\, V(\Pi) = 2\sum_{i=1}^{n}(t_i - t_{i-1})^2$. You may use that $\mathbb{V}\text{ar}\,(X^2) = 2\sigma^4$ if X has a normal distribution with zero mean and variance σ^2.

   (c) Show that $\mathbb{V}\text{ar}\, V(\Pi) \to 0$ if the Π come from a sequence of partitions whose mesh tend to zero.

   (d) Show that $\mathbb{E}\,(V(\Pi) - T)^2 \to 0$ if the Π come from a sequence of partitions whose mesh tend to zero and conclude that $[W, W](T) = T$.

12. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use the filtration generated by W. Let $\gamma \in \mathbb{R}$ and consider $Z(t) = \exp(\gamma W(t) - \frac{1}{2}\gamma^2 t)$ for $t \geq 0$. Let $T > 0$ be fixed. We define a new probability measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by $\widetilde{\mathbb{P}}(A) = \mathbb{E}\,[\mathbf{1}_A Z(T)]$ for $A \in \mathcal{F}(T)$. Note that under $\widetilde{\mathbb{P}}$ the process W is not a Brownian motion anymore.

   (a) Show that the process $Z = \{Z(t), t \geq 0\}$ is a martingale under $\mathbb{P}$. What is $\mathbb{E}\, Z(t)$?

   (b) Show that for $t > s$ one has

$$W(t)Z(t) - W(s)Z(s) = \int_s^t (\gamma W(u) + 1) Z(u)\, \mathrm{d}W(u) + \gamma \int_s^t Z(u)\, \mathrm{d}u.$$

   (c) Show that $\mathbb{E}\,[\int_s^t Z(u)\, \mathrm{d}u | \mathcal{F}(s)] = (t - s)Z(s)$.

   (d) Compute the conditional expectations $\mathbb{E}\,[W(t)Z(t)|\mathcal{F}(s)]$ and (under $\widetilde{\mathbb{P}}$!!) $\widetilde{\mathbb{E}}\,[W(t)|\mathcal{F}(s)]$ for $s < t < T$.

   (e) Let $Y(t) = W(t) - bt$ ($t \in [0, T]$), for some $b \in \mathbb{R}$. What is the quadratic variation $[Y, Y](T)$ under the new measure $\widetilde{\mathbb{P}}$?

   (f) Show that $Y(t)$ is a Brownian motion under $\widetilde{\mathbb{P}}$ iff $b = \gamma$.

13. Let $N = \{N(t) : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. We use the filtration defined by $\mathcal{F}(t) = \sigma(N(s), s \leq t)$, $t \geq 0$. From the theory it is known that for all $t \geq 0$

$$\mathbb{E}\, \exp(uN(t) = \exp((e^u - 1)\lambda t).$$

(a) Show that for $0 \leq t \leq T$ it holds that

$$M_T(t) := \mathbb{E}\left[\exp(uN(T))|\mathcal{F}(t)\right] = \exp((e^u - 1)\lambda(T - t))\exp(uN(t)).$$

(b) Show (from the definition of $M_T(t)$) that $\{M_T(t), t \geq 0\}$ is a martingale.

In the remainder of this exercise we keep T fixed and consider $t \in [0, T]$ as the time parameter.

(c) Write $M_T(t) = \exp((e^u - 1)\lambda T)\hat{M}_T(t)$ with

$$\hat{M}_T(t) = \exp(uN(t) - (e^u - 1)\lambda t).$$

Show that $\Delta\hat{M}_T(t) = (e^u - 1)\hat{M}_T(t-)\Delta N(t)$.

(d) Between jump times of N we can differentiate $\hat{M}_T(t)$ w.r.t. t. Show that in this case one has $\frac{\mathrm{d}}{\mathrm{d}t}\hat{M}_T(t) = -(e^u - 1)\hat{M}_T(t)\lambda$.

(e) Show the integral representation

$$\hat{M}_T(t) = 1 + (e^u - 1)\int_0^t \hat{M}_T(s-)(\mathrm{d}N(s) - \lambda\,\mathrm{d}s).$$

(f) Explain why $\{\hat{M}_T(t) : t \in [0, T]\}$ is a martingale.

(g) Give an integral representation for $M_T(t)$.

14. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the process $X = \{X(t) : t \geq 0\}$ that solves the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = (2 - 2X(t))\,\mathrm{d}t + 2\,\mathrm{d}W(t),$$

and assume that $X(0) = x_0$ for some $x_0 \in \mathbb{R}$. Let $Y(t) = e^{2t}X(t)$.

(a) Give an SDE that $Y(t)$ satisfies and write $Y(t)$ as the sum of an Itô integral and 'smooth' terms.

(b) Compute the quadratic variations $[X, X](t)$ and $[Y, Y](t)$.

(c) Compute $X(t)$ from $Y(t)$ explicitly in an expression that contains an Itô integral.

(d) Recognize that $X(t)$ is of the form $X(t) = f(t, Z(t))$ for some function $f(t, z)$ and use this to verify that $X(t)$ is indeed the solution to the given SDE.

15. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}(t) = \sigma(W(s) : s \leq t)$, $t \geq 0$.

Consider the process $S = \{S(t) : t \geq 0\}$ that solves the stochastic differential equation (SDE) of Black-Scholes type

$$\mathrm{d}S(t) = S(t)\mu(t)\,\mathrm{d}t + S(t)\sigma(t)\,\mathrm{d}W(t),$$

where μ and σ are suitable ordinary (non-random) functions, all operations further needed are well defined, and assume that $S(0) = s_0$ for some $s_0 \in \mathbb{R}$. In addition, we are given a function r (the short rate) that we use to define $D(t) = \exp(-R(t))$ for $R(t) = \int_0^t r(s)\,\mathrm{d}s$, $t \geq 0$. Let $\tilde{S}(t) = D(t)S(t)$.

(a) Show that $\mathrm{d}\tilde{S}(t) = \tilde{S}(t)(\mu(t) - r(t))\,\mathrm{d}t + \tilde{S}(t)\sigma(t)\,\mathrm{d}W(t)$.

(b) Let $\Theta(t) = \frac{r(t)-\mu(t)}{\sigma(t)}$ and $Z(t) = \exp(\int_0^t \Theta(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \Theta(s)^2\,\mathrm{d}s)$, $t \geq 0$. Show that $\{Z(t) : t \geq 0\}$ is a martingale.

(c) Let $\tilde{W}(t) = W(t) - \int_0^t \Theta(s)\,\mathrm{d}s$, $t \geq 0$. Give a SDE for $\tilde{S}(t)$ in terms of $\tilde{W}$.

(d) Show that $\tilde{S}(t) = s_0 \exp(\int_0^t \sigma(s)\,\mathrm{d}\tilde{W}(s) - \frac{1}{2}\int_0^t \sigma(s)^2\,\mathrm{d}s)$.

(e) Fix $T > 0$ and define a new probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by $\tilde{\mathbb{P}}(A) = \mathbb{E}\,[\mathbf{1}_A Z(T)]$ for $A \in \mathcal{F}(T)$. Theory guarantees that $\tilde{\mathbb{P}}$ is a measure. Why is it a *probability* measure?

(f) It is known that $\{\tilde{W}(t) : 0 \leq t \leq T\}$ is a martingale under $\tilde{\mathbb{P}}$. Show that $\{\tilde{W}(t)Z(t) : 0 \leq t \leq T\}$ is a martingale under $\mathbb{P}$.

16. Let $N = \{N(t) : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, and let, for $u \in \mathbb{R}$ and $t \geq 0$,

$$Z(t) = \exp(uN(t) - \lambda t(e^u - 1)) = \exp(uN(t))\exp(-\lambda t(e^u - 1)).$$

We use the filtration defined by $\mathcal{F}(t) = \sigma(N(s), s \leq t)$, $t \geq 0$.

(a) Show $\mathbb{E}\,[\exp(uN(t))|\mathcal{F}(s)] = \exp(uN(s))\exp\big((e^u - 1)\lambda(t-s)\big)$. *Hint: use the independent increment property of N as well as the MGF of the Poisson distribution or an explicit computation.*

(b) Show, use the previous item, that $\{Z(t) : t \geq 0\}$ is a martingale.

(c) Let $X(t) = \exp(uN(t))$. Show that $\mathrm{d}X(t) = (e^u - 1)X(t-)\mathrm{d}N(t)$, for $t \geq s$.

