Violating volatilities

Enno Veerman

July 7th, 2007

Consider n-dimensional affine square root SDE:

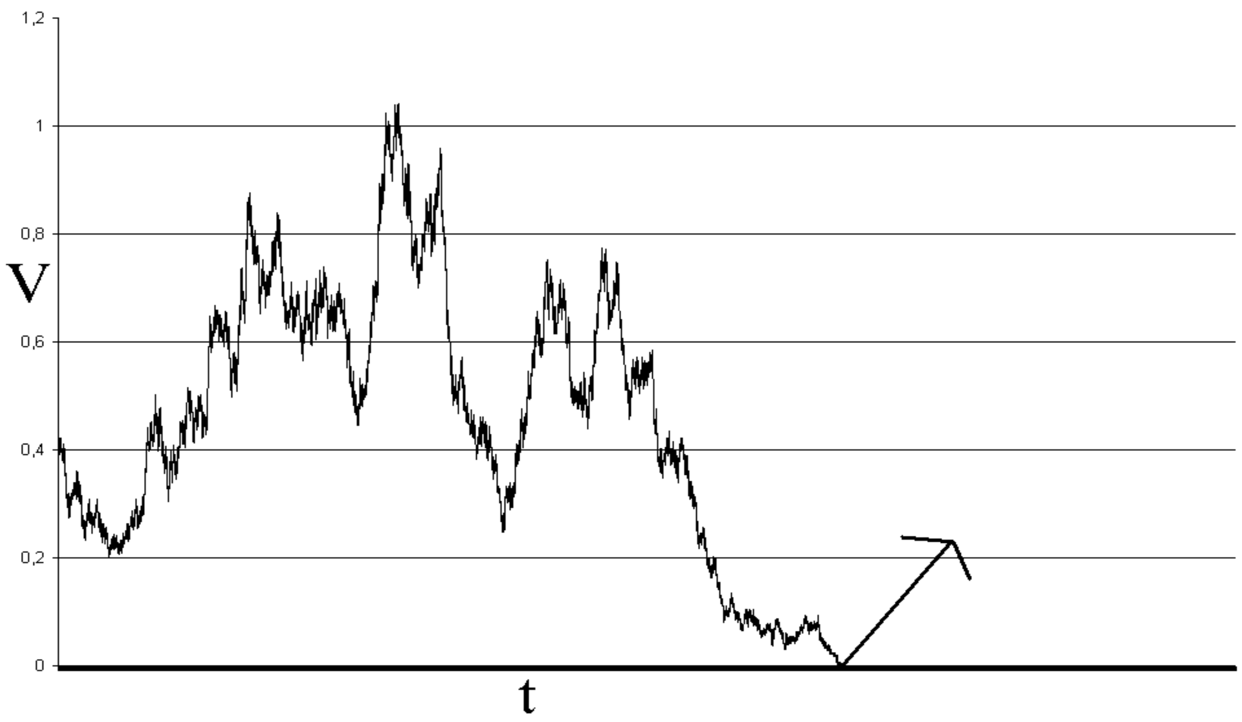
$$dX_t = (\mathbf{a}X_t + b)dt + \Sigma \begin{pmatrix} \sqrt{V_{1t}} & & 0 \\ & \ddots & \\ 0 & & \sqrt{V_{nt}} \end{pmatrix} dW_t$$

$$V_{it} := v_i(X_t) := \alpha_i + \beta_i^\top X_t.$$

- In the literature there are conditions imposed to assure that the volatilities V_i stay positive.
- What happens if we violate these conditions?
- Do the volatilities then get negative eventually?
- If so, is that a problem? (instead of $\sqrt{V_t}$ maybe we can use $\sqrt{V_t \vee 0}$ or $\sqrt{|V_t|}$)

Sufficient condition for general case: look at SDE for V_i; whenever a volatility V_i becomes zero, it should hold that

- the diffusion part of dV_{it} becomes zero;
- The drift part becomes positive.



Consider two-dimensional square root SDE with one volatility process

$$dX_t = (\mathbf{a}X_t + b)dt + \Sigma\sqrt{V_t}\, dW_t$$
$$X_0 = x_0$$

$$V_t := v(X_t) := \alpha + \beta^\top X_t$$

To see whether $V_t \geq 0$ for all t, we look at the SDE for V_t:

$$dV_t = \beta^\top dX_t = \beta^\top(\mathbf{a}X_t + b)dt + \beta^\top\Sigma\sqrt{V_t}\, dW_t$$

If $dV_t \geq 0$ whenever $V_t = 0$, then V_t can never become negative.

