Problem Set 6

March 9, 2009

Please hand in Problem 1 by March 16, 2pm.

Problem 1

The expansion of the free massless scalar X living on a Lorentzian cylinder is

$$X(\sigma,\tau) = x + 4p\tau + i\sum_{n\neq 0}\frac{1}{n}(\alpha_n e^{2in\sigma} + \tilde{\alpha}_n e^{-2in\sigma})e^{-2in\tau}, \tag{1}$$

where $\sigma \cong \sigma + \pi$. This is obtained by setting $l_s = 2$ (or $\alpha' = 2$) in the expansion formula for the X^μ fields for closed string (BBS (2.40), (2.41)).

(a) (2.5 pts) Write down the expansion formula on a Euclidean cylinder. First do a Wick rotation $\tau \to -i\tau$ and then express the result in terms of the complex coordinates $\zeta = 2(\tau - i\sigma)$, $\bar{\zeta} = 2(\tau + i\sigma)$.

(b) (2.5 pts) Derive the following expansion formula for a complex z-plane, by defining $z = e^{\zeta}$, $\bar{z} = e^{\bar{\zeta}}$:

$$X(z,\bar{z}) = x - ip\log|z|^2 + i\sum_{n\neq 0}\frac{1}{n}(\alpha_n z^{-n} + \tilde{\alpha}_n \bar{z}^{-n}). \tag{2}$$

(c) (5 pts) The commutation relations for $x, p, \alpha_n, \tilde{\alpha}_n$ are given by

$$[x,p] = i, \qquad [\alpha_m,\alpha_n] = [\tilde{\alpha}_m,\tilde{\alpha}_n] = m\delta_{m+n,0} \tag{3}$$

with all other commutators vanishing. We define the creation-annihilation normal ordering by

$$\begin{aligned} :xp: &= :px: = xp, \\ :\alpha_m\alpha_{-n}: &= :\alpha_{-n}\alpha_m: = \alpha_{-n}\alpha_m, \\ :\tilde{\alpha}_m\tilde{\alpha}_{-n}: &= :\tilde{\alpha}_{-n}\tilde{\alpha}_m: = \tilde{\alpha}_{-n}\tilde{\alpha}_m \end{aligned} \tag{4}$$

where $m, n > 0$. Namely, it places all lowering operators ($\alpha_n, \tilde{\alpha}_n$ with $n > 0$) to the right of all raising operators ($\alpha_n, \tilde{\alpha}_n$ with $n < 0$). We include p with the lowering operators and x with the raising operators. Show that the following relation holds

$$X(z,\bar{z})X(w,\bar{w}) = :X(z,\bar{z})X(w,\bar{w}): - \log|z-w|^2 \tag{5}$$

if $|z| > |w|$. Note the identity

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x), \qquad |x| < 1. \tag{6}$$

Problem 2

Consider the 2-point function of primary fields in a 2-dimensional CFT:

$$G(z_1, z_2) = \langle \Phi_1(z_1) \Phi_2(z_2) \rangle. \tag{7}$$

Let the conformal dimensions of Φ_1, Φ_2 be $(h_1, \bar{h}_1)$ and $(h_2, \bar{h}_2)$. Namely, under the infinitesimal conformal transformation $z \to z + \epsilon(z)$, the fields $\Phi_{1,2}$ transform as

$$\delta_\epsilon \Phi_i(z) = [\epsilon(z)\partial + h_i \partial\epsilon(z)]\Phi_i(z), \qquad i = 1, 2. \tag{8}$$

(Actually there is also an antiholormorphic part depending on $\bar{\epsilon}(\bar{z})$, but in this problem we focus on the holomorphic part only.)

(a) Show that the invariance of the 2-point function under the conformal transformation implies the following equation:

$$[\epsilon(z_1)\partial_1 + h_1 \partial\epsilon(z_1) + \epsilon(z_2)\partial_2 + h_2 \partial\epsilon(z_2)]G(z_1, z_2) = 0. \tag{9}$$

(b) By setting $\epsilon(z) = 1$ in (9), show that $G(z_1, z_2)$ is a function of $x = z_1 - z_2$ only.

(c) By setting $\epsilon(z) = z$ in (9), show that $G(x)$ takes the following form:

$$G(x) = \frac{C}{x^{h_1 + h_2}} \tag{10}$$

with C a constant.

(d) By setting $\epsilon(z) = z^2$ in (9), show that $G(x)$ vanishes unless $h_1 = h_2$.

So, in this problem, we have shown

$$G(z_1, z_2) = \langle \Phi_1(z_1) \Phi_2(z_2) \rangle = \begin{cases} \dfrac{C}{(z_1 - z_2)^{2h}} & (h_1 = h_2 = h) \\ 0 & (h_1 \neq h_2) \end{cases} \tag{11}$$