Problem Set 11

May 4, 2009

Please hand in Problem 2 by May 11, 2pm.

Problem 1

All gravitational theories contain vacuum solutions describing gravitational waves. In this exersise we will derive a solution describes a gravitational wave travelling in the x_1 direction (Brinkmann (1923)).

Consider the line element

$$ds^2 = dx^+ dx^- + (H(x^i, x^+) - 1)(dx^+)^2 + \sum_{i=2}^{D-1} dx^i dx^i \quad (1)$$

where $x^\pm = x^1 \pm t$ and $H(x^i, x^+)$ is a function of x^i and x^+. Show that Einstein equations $(\mu, \nu = \{+, -, i\})$,

$$R_{\mu\nu} = 0, \quad (2)$$

imply

$$\partial^i \partial_i H = 0 \quad (3)$$

Useful formulae:

$$\begin{aligned}
\Gamma^\kappa_{\mu\nu} &= \frac{1}{2} g^{\kappa\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\
R_{\mu\nu\kappa}{}^\lambda &= \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\mu\kappa} - (\mu \leftrightarrow \nu) \\
R_{\mu\nu} &= R_{\mu\kappa\nu}{}^\kappa
\end{aligned}$$

Problem 2

Consider a gravitational wave in 10 dimensions travelling along the x_1 direction with H independent of x^+. Then the metric (1) does not depend on x^1 and we may compactify this direction.

(1) [4 pts] T-dualize along x^1 to obtain the solution describing the long range fields produced by a fundamental string,

$$\begin{aligned} ds^2 &= H^{-1}(-dt^2 + (dx^1)^2) + \sum_{i=2}^{9} dx^i dx^i \\ B_{01} &= H^{-1} - 1 \\ e^{-2\phi} &= H \end{aligned} \tag{4}$$

This is a solution of both IIA and IIB supergravity.

(2) [4 pts] In the case of IIB supergravity, S-dualize this solution to obtain the D1-brane solution

$$\begin{aligned} ds^2 &= H^{-1/2}(-dt^2 + (dx^1)^2) + H^{1/2} \sum_{i=2}^{9} dx^i dx^i \\ C_{01} &= H^{-1} - 1 \\ e^{-2\phi} &= H^{-1} \end{aligned} \tag{5}$$

Note that the solutions in (4) and (5) are in the string frame. Recall that S-duality acts as follows: (i) the metric in the Einstein frame is invariant, (ii) $B_{\mu\nu}$ becomes $C_{\mu\nu}$ and vice versa, (iii) $C^+_{\mu_1\mu_2\mu_3\mu_4}$ is invariant, (iv) the dilaton ϕ becomes $-\phi$. The Einstein frame metric g_E is related to the string frame metric g_S by $g_E = e^{-\phi/2} g_S$.

(3) [2 pts] Let us now consider the x^2 direction to be periodic as well and T-dualize along x^2 to obtain the D2 solution of IIA supergravity,

$$\begin{aligned} ds^2 &= H^{-1/2}(-dt^2 + (dx^1)^2 + (dx^2)^2) + H^{1/2} \sum_{i=3}^{9} dx^i dx^i \\ C_{012} &= H^{-1} - 1 \\ e^{-2\phi} &= H^{-1/2} \end{aligned} \tag{6}$$

Problem 3

The M2 brane solution of 11d supergravity is given by

$$\begin{aligned} ds^2 &= H^{-2/3}(-dt^2 + (dx^1)^2 + (dx^2)^2) + H^{1/3} \sum_{i=3}^{10} dx^i dx^i \\ A_{012} &= H^{-1} - 1 \end{aligned}$$

(a) Consider the x^2 direction as the M-theory direction. Show that the M2 solution reduces to the fundamental string solution (4) of IIA supergravity.

(b) Consider the x^3 direction as the M-theory direction. Show that the M2 solution reduces to the D2 solution (6) of IIA supergravity.