(d) Show, apply the rule $\mathrm{d}\big(X(t)Y(t)\big) = X(t-)\,\mathrm{d}Y(t) + Y(t-)\,\mathrm{d}X(t)$ for X a function that may jump and Y a differentiable function, that

$$\mathrm{d}Z(t) = Z(t-)(e^u - 1)\mathrm{d}M(t),$$

for some process (which one?) $M(t)$.

(e) Argue from the previous item that $\{Z(t) : t \geq 0\}$ is a martingale. What is $\mathbb{E}\,Z(t)$?

17. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the process $X = \{X(t) : t \geq 0\}$ that solves the stochastic differential equation (SDE)

$$\mathrm{d}X(t) = (1 - X(t))\,\mathrm{d}t + 2\,\mathrm{d}W(t),$$

and assume that $X(0) = x_0$ for some $x_0 \in \mathbb{R}$. Let $Y(t) = e^t X(t)$.

(a) Give an SDE that $Y(t)$ satisfies and write $Y(t)$ as the sum of an Itô integral and 'smooth' terms.

(b) Compute the quadratic variations $[X, X](t)$ and $[Y, Y](t)$.

(c) Compute $X(t)$ from $Y(t)$ explicitly in an expression that contains an Itô integral.

(d) Recognize that $X(t)$ is of the form $X(t) = f(t, Z_t)$ for some function $f(x, z)$ and use this to verify that $X(t)$ is indeed the solution to the given SDE.

(e) What is the distribution of $X(t)$?

(f) Suppose we replace x_0 with a random variable $X(0)$, independent of the process W, having a $N(\mu, \sigma^2)$ distribution. What is now the distribution of $X(t)$.

(g) How to choose, in addition to the above, μ and σ^2 such that the distributions of the $X(t)$ are all the same?

18. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let

$$Z(t) = \exp(\frac{1}{2}W(t)^2 - \frac{1}{2}\int_0^t (W(s)^2 + 1)\,\mathrm{d}s),$$

for $t \in [0, T]$.

(a) Show that the quadratic variation process of $\frac{1}{2}W(t)^2$ is equal to $\int_0^t W(s)^2\,\mathrm{d}s$.

(b) $Z(t)$ is of the form $f(X(t))$. Show that $\mathrm{d}Z(t) = a(t)Z(t)\mathrm{d}W(t)$ for some adapted process $a(t)$ and identify it. What is $\mathbb{E}\,Z(T)$?

Define a new probability measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by $\widetilde{\mathbb{P}}(F) = \mathbb{E}\,[\mathbf{1}_F Z(T)]$ and put $\widetilde{W}(t) = W(t) - \int_0^t a(s)\,\mathrm{d}s$.

(c) Show that the *product* $\widetilde{W}(t)Z(t)$, for $t \in [0, T]$, is a martingale under $\mathbb{P}$.

(d) Compute $\widetilde{\mathbb{E}}\,[\widetilde{W}(t)|\mathcal{F}(s)]$ for $0 \leq s \leq t \leq T$.

(e) What is the distribution of $\widetilde{W}(T) - \widetilde{W}(t)$ under $\widetilde{\mathbb{P}}$?

19. Let $N = \{N(t) : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, and let Q be the compound Poisson process with $Q(t) = \sum_{i=1}^{N(t)} Y_i$. The usual independence assumptions are in force.

    Assume that $\mathbb{P}(Y_i = 1) = p$ and $\mathbb{P}(Y_i = -1) = 1 - p$. Let $N_+(t) = \sum_{i=1}^{N(t)} \mathbf{1}_{\{Y_i=1\}}$, $N_-(t) = \sum_{i=1}^{N(t)} \mathbf{1}_{\{Y_i=-1\}}$. Note that $N_+(t) + N_-(t) = N(t)$.

    (a) N_+ and N_- are independent Poisson processes according to the theory. What are their intensities?

    (b) Show that for integers $k, m \geq 0$

    $$\mathbb{P}(N_+(t) = k, N_-(t) = m) = \binom{k+m}{k} p^k (1-p)^m \, \mathbb{P}(N(t) = k+m).$$

    (c) Give an alternative expression for $Q(t)$ involving $N_+(t)$ and $N_-(t)$.

