2PI effective action for gauge theories: Renormalization

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2PI in a nutshell

The 2PI effective action

An approximation scheme in Quantum Field Theory:

- Good convergence properties.
- It captures certain non-perturbative aspects.
- Wide range of applications in- and out-of-equilibrium.

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How solid are its theoretical grounds?

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How solid are its theoretical grounds?

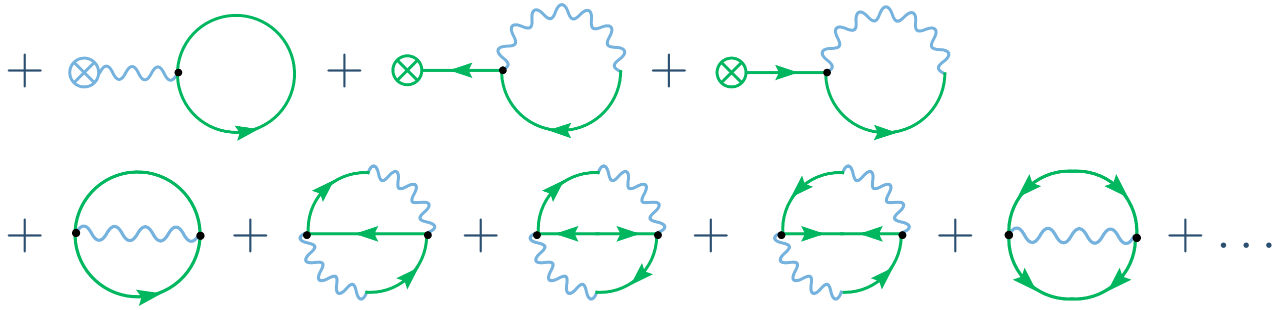
QED in the covariant gauge with gauge fixing parameter ξ

$$S[\varphi] = \int d^d x \left\{ \bar{\psi} \left[i\partial\!\!\!/ - eA\!\!\!/ - m \right] \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right\} \underbrace{- \frac{1}{\xi} \int d^d x \, (\partial^\mu A_\mu)^2}_{S_{\mathrm{gf}}[A]}$$

The 2PI effective action

The 2PI effective action for Quantum Electrodynamics

$$\Gamma_{\mathrm{2PI}}[\varphi,\mathcal{G}] = S[\varphi] + \frac{i}{2}\mathrm{Str}\,\log\mathcal{G}^{-1} + \frac{i}{2}\mathrm{Str}\,\mathcal{G}_0^{-1}\,\mathcal{G}$$



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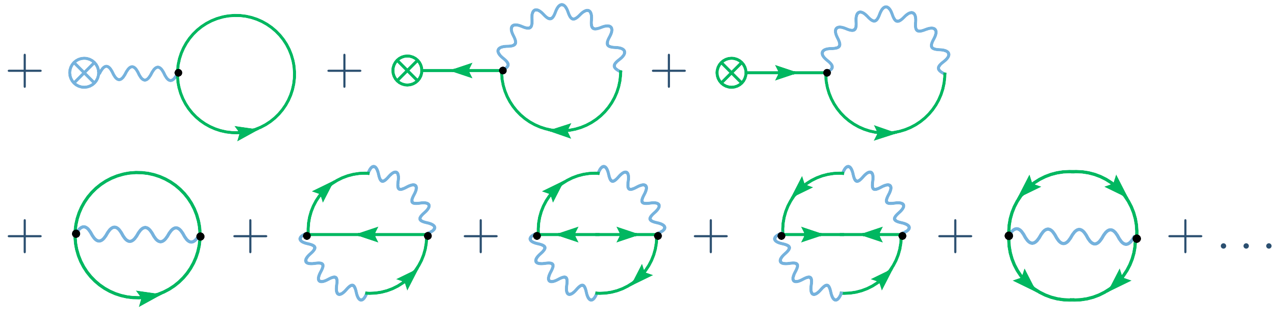


$$+ \ldots$$

Recall the superfield notations (talk by Julien Serreau)

$$\varphi \equiv \begin{pmatrix} A \\ \psi \\ \bar{\psi}^t \end{pmatrix} \quad \text{and} \quad \mathcal{G} \equiv \langle \varphi\varphi^t \rangle_c \equiv \begin{pmatrix} G & K^t & \bar{K} \\ K & F & D \\ \bar{K}^t & -D^t & \bar{F} \end{pmatrix}$$