Sufficient condition:

For all $x \in \mathbb{R}^2$ such that $v(x) = 0$ it holds that $\beta^\top(\mathbf{a}x + b) \geq 0$.

Why do we need the conditions from the literature?

- to prove volatilities stay positive
- to rewrite the SDE for X in **Canonical form**, which can be used
  - to prove pathwise uniqueness for the SDE (which implies existence of a strong solution)
  - to prove that in an **affine term structure model**, the bond-price equals $D_{t,T} = \exp(A(T-t) + B(T-t)^\top X_t)$,

Canonical representation

n-dimensional affine square root SDE with $m \le n$ "independent" volatilities:

$$
\begin{aligned}
dV_{1t} &= (\mathbf{a}_{11}V_{1t} + \mathbf{a}_{12}V_{2t} + \ldots + \mathbf{a}_{1m}V_{mt} + b_1)dt + \sqrt{V_{1t}}dW_{1t} \\
dV_{2t} &= (\mathbf{a}_{21}V_{1t} + \mathbf{a}_{22}V_{2t} + \ldots + \mathbf{a}_{2m}V_{mt} + b_2)dt + \sqrt{V_{2t}}dW_{2t} \\
&\vdots \\
dV_{mt} &= (\mathbf{a}_{m1}V_{1t} + \mathbf{a}_{m2}V_{2t} + \ldots + \mathbf{a}_{mm}V_{mt} + b_m)dt + \sqrt{V_{mt}}dW_{mt}
\end{aligned}
$$

where $\mathbf{a}_{ij} \ge 0$ for $i \ne j$ and $b_i \ge 0$.

The remaining volatilities V_j with $j > m$ are linear combinations of these ("dependence"):

$$V_{jt} = \alpha_j + \sum_{i=1}^{m} \beta_{ji} V_{it}$$

with $\alpha_j \geq 0$ and $\beta_{ji} \geq 0$, so that $V_{jt} \geq 0$ since $V_{it} \geq 0$ for $i \leq m$.

Short rate term structure model

A zero coupon bond is a contract which guarantees a payment of one unit of money at a given time T in the future. The bond price at time t is defined to be

$$D_{t,T} = \mathrm{E}(e^{-\int_t^T r_s ds} | \mathcal{F}_t),$$

with $r_t = r(X_t)$, the short rate, is a function of a state factor X, which satisfies a certain SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

In an **affine** term structure model this SDE is an **affine square root SDE**, and r is an **affine** transformation of X:

$$r_t = r(X_t) = \delta_0 + \delta^\top X_t.$$

Term structure equation

- State factor X satisfies SDE $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$;
- Short rate $r_t = r(X_t)$ is a function of X;
- We want to determine the bond price $F(t,x) = \mathrm{E}(e^{-\int_t^T r_s ds} | X_t = x)$.

Standard trick: differentiate the **martingale**

$$\mathrm{E}(e^{-\int_0^T r_s ds} | X_t) = e^{-\int_0^t r_s ds} F(t, X_t) =: R_t F(t, X_t) = RF(\text{shorthand notation})$$

$$\begin{aligned} dRF &= RdF + FdR \\ &= \ldots (\text{Itô calculus}) \\ &= R \underbrace{\left(-rF + F_t + F_x\mu + \frac{1}{2}F_{xx}\sigma^2\right)}_{0} dt + RF_x\sigma dW. \end{aligned}$$

So the bond price F satisfies the PDE

$$-rF + F_t + F_x\mu + \frac{1}{2}F_{xx}\sigma^2 = 0$$
$$F(T,x) = 1$$
$$(t,x) \in [0,T) \times \mathcal{D},$$

under the assumption that F is smooth.

For an affine term structure model this PDE can be solved by

$$F(t,x) = \exp(A(T-t) + B(T-t)^\top x),$$

where A and B on their turn satisfy the (Riccati) ODE's

$$A' = b^\top B - \frac{1}{2}\sum_i\sum_j\sum_k B_i B_j \Sigma_{ik}\Sigma_{jk}\alpha_k - \delta_0, \quad A(0) = 0$$

$$B' = \mathbf{a}^\top B - \frac{1}{2}\sum_i\sum_j\sum_k B_i B_j \Sigma_{ik}\Sigma_{jk}\beta_k - \delta, \quad B(0) = 0.$$

The above argument doesn't seem to need that the volatilities stay positive. Moreover, we still have to prove that F is smooth. Under the assumption that F is smooth, we have proved that

$$F(t, x) = \exp(A(T - t) + B(T - t)^\top x),$$

which is smooth, but that doesn't mean that the assumption necessarily holds true.