    (d) Suppose that the process Q is a martingale. Show that $p = \frac{1}{2}$.

    (e) Let Y be a process such that $Y(S^n) = Y_n$, where the S_n are the arrival times of N. Show that $\int_0^t Y(s)\,\mathrm{d}Q(s) = N(t)$.

    (f) Show that $Y(s) := \Delta Q(s)$ is an example of such a process. Even when Q is a martingale, the integral in the previous item (for this choice of $Y(s)$) item is not. Explain.

20. Consider the function $f(t) = t^2$ for $t \in [-1, 1]$. Let $\Pi_+ = \{t_0, \ldots, t_n\}$ be a partition of $[0, 1]$ with $0 \leq t_1 < \cdots < t_n = 1$ and let $\Pi_- = \{s_1, \ldots, s_m\}$ be a partition of $[-1, 0)$ with $-1 = s_1 < \cdots < s_m < 0$. Then $\Pi := \Pi_+ \cup \Pi_-$ is a partition of $[-1, 1]$, whose negative elements are the s_i and the positive elements are the t_j.

    (a) Let $V_1(f, \Pi)$ be the first order variation of f over the interval $[-1, 1]$ for the partition Π. Show that $V_1(f, \Pi) = 2 - s_m^2 - t_1^2 + |s_m^2 - t_1^2|$. What is the limit of $V_1(f, \Pi)$ if one considers a sequence of such partitions Π, whose mesh tend to zero?

    (b) Let $V_2(f, \Pi_+)$ be the quadratic variation of f over the interval $[0, 1]$ with respect to the partition Π_+. Show that $V_2(f, \Pi_+) = \sum_{k=1}^n (t_k - t_{k-1})^2 (t_k + t_{k-1})^2$ and $V_2(f, \Pi_+) \leq 4 \sum_{k=1}^n (t_k - t_{k-1})^2$. What is the limit of $V_2(f, \Pi_+)$ if one considers a sequence of such partitions Π_+, whose mesh tend to zero?

    (c) Replace the function f above by a Brownian motion W on $[0, 1]$, so $f(t) = W(t)$ for $t \in [0, 1]$, and put $f(t) = W(-t)$ for $t \in [-1, 0)$. What are the limits of $V_1(W, \Pi)$ and $V_2(W, \Pi)$ if one considers a sequence of such partitions Π, whose mesh tend to zero? *No computations need to be given for this question; a short answer suffices!*

21. Let W be a Brownian motion, defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Put $X(t) = W(t)^2$, $t \geq 0$. Then X is an Itô process. Write $X(t)$ as the sum of an Itô integral $I(t)$ and a 'smooth' process $R(t)$.

   (b) Compute the quadratic variation process $[X, X]$.

   (c) Put $Y(t) = W(t)^4$. Then also Y is an Itô process. Use this to compute $\mathbb{E}\, W(t)^4 = 3t^2$.

   (d) Use the two previous items to compute $\mathbb{V}\mathrm{ar}\,(W(t)^2)$.

22. Let W be a Brownian motion, defined on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a Black-Scholes model for the price $S(t)$ of some asset,

$$\mathrm{d}S(t) = S(t)\big(\beta\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\big),\ S(0) = 1.$$

Let $r > 0$ be the interest rate and let $D(t) = e^{-rt}$ be the discount factor at time t. Put $\hat{S}(t) = D(t)S(t)$ for $t \geq 0$.

   (a) Show that the process $\hat{S}$ satisfies the SDE

$$\mathrm{d}\hat{S} = \hat{S}\big((\beta - r)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t)\big).$$

   (b) Let

$$Z(t) = \exp\big(-\frac{\beta - r}{\sigma}W(t) - \alpha t\big),\ t \geq 0.$$

   Use the Itô formula to give an SDE that Z satisfies and determine the value of α that makes Z a martingale. *This value will be kept in the questions below.*

   (c) Show that the process M defined by $M(t) = \hat{S}(t)Z(t)$ is a martingale.

   (d) Let $T > 0$. What is $\mathbb{E}\, Z(T)$?

We will use $Z(T)$ to define a new probability measure $\mathbb{Q}$ on $\mathcal{F}(T) = \sigma\big(W(s),\ s \leq T\big)$ by $\mathbb{Q}(F) = \mathbb{E}\,(\mathbf{1}_F Z(T))$.