The 2PI effective action

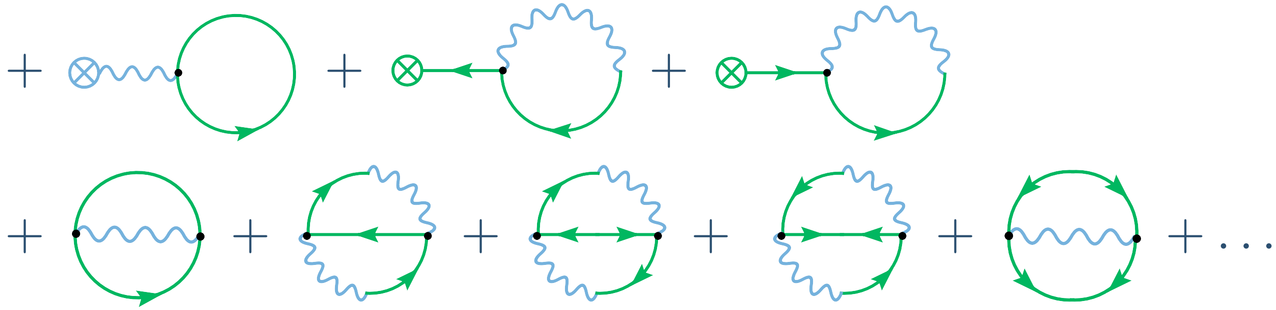
2PI in a nutshell
● 2PI Effective action
● 2PI Vertices
● 2PI WT identities

Renormalization

Summary

The 2PI effective action for Quantum Electrodynamics

$$\Gamma_{2\text{PI}}[\varphi,\mathcal{G}] \;=\; S[\varphi] + \frac{i}{2}\text{Str}\,\log\mathcal{G}^{-1} + \frac{i}{2}\text{Str}\,\mathcal{G}_0^{-1}\,\mathcal{G}$$



The physics is obtained from a variational principle

$$\Gamma[\varphi] = \Gamma_{2\text{PI}}[\varphi,\bar{\mathcal{G}}[\varphi]] \quad \text{where} \quad 0 = \left.\frac{\delta\Gamma_{2\text{PI}}}{\delta\mathcal{G}}\right|_{\bar{\mathcal{G}}[\varphi]}$$

A given truncation of $\Gamma_{2\text{PI}}[\varphi,\mathcal{G}]$ defines an approximation of $\Gamma[\varphi]$.

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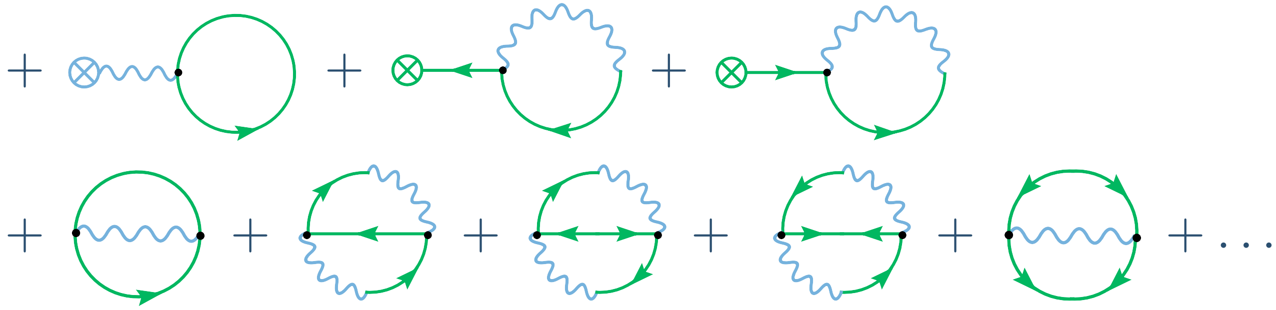

To which extent does $\Gamma[\varphi]$ reflect the basic properties of the theory?

