# Deconvolution for an atomic distribution

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## Model

Let $X_1, \ldots, X_n$ be i.i.d. observations, s.t.

$$X_i = Y_i + Z_i,$$

and let $Y'$s and $Z'$s be independent.

## Assumption on $Y$

$Y = UV$ in distribution, where $U$ has Bernoulli distribution with probability of zero $p < 1$ and $V$ has density $f$.

## Assumption on $Z$

$Z$ has the standard normal distribution.

**Aim**

Based on indirect observations $X_1, \ldots, X_n$, estimate the probability $p$ and the unknown density $f$.

**Related problem**

Classical deconvolution problem: $Y$ is assumed to have a density.

**Motivation**

Let $\eta_t = \xi_t + B_t$, where $\xi = (\xi_t)_{t \geq 0}$ is a compound Poisson process with intensity $\lambda$ and $B = (B_t)_{t \geq 0}$ is a Brownian motion independent of $\xi$. Assume that $\eta$ is observed at time points $1, 2, \ldots, n$ and we want to characterise the distribution of $\xi_1$.

Passing to increments $\eta_1, \eta_2 - \eta_1, \ldots$ reduces the problem to the deconvolution for an atomic distribution.

$\eta_i$'s can be interpreted as measurements of some quality characteristic of interest corrupted by noise $Z_i$.

**Nonparametric Estimation**

Tools: kernel smoothing and Fourier inversion.

**Case of known $p$**

Assume that $p$ is known.

We have

$$\phi_X(t) = [p + (1-p)\phi_f(t)]e^{-t^2/2},$$

where $\phi_X$ and $\phi_f$ are the ch.f.'s of $X$ and $V$, respectively.

Solving for $\phi_f$, we get

$$\phi_f(t) = \frac{\phi_X(t) - pe^{-t^2/2}}{(1-p)e^{-t^2/2}}.$$

Assuming that $\phi_f$ is integrable, by Fourier inversion we obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{\phi_X(t) - pe^{-t^2/2}}{(1-p)e^{-t^2/2}}dt.$$

**Step 1: Estimator of $\phi_X$**

Let $\phi_{emp}$ denote an empirical characteristic function,

$$\phi_{emp}(t) = \frac{1}{n} \sum_{j=1}^{n} e^{itX_j}.$$

**Step 2: Plug-in type estimator for $f$**

Define an estimator $f_{nh}$ of $f$ as

$$f_{nh}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_{emp}(t) - pe^{-t^2/2}}{(1-p)e^{-t^2/2}} \phi_w(ht)dt,$$

where $w$ denotes a *kernel function* with characteristic function $\phi_w$ and $h$ denotes a positive number, the *bandwidth*.

We assume that $\phi_w$ has a compact support on $[-1, 1]$. An extra smoothing (in terms of $\phi_w$) is required to ensure the integrability of an integrand in the definition of $f_{nh}$.

**Estimator $f_{nh}$ is asymptotically unbiased**

We have

$$\mathsf{E}[f_{nh}(x)] - f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_f(t)(\phi_w(ht) - 1) dt.$$

This expression coincides with the bias of an ordinary kernel density estimator based on observations from $f$ and it is known that under sufficient smoothness assumptions on $f$ it asymptotically vanishes.

**Estimator for $p$**

$p$ is identifiable, since $\phi_Y(t) \to p$ as $t$ tends to infinity.

This relation, however, cannot be used as a hint for the construction of a meaningful estimator of $p$, because of the oscillating behaviour of $\phi_{emp}(t)$ as $t \to \infty$.

As an estimator of $p$ we propose

$$p_{ng} = \frac{g}{2} \int_{-1/g}^{1/g} \frac{\phi_{emp}(t)\phi_k(gt)}{e^{-t^2/2}} dt,$$

where the number $g > 0$ denotes the bandwidth and $\phi_k$ denotes a Fourier transform of a kernel $k$.

The definition is motivated by the fact that

$$\begin{aligned}\lim_{g\to 0} \frac{g}{2} \int_{-1/g}^{1/g} \frac{\phi_X(t)}{e^{-t^2/2}} dt &= \lim_{g\to 0} \frac{g}{2} \int_{-1/g}^{1/g} \phi_Y(t) dt \\ &= \lim_{g\to 0} \frac{g}{2} \int_{-1/g}^{1/g} (p + (1-p)\phi_f(t)) dt = p.\end{aligned}$$

**General case: unknown $p$**

We define an estimator of $f$ as

$$f^*_{nhg}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{\phi_{emp}(t) - \widehat{p}_{ng} e^{-t^2/2}}{(1-\widehat{p}_{ng}) e^{-t^2/2}} \phi_w(ht) dt,$$

where

$$\widehat{p}_{ng} = \min(p_{ng}, 1 - \epsilon_n).$$

Here $0 < \epsilon_n < 1$ and $\epsilon_n \downarrow 0$ at a suitable rate, which will be specified below. The truncation in is introduced due to technical reasons.

## Main goal

The main goal is to derive the asymptotic normality of $f^*_{nhg}$ at a fixed point $x$.

## More conditions

**Condition 1.** *Let the true density $f$ be such that $u^{2+\gamma}\phi_f(u)$ is integrable. Here $\gamma$ is some strictly positive number.*

**Condition 2.** *Let $\phi_w$ be real valued, symmetric and have support $[-1,1]$. Let $\phi_w(0) = 1$ and let*

$$\phi_w(1-t) = At^{\alpha} + o(t^{\alpha}), \qquad \text{as} \qquad t \downarrow 0 \tag{1}$$

*for some constants $A$ and $\alpha \geq 0$. Moreover, we assume that $\alpha < \gamma/2$.*

An example of such a kernel is a sinc kernel, $w(x) = \frac{\sin x}{\pi x}$. Its characteristic function is given by $\phi_w(t) = 1_{[-1,1]}(t)$.