   (e) Compute $\mathbb{E}_{\mathbb{Q}}\hat{S}(T)$.

   (f) Compute $\mathbb{E}_{\mathbb{Q}}[\hat{S}(T)|\mathcal{F}(t)]$ for $t \leq T$.

23. Let N_1 and N_2 be two independent Poisson processes, with intensities λ_1 and λ_2 respectively. Consider the process Q given by $Q(t) = N_1(t) - N_2(t)$.

(a) Sketch a typical trajectory of Q, say with 5 jumps of N_1 and 3 jumps of N_2 before some time $T > 0$ (or choose different numbers of jumps, if you prefer).

(b) What is $\mathbb{E}\, Q(t)$?

One may view Q as a compound Poisson process. With $N(t) = N_1(t) + N_2(t)$, Q can be represented by

$$Q(t) = \sum_{k=1}^{N(t)} Y_k,$$

where the Y_k form an *iid* sequence of random variables, assuming the values -1 and $+1$, which is *also independent of* $N(t)$. Let $\beta = \mathbb{E}\, Y_k$.

(c) Compute β in terms of λ_1 and λ_2.

(d) Compute $\mathbb{P}(Y_k = 1)$ and $\mathbb{P}(Y_k = -1)$ in terms of λ_1 and λ_2.

Let $T_1, T_2, \ldots$ be the jump times of N and define the process Y by $Y(s) = \sum_{k\geq 1} Y_k \mathbf{1}_{[T_{k-1},T_k)}(s)$, so $Y(s) = Y_k$ if $s \in [T_{k-1}, T_k)$.

(e) Argue that $s \in [T_{k-1}, T_k)$ iff $N(s) = k - 1$ and, exploiting the independence of the Y_k and $N(s)$, show that $\mathbb{E}\, Y(s) = \beta$.

(f) Show $Y(s-) = Y_k$ if $s \in (T_{k-1}, T_k]$ and $Q(t) = \int_0^t Y(s-)\, \mathrm{d}N(s)$.

(g) Compute, using the previous two questions, $\mathbb{E}\, Q(t)$ again.

24. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T > 0$ be fixed. We are interested in $\int_0^T W(t)^2\, \mathrm{d}W(t)$. Let $S(\Pi) = \sum_{i=0}^{n-1} W(t_i)^2(W(t_{i+1}) - W(t_i))$, where $\Pi = \{t_0, \ldots, t_n\}$ is a partition of $[0, T]$ with $t_0 = 0$ and $t_n = T$.

(a) Show that

$$\begin{aligned} 3S(\Pi) = &\sum_{i=0}^{n-1} (W(t_{i+1})^3 - W(t_i)^3) \\ &- 3\sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^2 W(t_i) \\ &- \sum_{i=0}^{n-1} (W(t_{i+1}) - W(t_i))^3. \end{aligned}$$

(b) Write, in order of appearance, $S(\Pi) = \mathrm{I} - \mathrm{II} - \mathrm{III}$. Show that the term III converges (almost surely) to zero for a sequence of partitions whose mesh tend to zero. *Hint:* $|W(t_{i+1}) - W(t_i)| \leq \max_i |W(t_{i+1}) - W(t_i)|$.

(c) The term I has a trivial limit. Which one?

(d) Argue (relying on known results) that II converges (in the L^2 sense) to a Lebesgue integral (for the same sequence of partitions). Which integral is it?

(e) Apply the Itô formula to $W(T)^3$ and verify your answers above.

(f) Compute the quadratic variation of W^3 over the interval $[0, T]$, (i.e. write it as an integral).

25. Let $N = \{N(t), t \geq 0\}$ be a Poisson process with intensity λ. Let $\alpha, \beta > 0$ and let $Z = \{Z(t) = \exp(\alpha N(t) - \beta t),\ t \geq 0\}$. We use $\mathcal{F}(t) = \sigma(N(s) : s \leq t)$.

(a) Derive for $t \geq 0$, using properties of the Poisson distribution,

$$\mathbb{E}\,[\exp(\alpha N(t)) = \exp((e^\alpha - 1)\lambda t).$$

(b) Show, using properties of the Poisson process, for $t > s$

$$\mathbb{E}\,[\exp(\alpha N(t))|\mathcal{F}(s)] = \exp(\alpha N(s))\exp((e^\alpha - 1)\lambda(t - s)).$$

(c) If α is given, show that Z is a martingale if $\beta = (e^\alpha - 1)\lambda$. Show that in this case $\mathbb{E}\,Z(t) = 1$ for all $t \geq 0$.