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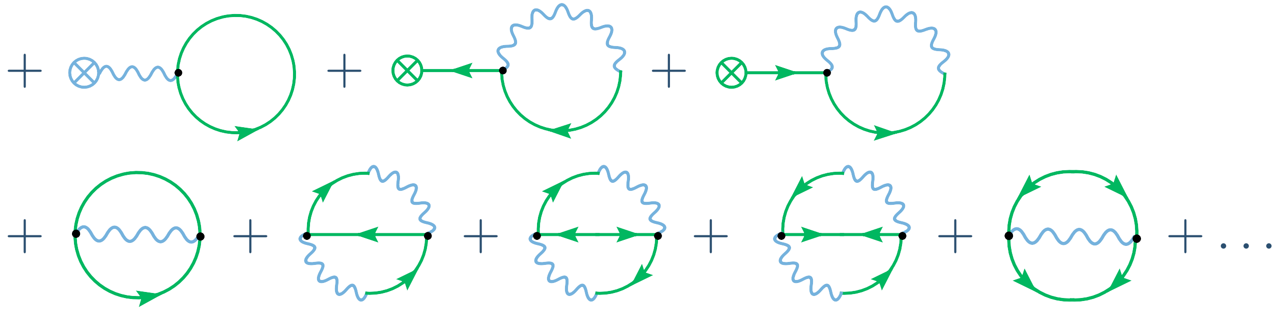

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Symmetry?

Renormalizability?

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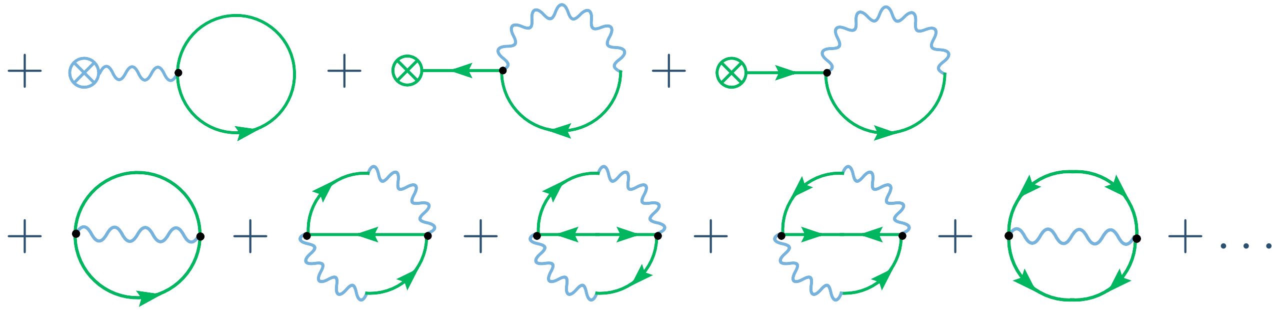

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✓ Ward-Takahashi identities.

? Complete gauge-fixing independence.

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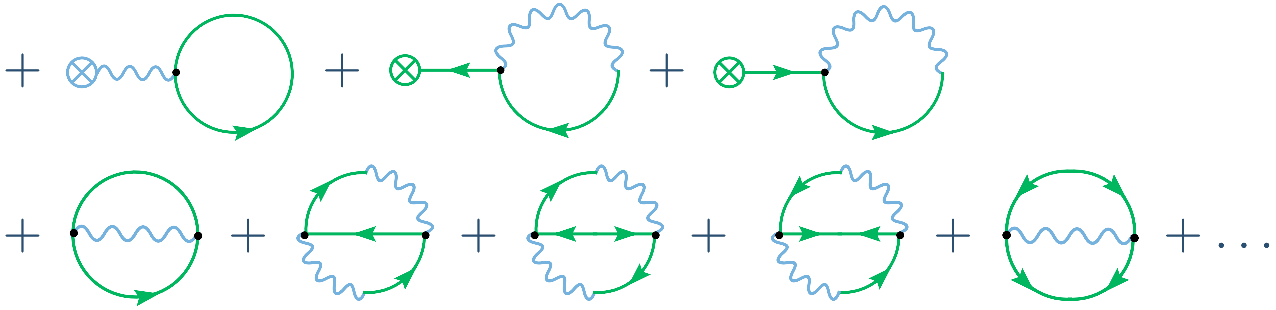

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Renormalizability?

