Problem Set 2

February 9, 2009

Please hand in by February 16, 2pm.

1 Problem 2

In this problem we want to derive the mode expansion of the field $X^\mu(\sigma,\tau)$ (equations 2.40–2.41 of BBS) and the Poisson brackets satisfied by the modes (equations 2.51–2.52).

Mode expansion:

As we showed in class, if we pick a gauge in which $h_{\alpha\beta} = \eta_{\alpha\beta}$, then the field $X^\mu(\sigma,\tau)$ satisfies the wave equation:

$$\Box X^\mu(\sigma,\tau) = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu(\sigma,\tau) = 0. \quad (1)$$

The wave equation is a separable partial linear differential equation in terms of the variables σ, τ, so it has solutions of the form:

$$X^\mu(\sigma,\tau) = f(\sigma)\, g(\tau). \quad (2)$$

a) [1 pt] Applying this ansatz into the wave equation, show that the two functions must satisfy:

$$\frac{\partial^2 f(\sigma)}{\partial\sigma^2} = cf(\sigma), \qquad \frac{\partial^2 g(\tau)}{\partial\tau^2} = cg(\tau), \quad (3)$$

where c is an arbitrary constant.

b) [1 pt] Because the σ direction is compact, we have to make sure that the solution (2) satisfies the correct boundary condition:

$$X^\mu(\sigma+\pi,\tau) = X^\mu(\sigma,\tau). \quad (4)$$

Write the most general solution of the first equation in (3), and impose the boundary condition, to show that the constant c must take the values:

$$c = -4m^2, \qquad m \in \mathbb{Z}. \quad (5)$$

c) [2 pts] By taking linear combinations of solutions of the form (2) we can construct the most general solution of the wave equation. Show that it takes the form:

$$\begin{aligned} X^\mu(\sigma,\tau) &= X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma) \\ X^\mu_R &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-2in(\tau-\sigma)} \\ X^\mu_L &= \frac{1}{2}x^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau+\sigma)} \end{aligned} \tag{6}$$

where we have introduced the factors of l_s for convenience.

d) [2 pts] Notice that to find the mode expansion of the wave equation, we had to solve a differential equation of the form:

$$\frac{\partial^2 f}{\partial \sigma^2} = cf(\sigma) \tag{7}$$

with certain boundary conditions:

$$f(0) = f(\pi). \tag{8}$$

This is a special case of the more general eigenvalue problem:

$$\mathcal{L}f(\sigma) = cf(\sigma), \tag{9}$$

where $\mathcal{L}$ is a linear second order differential operator acting on the space of functions satisfying the boundary conditions (8). In our case the linear operator was $\mathcal{L} = \frac{\partial^2}{\partial\sigma^2}$. The constant c plays the role of the eigenvalue.

Let us recall some facts about second order differential equations and general properties of their solutions. One can show that if the operator $\mathcal{L}$ is hermitian, which is true for the case $\mathcal{L} = \frac{\partial^2}{\partial\sigma^2}$, then its eigenfunctions $f_n(\sigma)$,

$$\mathcal{L}f_n(\sigma) = c_n f_n(\sigma), \tag{10}$$

constitute a good basis for all (smooth enough) functions satisfying the boundary conditions (8).

This means the general function $g(\sigma)$ satisfying (8) can be written as:

$$g(\sigma) = \sum_n c_n f_n(\sigma). \tag{11}$$

The coefficients c_n can be easily computed by projecting both sides of the equation on the basis of eigenfunctions and using the orthogonality relation (14):

$$c_n = \int_0^\pi d\sigma f_n^*(\sigma) g(\sigma). \tag{12}$$

The completeness of the basis can also be expressed by the fact that the Dirac $\delta(\sigma)$ function can be given in terms of the orthonormal eigenfunctions $f_n(\sigma)$ by the formula

$$\delta(\sigma-\sigma') = \sum_n f_n(\sigma) f_n^*(\sigma'). \tag{13}$$

Recall that the eigenfunctions are orthonormal if they satisfy

$$\int_0^\pi d\sigma\, f_m^*(\sigma) f_n(\sigma) = \delta_{mn}. \tag{14}$$

In this homework we will assume as given the general theory reviewed here.

Now consider the special case $\mathcal{L} = \frac{\partial^2}{\partial\sigma^2}$, and (i) find its orthonormal eigenfunctions $f_n(\sigma)$ and (ii) show the Dirac function can be expressed as:

$$\delta(\sigma - \sigma') = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} e^{2in(\sigma-\sigma')}. \tag{15}$$

Poisson brackets:

Before we showed that the field $X^\mu(\sigma,\tau)$ has the expansion in terms of modes (6). The conjugate momentum to $X^\mu(\sigma,\tau)$ is given by:

$$P^\mu(\sigma,\tau) = \frac{\partial S}{\partial \dot{X}^\mu} = \frac{1}{\pi l_s^2} \dot{X}^\mu \tag{16}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$.

In this problem we want to start with the classical Poisson brackets for the field $X^\mu(\sigma,\tau)$ and its conjugate momentum $P^\mu(\sigma,\tau)$:

$$\begin{aligned} &[P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] = [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = 0 \\ &[P^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = \eta^{\mu\nu}\delta(\sigma-\sigma') \end{aligned} \tag{17}$$

and derive the Poisson brackets for the modes:

$$\begin{aligned} &[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = im\eta^{\mu\nu}\delta_{m+n,0} \\ &[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0 \\ &[p^\mu, x^\nu] = \eta^{\mu\nu} \end{aligned} \tag{18}$$

This can be done as follows:

e) [1 pt] Using (6) and (16) write the expansion of $P^\mu(\sigma,\tau)$ in terms of the modes $p^\mu, \alpha_m^\mu, \tilde{\alpha}_m^\mu$.

f) [2 pts] Substitute the mode expansions for $X^\mu(\sigma,\tau)$ and $P^\mu(\sigma,\tau)$ in the Poisson brackets (17). Simplify the resulting equations by projecting on the eigenfunction basis of the operator $\frac{\partial^2}{\partial\sigma^2}$. This is equivalent with comparing the coefficients of $e^{2i(m\sigma+n\sigma')}$ on both sides (why?).

g) [1 pt] Your results in f) should hold for any τ. Using this fact, solve for the Poisson brackets of the modes $x^\mu, p^\mu, \alpha_m^\mu, \tilde{\alpha}_m^\mu$ to get the final expressions (18).