In the case that Z is a martingale we use $Z(T)$ to define a new probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by $\tilde{\mathbb{P}}(F) = \mathbb{E}\,[\mathbf{1}_F Z(T)]$. Let $\tilde{\mathbb{E}}$ denote the corresponding expectation. Then for every random variable Y that is $\mathcal{F}(T)$-measurable and for which the expectations exist, one has $\tilde{\mathbb{E}}\,Y = \mathbb{E}\,[Y Z(T)]$.

(d) Show by direct computation that $\tilde{\mathbb{E}}\,N(T) = e^\alpha \lambda T$. *You may use* $\sum_{k=0} k x^k/k! = x e^x$.

(e) Deduce from the previous item and a property of the process Z that $\tilde{\mathbb{E}}\,N(t) = e^\alpha \lambda t$.

(f) Show

$$\tilde{\mathbb{E}}\,[N(t)|\mathcal{F}(s)] = N(s) + \mathbb{E}\,[(N(t) - N(s))\frac{Z(t)}{Z(s)}|\mathcal{F}(s)].$$

(g) Show $\mathbb{E}\,[(N(t) - N(s))\frac{Z(t)}{Z(s)}|\mathcal{F}(s)] = e^\alpha \lambda(t - s)$.

(h) Show that $\tilde{M}(t) = N(t) - e^\alpha \lambda t$ defines a martingale under $\tilde{\mathbb{P}}$ for $t \leq T$.

(i) What kind of a process is N under $\tilde{\mathbb{P}}$?

26. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Below we work with the filtration generated by W. Fix $T > 0$. Consider the martingale $M = \{M(t), 0 \leq t \leq T\}$, defined by $M(t) = \mathbb{E}[\sin(W(T))|\mathcal{F}(t)]$, $t \leq T$.

   (a) Show that M is indeed a martingale.

   (b) Show that $M(t) = \exp(-\frac{1}{2}(T-t))\sin(W(t))$. *Hint:* You may need three facts, the formula $\sin(x+y) = \sin x \cos y + \cos x \sin y$, the relation $\mathbb{E}[\cos(W(u))] = \exp(-\frac{1}{2}u)$ and the fact that $\mathbb{E}\, g(Z) = 0$ if the function g is bounded and satisfies $g(-x) = -g(x)$ and the random variable Z has a density f that satisfies $f(-x) = f(x)$.

   (c) Write M as an Itô process and verify the martingale property again.

27. Let $N = \{N_t : t \geq 0\}$ be a Poisson process with intensity $\lambda \geq 0$.

   (a) Show that $\int_0^t N(s-)\,\mathrm{d}N(s) = \frac{1}{2}N(t)(N(t)-1)$. *Hint:* $\sum_{k=0}^{n-1} k = \frac{1}{2}n(n-1)$.

   (b) Show directly, using the distribution of $N(t)$, that $\mathbb{E}\, N(t)(N(t)-1) = \lambda^2 t^2$.

   (c) It is easier to compute the above expectation from (a) using $N(s) = \lambda s + M(s)$. Compute the expectation by exploiting this decomposition.

28. Let $N = \{N(t) : t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$, and let Q be the compound Poisson process with $Q(t) = \sum_{i=1}^{N(t)} Y_i$. The usual independence assumptions are in force.

   Assume that $\mathbb{P}(Y_i = 1) = p$ and $\mathbb{P}(Y_i = -1) = 1 - p$. Let $N_+(t) = \sum_{i=1}^{N(t)} \mathbf{1}_{\{Y_i=1\}}$, $N_-(t) = \sum_{i=1}^{N(t)} \mathbf{1}_{\{Y_i=-1\}}$. Note that $N_+(t) + N_-(t) = N(t)$.

   (a) N_+ and N_- are independent Poisson processes according to the theory. What are their intensities?