⇒ Vertices.

2PI and 2PI-resummed vertices

In a given approximation, two distinct ways to define vertices

$$\boxed{\text{2PI-resummed vertices}}\quad \frac{\delta^n\Gamma}{\delta\varphi_n\cdots\delta\varphi_1} = \frac{i\,\delta^{n-2}\bar{\mathcal{G}}^{-1}_{12}}{\delta\varphi_n\cdots\delta\varphi_3}\quad \boxed{\text{2PI vertices}}$$

In the exact theory, these two definitions coincide.

2PI and 2PI-resummed vertices

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In a given approximation, they become different.

Approximation artefact: at a given order of approximation

$$\frac{\delta^n\Gamma}{\delta\varphi_n\cdots\delta\varphi_1} - \frac{i\,\delta^{n-2}\bar{\mathcal{G}}^{-1}_{12}}{\delta\varphi_n\cdots\delta\varphi_3} = \mathcal{O}\big(\text{higher order contributions}\big)$$

Ward-Takahashi identities

At any approximation order, the 2PI effective action is gauge invariant

$$\delta_\alpha \left(\Gamma_{2\mathrm{PI}}[\varphi,\mathcal{G}] - S_{\mathrm{gf}}[A]\right) = 0$$

under a generalized gauge transformation

$$\delta_\alpha \varphi = \begin{pmatrix} -(1/e)\partial\alpha \\ ie\,\alpha\,\psi \\ -ie\,\alpha\,\bar{\psi}^t \end{pmatrix} \quad \text{and} \quad \delta_\alpha \mathcal{G} = \begin{pmatrix} \boxed{\delta_\alpha G = 0} & \delta_\alpha K^t & \delta_\alpha \bar{K} \\ \delta_\alpha K & \delta_\alpha F & \delta_\alpha D \\ \delta_\alpha \bar{K}^t & -\delta_\alpha D^t & \delta_\alpha \bar{F} \end{pmatrix}$$

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WT identity for the 2PI-resummed four-photon vertex [UR & J. Serreau, JHEP 0711:097 (2007)]

$$\partial^\mu\,\frac{\delta^4\Gamma}{\delta A^\mu\delta A^\nu\delta A^\rho\delta A^\sigma}=0$$

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WT identity for the 2PI-resummed two-photon vertex [UR & J. Serreau, JHEP 0711:097 (2007)]

$$\partial^\mu \frac{\delta^2\Gamma}{\delta A^\mu \delta A^\nu} = i\partial^\mu G^{-1}_{0,\mu\nu}$$

BUT! 2PI two-photon vertex unconstrained [UR & J. Serreau, JHEP 0711:097 (2007)]

$$i\partial^\mu \bar{G}^{-1}_{\mu\nu} = i\partial^\mu G^{-1}_{0,\mu\nu} + \mathcal{O}(\text{higher order contributions})$$

Renormalization

Renormalization factors

QED is renormalizable: one can redefine the fields

$$A_b^\mu \equiv Z_3^{1/2}(d)A^\mu \qquad \psi_b \equiv Z_2^{1/2}(d)\psi \qquad \bar\psi_b \equiv Z_2^{1/2}(d)\bar\psi$$

as well as the parameters of the theory

$$Z_2(d)m_b \equiv Z_0(d)m \qquad Z_2(d)Z_3^{1/2}(d)e_b \equiv Z_1(d)e \qquad \frac{Z_3(d)}{\xi_b} = \frac{Z_4(d)}{\xi}$$

in order to remonve all UV divergences from the vertices, as $d \to 4$.

