Perturbation theory at three loops with finite quark masses

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Outline

Motivation
QCD phase diagram
Perturbation theory
Finite mass effects at $T = 0$

Diagrammatic rules for $T = 0$ computations
Setup
Alternative formulation
Example: Three-loop pressure of QCD with massive quarks

Results

Conclusions
Summary

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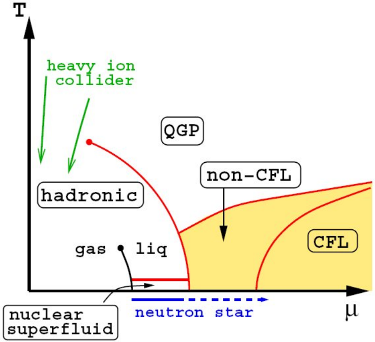
Basics of QCD thermodynamics

- Fundamental challenge in thermal QCD: Determine the phase diagram and equation of state
  - Lattice QCD most important tool at $T \sim T_c$ and small μ
  - Weak coupling methods required to bridge the gap to asymptopia and/or $\mu \neq 0$



Status of perturbation theory

- Expansion of pressure known to
  - High T, $\mu \lesssim T/g$: $\mathcal{O}(g^6 \ln g)$ (Kajantie, Laine, Schröder Rummukainen; AV)
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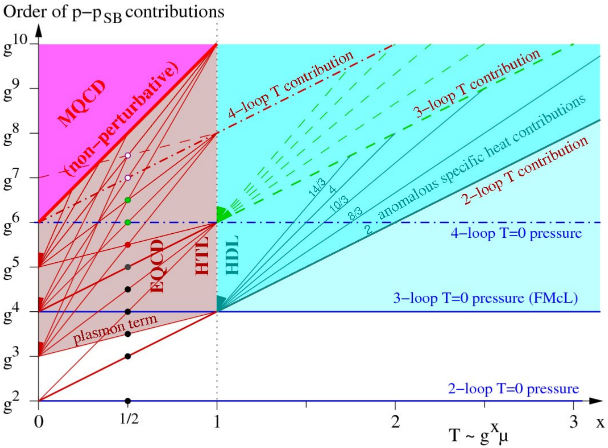

(Ipp, Kajantie, Rebhan, AV)

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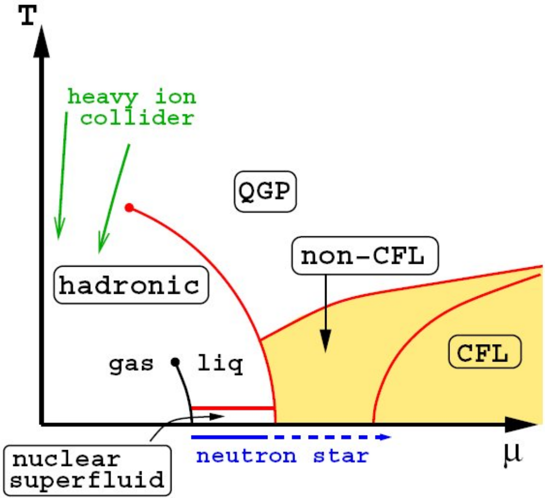
  - Indications that at high T Stefan-Boltzmann mass dependence sufficient
- Rest of the talk: How to go to three loops at $m_q \neq 0$?
  - Perhaps more importantly: Why??

Motivation, part I



Motivation, part I

- At $T \lesssim 100$MeV and $\mu_B \gtrsim 1$GeV, unpaired quark matter disfavored $\Rightarrow$ Color Superconductivity
  - At asymptotically high μ_B, physical phase CFL
- FAQ: When does CFL break down? What replaces it?
  - Well-known problem: 1GeV too low an energy for weak coupling methods to be useful
- More tractable question: At what μ_B does CFL *have to* break down, *i.e.* becomes disfavored wrt unpaired qm?
  - Addressable through a perturbative computation of u, d and s quark Fermi momenta, and comparison to the CFL gap
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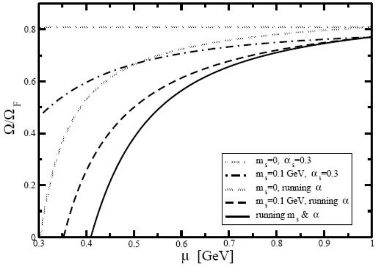
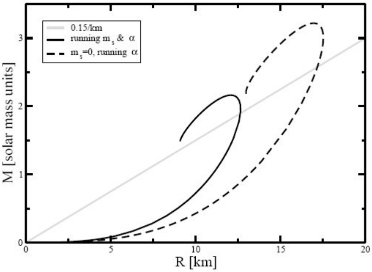
Motivation, part II

- Equation of state of unpaired $T = 0$ quark matter needed for quark star physics
  - Quark mass effects important (Fraga, Romatschke)
  - At two-loop level renormalization group running of g, m_q again sizable





