Measure theory and stochastic processes
Additional exercises

1. Let $\mathcal{A} = \{A_1, A_2, A_3\}$ be non-empty sets that form a partition of a set Ω. Write down all elements of $\sigma(\mathcal{A})$. Let B_1, B_2 be two subsets of Ω such that $B_1 \cap B_2$ and $(B_1 \cup B_2)^c$ are non-empty. Write down all elements of $\sigma(\{B_1, B_2\})$.

2. Let Ω be a nonempty set and let for each i in some (index) set I $\mathcal{F}_i$ be a σ-algebra on Ω. Let $\mathcal{C}$ be some collection of subsets of Ω. In alternative wordings compared to Section A.2, but in content the same, we define $\sigma(\mathcal{C})$ to be the smallest σ-algebra that contains $\mathcal{C}$, i.e. the intersection of all σ-algebras that contain $\mathcal{C}$.

   (a) Show that $\bigcap_{i \in I} \mathcal{F}_i$ (the intersection of all σ-algebras $\mathcal{F}_i$) is a σ-algebra.

   (b) Why is there is at least one σ-algebra that contains $\mathcal{C}$?

   (c) Here we take $\Omega = \mathbb{R}$. Argue that $\mathcal{B}(\mathbb{R})$ is equal to $\sigma(\mathcal{C})$, where $\mathcal{C} = \{(-\infty, a],\, a \in \mathbb{R}\}$.

   (d) Consider a function $X : \Omega \to \mathbb{R}$. Let $\mathcal{C}$ be a collection of subsets of $\mathbb{R}$ that is such that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$. Suppose that all sets $\{X \in C\}$ (for $C \in \mathcal{C}$) belong to a σ-algebra $\mathcal{F}$ on Ω. Show that X is a random variable (Definition 1.1.5).

   (e) Suppose that for all $a \in \mathbb{R}$ the set $\{X \leq a\}$ is an element of $\mathcal{F}$. Show that X is random variable.

   (f) Suppose that for all $a \in \mathbb{R}$ the set $\{X < a\}$ is an element of $\mathcal{F}$. Is X a random variable?

3. Let μ_X be the distribution of a random variable X, see Definition 1.2.3. Show that μ_X is probability measure on the Borel sets of $\mathbb{R}$.

4. Assume that the random variable X takes on the different values $x_0, x_1, \ldots$ in $\mathbb{R}$ and that $\mathbb{E}\,|X| < \infty$. Show that $\mathbb{E}\,X = \sum_{k=0}^{\infty} x_k \mathbb{P}(X = x_k)$. Special case: X is such that $\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^{|k|}}{2(|k|)!}$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\mathbb{P}(X = 0) = e^{-\lambda}$. What is $\mathbb{E}\,X$?

5. Consider the setting of Theorem 1.6.1. Show that $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent iff $\mathbb{P}(Z > 0) = 1$.

6. Show that $\sigma(X)$ as defined in Definition 2.1.3 is indeed a σ-algebra and that $\sigma(X) \subset \mathcal{F}$ if X is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ (*there is almost nothing to prove*).

7. Let X be a *nonnegative* random variable. Show that $\int X \mathrm{d}\mathbb{P} \geq \frac{\mathbb{P}(X > 1/n)}{n}$. Assume further that $\int X \mathrm{d}\mathbb{P} = 0$. Show that it follows that $\mathbb{P}(X = 0) = 1$.

8. Show (use the previous exercise) that the Radon-Nikodym derivative Z of Theorem 1.6.7 satisfies $Z \geq 0$ $\mathbb{P}$-a.s. (integrate over the set $\{Z < 0\}$). Use the equivalence of $\tilde{\mathbb{P}}$ and $\mathbb{P}$ to show that even $Z > 0$ $\mathbb{P}$-a.s. Show also that for a possibly different Z' satisfying the assertions of Theorem 1.6.7 one has that $\mathbb{P}(Z > Z') = 0$ and therefore $\mathbb{P}(Z = Z') = 1$.

9. Show (use the previous exercise) that the random variable Y of Theorem B.1 is a.s. nonnegative. Alternative, you can modify the proof of Theorem B.1 with the integrand $\frac{X+a}{\mathbb{E}\,X+a}$ for arbitrary rational $a > 0$ instead of $\frac{X+1}{\mathbb{E}\,X+1}$. This yields the existence of $\mathcal{G}$-measurable random variables Y_a. Show that they are a.s. all the same. So we can define an a.s. limit of them, Y say. Show that it follows that $Y \geq 0$ a.s.

10. Let $\Pi = \{A_1, \ldots, A_n\}$ be a *partition* of Ω, i.e. the A_i are non-empty, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_1 \cup \cdots \cup A_n = \Omega$. Let $\mathcal{G} = \sigma(\Pi)$ and $X : \Omega \to \mathbb{R}$. Show that X is constant on each A_i iff X is $\mathcal{G}$-measurable. If X is constant on the whole set Ω, what is $\sigma(X)$?

11. Let $(Z_t)_{t \geq 0}$ be a sequence of independent random variables, also independent of another random variable X_0. Assume that the following recursion hold for some 'good' measurable functions.

$$X_{t+1} = f(X_t, Z_t),\ t \geq 0.$$

Find a filtration to which the sequence (X_t) is adapted and that (X_t) is a (discrete time) Markov process (w.r.t. this filtration).

12. Use moment generating functions to show that $W(u) - W(t)$ and $W(t) - W(s)$ are independent random variables if $s < t < u$ (of course W is a standard Brownian motion). Compute also the conditional MGF $\mathbb{E}\left[\exp(uW(t))|\mathcal{F}(s)\right]$ for $s < t$, where $\{\mathcal{F}(s)\}_{s \geq 0}$ is a filtration for the Brownian motion. What is the conditional distribution of $W(t)$ given $\mathcal{F}(s)$?