Exact theory

In terms of renormalized fields, the 2PI effective action gets the additional contribution

$$\begin{aligned}\delta\Gamma_{\rm 2PI} &= \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x)\\ &+\frac{\delta Z_3}{2}\int_x \Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y}+\cdots\end{aligned}$$

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The 2PI-resummed photon propagator

gets the additional contribution

$$\begin{aligned}\frac{\delta^2\Gamma}{\delta A^\mu\delta A^\nu} &= \delta Z_3\big(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu\big)\\ &+\cdots\end{aligned}$$

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$$\frac{\delta^2\Gamma}{\delta A^\mu\delta A^\nu}=i\bar G^{-1}_{\mu\nu}$$

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$$\delta\Gamma_{2\mathrm{PI}} = \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2 - \partial_x^\mu\partial_x^\nu\Big)A_\nu(x) + \frac{\delta\bar{Z}_3}{2}\int_x \Big[\Big(g^{\mu\nu}\partial_x^2 - \partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y} + \cdots$$

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$$\frac{\delta^2\Gamma}{\delta A^\mu\delta A^\nu} = \delta Z_3\big(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\big) + \cdots$$

$$\delta Z_3 \neq \delta\bar{Z}_3$$

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$$\frac{\delta^2\Gamma}{\delta A^\mu\delta A^\nu} = \delta Z_3\big(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu\big) + \cdots$$

$$\delta Z_3 \neq \delta\bar{Z}_3,\ \delta\bar{Z}_L,\ \delta\bar{M}^2$$

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$$\begin{aligned}\bar{G}^{-1}_{\mu\nu} &= \delta\bar{Z}_3\big(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu\big)\\ &+\ i\,\delta\bar{Z}_L\,\partial_\mu\partial_\nu+i\,\delta\bar{M}^2 g_{\mu\nu}+\cdots\end{aligned}$$

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Approximations

In terms of renormalized fields, the 2PI effective action gets the additional contribution

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All these new contributions do not affect the gauge invariance of the 2PI effective action

$$\delta_\alpha G=0\;\Rightarrow\;\delta_\alpha\Big(\Gamma_{\rm 2PI}+\delta\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

$\Rightarrow$ new counterterms allowed by symmetry [UR & J. Serreau, in preparation]

Approximations

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&+\frac{\delta\bar Z_3}{2}\int_x\Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y}+\cdots\\
&+\frac{\delta\bar Z_L}{2}\int_x\Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y}+\frac{\delta\bar M^2}{2}\int_x G_\mu^\mu(x,x)\\
&+\frac{\delta\bar g_1}{8}\int_x G_\mu^\mu(x,x)G_\nu^\nu(x,x)+\frac{\delta\bar g_2}{4}\int_x G^{\mu\nu}(x,x)\,G_{\mu\nu}(x,x)
\end{aligned}
$$

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Approximations

2PI in a nutshell

Renormalization
- Renormalization factors I
- Renormalization factors II
- Renormalized invariance
- Renormalization conditions

Summary

In terms of renormalized fields, the 2PI effective action gets the additional contribution

$$\begin{aligned}
\delta\Gamma_{\rm 2PI} \; = \; & \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x)\\
& + \frac{\delta\bar Z_3}{2}\int_x\Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y}+\cdots\\
& + \frac{\delta\bar Z_L}{2}\int_x\Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y}+\frac{\delta\bar M^2}{2}\int_x G_\mu^\mu(x,x)\\
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\boxed{\text{not wanted!}} \; : \; & \int_x A_\mu(x)G^{\mu\nu}(x,y)A_\nu(x)\,,\quad \int_x A_\mu(x)A^\mu(x)G_\nu^\nu(x,x)
\end{aligned}$$

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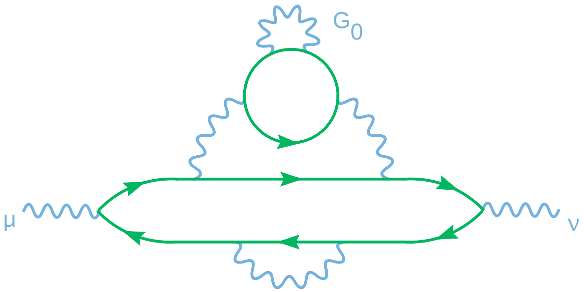
Four photon leg subgraphs