Alternatively, we can use the Feynman-Kaç approach, which represents solutions to particular PDE's (namely the Cauchy problem) as a (conditional) expectation.

Cauchy problem

Consider an n-dimensional SDE $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$. We define a differential operator $\mathcal{L}$ by:

$$\mathcal{L}u = \sum_i \mu_i \frac{\partial u}{\partial x_i} + \frac{1}{2} \sum_i \sum_j \sigma_i \sigma_j^\top \frac{\partial^2 u}{\partial x_i \partial x_j},$$

The *Cauchy problem* is the problem of finding a (unique) solution u for the backward partial differential equation

$$u_t + \mathcal{L}u = ku - g, \quad u(T, \cdot) = f.$$

If u solves the Cauchy problem and u can be stochastically represented as a conditional expectation (using X), then we say that u admits a *Feynman-Kaç representation*

In our case we have $u(t,x) = \exp(A(T-t) + B(T-t)^\top x)$.

$$u_t + \mathcal{L}u = ru, \quad u(T,\cdot) = 1.$$

Then u admits a *Feynman-Kaç representation*, i.e.
$u(t,x) = \mathrm{E}(\exp(-\int_t^T r(X_s)ds)|X_t = x)$, if u satisfies the growth condition

$$\sup_{t\leq T} |u(t,x)| \leq K(1+|x|^c), \text{ for some } K > 0,\ c \geq 2 \text{ and all } x \text{ in domain } \mathcal{D} \text{ of } X$$

This is not obvious from the formula

$$u(t,x) = \exp(A(T-t) + B(T-t)^\top x)$$

Suppose $r_t \geq 0$ almost surely for all t. Then

$$0 \leq \mathrm{E}[\exp(-\int_t^T r_s ds)|X_t = x] \leq 1, \text{ for all } x \in \mathcal{D}$$

Hence we expect that $0 \leq u(t,x) \leq 1$ for x in domain $\mathcal{D}$ of X, which is (more than) sufficient. Therefore, we have to prove that $A(t) \leq 0$ and $B(t)^\top x \leq 0$ for all t and all $x \in \mathcal{D}$.

$$A' = b^\top B + \frac{1}{2}\sum_i \sum_j \sum_k B_i B_j \Sigma_{ik}\Sigma_{jk}\alpha_k - \delta_0, \quad A(0) = 0$$

$$B' = \mathbf{a}^\top B + \frac{1}{2}\sum_i \sum_j \sum_k B_i B_j \Sigma_{ik}\Sigma_{jk}\beta_k - \delta, \quad B(0) = 0.$$

This is still not obvious.

Simplify ODE's by using canonical representation. Consider for example 2-dimensional affine square root SDE and canonical form

$$V_1 = (\mathbf{a}_{11}V_1 + \mathbf{a}_{12}V_2 + b_1)dt + \sqrt{V_1}dW_{1t}$$
$$V_2 = (\mathbf{a}_{21}V_1 + \mathbf{a}_{22}V_2 + b_2)dt + \sqrt{V_2}dW_{2t}$$

with $a_{ij} \geq 0$ for $i \neq j$ and $b_i \geq 0$. Take $r = \delta_0 + \delta_1 V_1 + \delta_2 V_2$, with $\delta_i > 0$ so that $r \geq 0$. Then ODE's for A and B reduce to

$$A' = b_1 B_1 + b_2 B_2 - \delta_0, \qquad A(0) = 0$$
$$B_1' = \mathbf{a}_{11}B_1 + \mathbf{a}_{21}B_2 + \frac{1}{2}B_1^2 - \delta_1, \quad B_1(0) = 0$$
$$B_2' = \mathbf{a}_{12}B_1 + \mathbf{a}_{22}B_2 + \frac{1}{2}B_2^2 - \delta_2, \quad B_2(0) = 0.$$

Now it is obvious that $A(t) \leq 0$, $B_1(t) \leq 0$ and $B_2(t) \leq 0$ for all t. Since $X = (V_1, V_2) \in [0, \infty) \times [0, \infty)$ we are done.

Conclusions

While the conditions from literature might be relaxed to assure positive volatilities (as the simulation suggests), they seem to be necessary

- to prove pathwise uniqueness of the SDE for X;
- to prove Feynman-Kaç formula

$$\mathrm{E}(e^{-\int_t^T r_s ds}|X_t) = \exp(A(T-t) + B(T-t)^\top X_t,$$

using the Canonical representation of the SDE for X.