**Condition 3.** *Let $\phi_k$ be real valued, symmetric and have support $[-1,1]$. Let $\phi_k$ integrate to 2 and let*

$$\phi_k(1-t) = At^{\alpha} + o(t^{\alpha}),$$
$$\phi_k(t) = Bt^{2+\gamma} + o(t^{2+\gamma}).$$

*as $t \downarrow 0$. Here $B$ is some constant, and $A, \gamma$ and $\alpha$ are as above.*

**Condition 4.** *Let the bandwidths $h$ and $g$ depend on $n$, $h = h_n$ and $g = g_n$, and let $h_n = ((1+\eta_n)\log n)^{-1/2}$, $g_n = ((1+\delta_n)\log n)^{-1/2}$, where $\eta_n$ and $\delta_n$ are such that $\eta_n \to 0, \delta_n \to 0$, $\eta_n - \delta_n > 0$, and $(\eta_n - \delta_n)\log n \to \infty$.*

An example of $\eta_n$ and $\delta_n$ in the definition above are

$$\eta_n = 2\frac{\log\log n}{\log n}, \quad \delta_n = \frac{\log\log n}{\log n}.$$

**Condition 5.** *Let* $\epsilon_n$ *be such that* $-\log \epsilon_n \ll \log n(\eta_n - \delta_n)$.

An example of such $\epsilon_n$ for $\eta_n$ and $\delta_n$ given above is $(\log \log n)^{-1}$.

**Principal results**

Our first goal is to derive the asymptotic normality of the estimator $f_{nh}(x)$.

**Theorem 1.** *Assume Conditions 1, 2, 4, suppose that* $p$ *is known and let* $\mathsf{E}\,[X^2] < \infty$. *Then, as* $n \to \infty$ *and* $h \to 0$,

$$\frac{\sqrt{n}}{h^{1+2\alpha}e^{1/2h^2}}\left(f_{nh}(x) - \mathsf{E}\left[f_{nh}(x)\right]\right) \xrightarrow{\mathcal{D}} N\left(0, \frac{A^2}{2\pi^2(1-p)^2}(\Gamma(\alpha+1))^2\right),$$

*where* $\Gamma(t) = \int_0^\infty v^{t-1}e^{-v}dv$.

## Remarks

In order to get a consistent estimator, $\sqrt{n}h^{-1-2\alpha}e^{-1/2h^2}$ has to diverge to infinity. Therefore the bandwidth $h$ has to be at least of order $(\log n)^{-1/2}$.

The asymptotic variance depends on an unknown $p$ : the larger $p$ is, the larger is the asymptotic variance. This reflects the fact that it is 'harder' to estimate $f$ for large values of $p$, since many of the 'signals' $Y_i$ are expected to be zero.

## Results concerning an estimator of $p$

**Theorem 2.** *Let the conditions of the previous theorem hold true. Then $p_{ng}$ is a consistent estimator of $p$,*

$$\mathsf{P}(|p_{ng} - p| > \epsilon) \to 0$$

*as $n \to 0$ and $g \to 0$. Here $\epsilon$ is an arbitrary positive number.*

$p_{ng}$ is not only consistent, but also asymptotically normal, as the following theorem demonstrates.

**Theorem 3.** *Let the conditions of the previous theorem hold true. Then we have*

$$\frac{\sqrt{n}}{g^{2+2\alpha}e^{1/2g^2}}(p_{ng} - \mathsf{E}\left[p_{ng}\right]) \xrightarrow{\mathcal{D}} N\left(0, \frac{A^2(\Gamma(1+\alpha))^2}{2}\right).$$

## Main theorem

We formulate the main theorem.

**Theorem 4.** *Assume that the Condition 1,2,3 and 4 hold true. Then, as $n \to \infty$ and $h \to 0$, $g \to 0$, we have*

$$\frac{\sqrt{n}}{h^{1+2\alpha} e^{\frac{1}{2h^2}}} \left(f^*_{nhg}(x) - \mathsf{E}\left[f^*_{nhg}(x)\right]\right) \xrightarrow{\mathcal{D}} N\left(0, \frac{A^2}{2\pi^2(1-p)^2}(\Gamma(\alpha+1))^2\right),$$

*where $\Gamma(t) = \int_0^\infty v^{t-1} e^{-v} dv$.*

## Possible generalisations

There are many possibilities to generalise the problem in question.

- One can assume that $Y$ has an atom not necessarily in zero, but in some other known point.

- One can suppose that the location of an atom is unknown.

- One can consider a more general model $X = aY + \sigma Z$, where $\sigma$ is some positive and unknown number.

- A more general noise distribution can be considered.

## Further applications

The results that we obtained can be used in decompounding problem under Gaussian noise: assume that we observe a process $\eta_t = \xi_t + B_t$ at equidistant time points $1, 2, \ldots,$ where $\xi$ is a compound Poisson process with intensity $\lambda$ and jump size density $f$, and $B$ is Brownian motion. Based on a discrete sample from $\eta$, we want to estimate the density $f$.

The process $\eta$ is a simple example of a Lévy process with Lévy density $\lambda f$.

**References**

This presentation is based on

Bert van Es, Shota Gugushvili, Peter Spreij, Deconvolution for an atomic distribution, to appear shortly on arxiv.org.