Problem Set 7

March 16, 2009

Please hand in Problem 1 by April 6, 2pm.

1 Problem 1

Consider a free boson X CFT with the XX OPE

$$X(z,\bar{z})X(w,\bar{w}) \sim -\ln|z-w|^2 \tag{1}$$

and the stress-energy tensor

$$T(z) = -\frac{1}{2}{:}\partial X \partial X(z){:}\,, \qquad \bar{T}(\bar{z}) = -\frac{1}{2}{:}\bar{\partial} X \bar{\partial} X(\bar{z}){:}\,. \tag{2}$$

(i) (7.5 pts) Derive the OPE of $T(z)$ and $\bar{T}(\bar{z})$ with $X(w,\bar{w})$, $\partial X(w,\bar{w})$, $\bar{\partial} X(w,\bar{w})$, $\partial^2 X(w,\bar{w})$, and ${:}\exp(i\sqrt{2}X)(w,\bar{w}){:}\,$.

(ii) (2.5 pts) What do these results imply for the conformal dimension $(h,\bar{h})$ in each case?

Note: if you want to draw hooks for contractions using LaTeX, try googling for `simplewick.sty` or `wick.sty`.

2 Problem 2

Let A and B be two free fields whose contractions with themselves and each other are c numbers. We denote by $\overbrace{\mathcal{F}\mathcal{G}}$ the contraction between two operators $\mathcal{F}, \mathcal{G}$ which are functions of A, B. Recall that

$$\mathcal{F}\mathcal{G} = {:}\mathcal{F}\mathcal{G}{:} + \overbrace{\mathcal{F}\mathcal{G}}. \tag{3}$$

(i) Show by recursion that

$$\overbracket{A(z) :B}^n(w)\!: = n\overbracket{A(z)B}(w)\!:B^{n-1}(w)\!: \tag{4}$$

(ii) Use this result to prove

$$\overbracket{A(z) :\exp B}(w)\!: = \overbracket{A(z)B}(w)\!:\exp B(w)\!: \tag{5}$$

(iii) By counting multiple contractions, show that

$$\begin{aligned}
&:\exp \overbracket{A(z)\!: :\exp B}(w)\!: \\
&= \sum_{m,n=0}^{\infty} \sum_{1\le k\le m,n} \frac{k!}{m!\,n!}\binom{m}{k}\binom{n}{k}\left(\overbracket{A(z)B}(w)\right)^k :A^{m-k}(z)B^{n-k}(w)\!: \\
&= \left[\exp\left(\overbracket{A(z)B}(w)\right) - 1\right] :\exp A(z)\exp B(w)\!:
\end{aligned} \tag{6}$$

(iv) Consider the free boson X as in Problem 1 and compute

$$\langle :\exp(iaX)(z)\!: :\exp(-iaX)(w)\!:\rangle. \tag{7}$$

Use the answer to determine the conformal weight of $:\exp(iaX)(z)\!:$.

3 Problem 3

Show that the correlation function containing one secondary field can be obtained from a correlation function of only primaries fields by acting with a differential operator (we ignore the anti-holomorphic part in what follows):

$$\begin{aligned}
&\langle \phi_1(w_1)\cdots\phi_n(w_n)(\hat{L}_{-k}\phi)(z)\rangle \\
&\qquad = \mathcal{L}_{-k}\langle \phi_1(w_1)\cdots\phi_n(w_n)\phi(z)\rangle
\end{aligned} \tag{8}$$

where

$$\mathcal{L}_{-k} = -\sum_{j=1}^{n}\left(\frac{(1-k)h_j}{(w_j-z)^k} + \frac{1}{(w_j-z)^{k-1}}\frac{\partial}{\partial w_j}\right) \tag{9}$$

Here ϕ_i are chiral primaries of weight h_i and

$$(\hat{L}_{-k}\phi)(w) = \oint \frac{dz}{2\pi i}\frac{1}{(z-w)^{k-1}}T(z)\phi(w) \tag{10}$$

is a descentant of the chiral primary ϕ.