   (b) Show that for integers $k, m \geq 0$

   $$\mathbb{P}(N_+(t) = k, N_-(t) = m) = \binom{k+m}{k} p^k (1-p)^m\, \mathbb{P}(N(t) = k+m).$$

   (c) Give an alternative expression for $Q(t)$ involving $N_+(t)$ and $N_-(t)$.

   (d) Suppose that the process Q is a martingale. Show that $p = \frac{1}{2}$.

   (e) Let Y be a process such that $Y(S^n) = Y_n$, where the S_n are the arrival times of N. Show that $\int_0^t Y(s)\,\mathrm{d}Q(s) = N(t)$.

(f) Show that $Y(s) := \Delta Q(s)$ is an example of such a process. Even when Q is a martingale, the integral in the previous item (for this choice of $Y(s)$) item is not. Explain.

29. Let $N = \{N(t), t \geq 0\}$ be a Poisson process with intensity λ.

   (a) Derive for any 'good' function $(t, k) \mapsto g(t, k)$ (i.e. the expressions below are assumed to make sense) the Itô-formula

$$g(t, N(t)) = g(0, 0) + \int_0^t g_t(s, N(s))\,\mathrm{d}s \\ + \int_0^t \big(g(s, N(s-) + 1) - g(s, N(s-))\big)\,\mathrm{d}N(s),$$

   where g_t stands for the 'first' partial derivative $\frac{\partial g}{\partial t}$.

   Let $h : \mathbb{N} = \{0, 1, 2, \ldots\} \to \mathbb{R}$ be a function such that $\mathbb{E}\,h(N(t))$ is well defined for every t. Fix $T > 0$ and let $g(t, k) = \mathbb{E}\,[h(N(T) - N(t) + k)]$ for $t \leq T$ and $k \in \mathbb{N}$.

   (b) Show that $g(t, N(t)) = \mathbb{E}\,[h(N(T))|\mathcal{F}(t)]$ for $t \leq T$.

   (c) Show that $M(t) := g(t, N(t))$ defines a martingale for $t \leq T$ w.r.t. filtration generated by N.

   (d) Show that it follows that

$$g_t(t, k) + \lambda(g(t, k + 1) - g(t, k)) = 0 \tag{1}$$

   for all $t \geq 0$ and $k \in \{0, 1, 2, \ldots\}$.

   (e) Compute $g(t, k)$ for the case (i) $h(m) = m$.

   (f) Compute $g(t, k)$ for the case (ii) $h(m) = m^2$. (Recall the formula $\mathbb{E}\,X^2 = \mathbb{V}\mathrm{ar}\,X + (\mathbb{E}\,X)^2$.)

   (g) Verify Equation (1) for both cases (i) and (ii).

   (h) What is in both cases (i) and (ii) the boundary condition $g(T, k)$? Give an explicit answer!

30. Let $W = \{W(t) : t \geq 0\}$ be a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Below we work with the filtration generated by W. Let a be an adapted process and let Z be a positive process with $Z(0) = 1$ satisfying the stochastic differential equation $\mathrm{d}Z(t) = a(t)Z(t)\,\mathrm{d}W(t)$. We assume that a is such that Z is well defined for all $t \geq 0$. Put $U(t) = \log Z(t)$.

   (a) Use the Itô-formula to derive a stochastic differential equation for $U(t)$.

(b) Show that $U(t) = \int_0^t a(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t a(s)^2\,\mathrm{d}s$.

Suppose that for some $T > 0$ it holds that $\mathbb{E}\,Z(T) = 1$. Define a probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}(T)$ by $\tilde{\mathbb{P}}(F) = \mathbb{E}\,[\mathbf{1}_F Z(T)]$. Let b be some adapted process satisfying $\int_0^T |b(s)|\,\mathrm{d}s < \infty$ and put $\tilde{W}(t) = W(t) - \int_0^t b(s)\,\mathrm{d}s$ for $t \in [0, T]$.

(c) Use the product rule to characterize that process b for which the process $\tilde{W}Z$ becomes a martingale under $\mathbb{P}$.

(d) Show, use the formula for computing a conditional expectation under a change of measure, that for the choice of b as in item (c), $\tilde{W}$ becomes a martingale under $\tilde{\mathbb{P}}$.

(e) Why can we conclude that $\tilde{W}$ is a Brownian motion under $\tilde{\mathbb{P}}$ if b is taken as in item (c)?