Outline

Dressed propagators

- Technical challenge: Find optimal way to perform perturbative calculations at $T = 0$, $(\mu, m_q) \neq 0$
  - $\mu = 0$: Integration-by-parts identities, vast literature
  - $m_q = 0$: Well-developed, simple machinery even at finite (T, μ) — $3d$ Fourier transforms
- Combination problematic $\Rightarrow$ Need to build "new" machinery
- After scalarization, end up with computing $d = 4 - 2\epsilon$ dim. integrals with μ-dep. massive fermion propagators

$$\frac{1}{(p_0 + i\mu)^2 + \mathbf{p}^2 + m^2}$$

- Obvious issue: Temporal integrations complicate life

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$$\frac{1}{(p_0 + i\mu)^2 + \mathbf{p}^2 + m^2} \rightarrow \frac{1}{p_0^2 + \mathbf{p}^2 + m^2} - 2\pi i \delta(p_0^2 + \mathbf{p}^2 + m^2)\, \theta_{\mu,\, p_0}\,,$$
$$\theta_{\mu,\, p_0} \equiv \theta(\mu - \mathrm{Im}\, p_0)\, \theta(\mathrm{Im}\, p_0),$$

Cutting rules

- Another formulation of the method: Cutting rules
  - After performing Lorentz algebra, cut up to n fermionic lines in a 'scalar graph' with n loops
- Cutting a line $\Leftrightarrow$ Integrate over three-momentum after
  - Placing cut line on shell
  - Replacing p_0 integral by weight factor
- In the end, perform renormalization and remaining $3d$ integrals
  - Integrands of $3d$ integrals $4 - 2\epsilon$ dim. $2n$-point functions at $\mu = 0$

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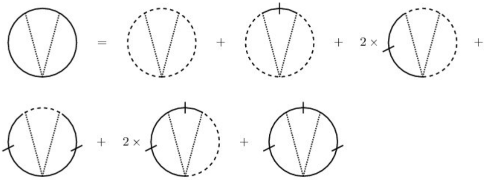
- E.g. at three loop order integrands are:
  - $n = 0$: 3-loop $4 - 2\epsilon$ dimensional massive vacuum bubbles
  - $n = 1$: 2-loop $4 - 2\epsilon$ dimensional massive 2-pt functions with external legs on shell
  - $n = 2$: 1-loop $4 - 2\epsilon$ dimensional massive 4-pt functions with external legs on shell
  - $n = 3$: Tree-level 6-pt functions with external legs on shell



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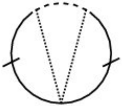
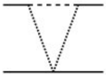
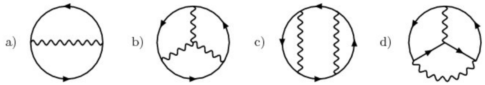


$$= \int \frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}} \frac{\theta(\mu - E(\boldsymbol{p}))}{2E(\boldsymbol{p})} \int \frac{d^{3-2\epsilon}r}{(2\pi)^{3-2\epsilon}} \frac{\theta(\mu - E(\boldsymbol{r}))}{2E(\boldsymbol{r})} \times \left\{ \quad \right\}$$



2PI graphs

- Main task: Evaluate the two- and three-loop 2PI graphs using the new technique



- Recipe
  - 1. Perform scalarization of all diagrams
  - 2. Cut the fermion lines
  - 3. Use IBP identities to reduce 2- and 4-pt functions to a small set of masters
  - 4. Evaluate masters
  - 5. Perform renormalization and external integrations
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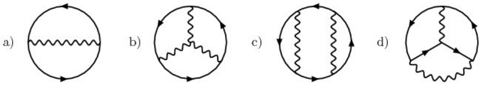

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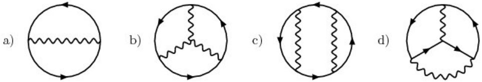

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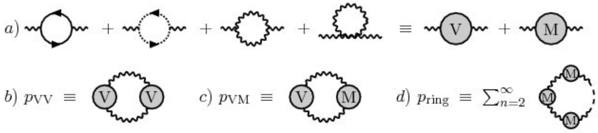

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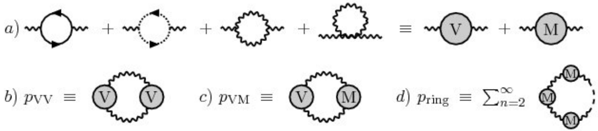

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No results yet... but coming up very soon.

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Summary

- Effects of finite quark masses in perturbative QCD computations interesting and potentially important especially at small T
- "New", efficient scheme developed for $T = 0$, finite μ calculations at finite masses
  - Use of existing $T = \mu = 0$ diagrammatic results essential
- Three-loop corrections to the equation of state of unpaired quark matter almost there
  - Effects on breakdown of CFL phase and quark star physics potentially important
  - Ask next speaker for results