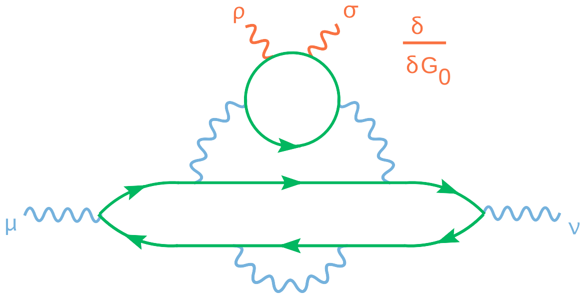
The 2PI photon propagator $\bar{G}^{-1}_{\mu\nu}$ contains subgraphs involving four photons legs

$$\bar{G}^{-1}_{\mu\nu} = \cdots +$$



$$+ \cdots$$

$$\frac{\delta}{\delta G_0^{\rho\sigma}}\left(\bar{G}^{-1}_{\mu\nu}\right) = \cdots +$$



$$+ \cdots \propto \bar{V}_{\mu\nu,\rho\sigma}$$

The same applies to the 2PI-resummed photon propagator $\delta^2\Gamma/\delta A^\mu \delta A^\nu$

$$\frac{\delta}{\delta G_0^{\rho\sigma}}\left(\frac{\delta^2\Gamma}{\delta A^\mu \delta A^\nu}\right) \propto \frac{\delta^2 \bar{G}^{-1}_{\rho\sigma}}{\delta A^\mu \delta A^\nu}$$

Renormalized 2PI effective action

2PI in a nutshell

Renormalization
- Renormalization factors I
- Renormalization factors II
- Renormalized invariance
- Renormalization conditions

Summary

A similar analysis on other 2PI and 2PI-resummed vertices leads to

$$
\begin{aligned}
\delta\Gamma_{\rm 2PI} = {} & \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x) \\
& + \delta Z_2\int_x \bar\psi(x)i\partial\!\!\!/_x\psi(x) - m\,\delta Z_0\int_x\bar\psi(x)\psi(x) \\
& - e\,\delta Z_1\int_x\bar\psi(x)A\!\!\!/(x)\psi(x) \\
& + \frac{\delta\bar Z_3}{2}\int_x\Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y} \\
& + \frac{\delta\bar Z_L}{2}\int_x\Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y} + \frac{\delta\bar M^2}{2}\int_x G^\mu_\mu(x,x) \\
& + \frac{\delta\bar g_1}{8}\int_x G^\mu_\mu(x,x)G^\nu_\nu(x,x) + \frac{\delta\bar g_2}{4}\int_x G^{\mu\nu}(x,x)\,G_{\mu\nu}(x,x) \\
& - \delta\bar Z_2\int_x \mathrm{tr}\,[i\partial\!\!\!/_x D(x,y)]_{x=y} + \delta\bar m\int_x \mathrm{tr}\,D(x,x) \\
& + e\,\delta\bar Z_1\int_x \mathrm{tr}\,[A\!\!\!/(x)D(x,x)] - e\,\delta\tilde Z_1\int_x\big[\bar\psi(x)K(x,x)+\bar K(x,x)\psi(x)\big]
\end{aligned}
$$

Renormalized 2PI effective action

A similar analysis on other 2PI and 2PI-resummed vertices leads to

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\begin{aligned}
\delta\Gamma_{\rm 2PI} = {} & \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x)\\
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& - e\,\delta Z_1\int_x \bar\psi(x)\not{A}(x)\psi(x)\\
& + \frac{\delta\bar Z_3}{2}\int_x \Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y}\\
& + \frac{\delta\bar Z_L}{2}\int_x \Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y} + \frac{\delta\bar M^2}{2}\int_x G^\mu_\mu(x,x)\\
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Renormalized gauge invariance

The unrenormalized 2PI effective action is gauge invariant

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

Renormalized gauge invariance

2PI in a nutshell

Renormalization
- Renormalization factors I
- Renormalization factors II
- Renormalized invariance
- Renormalization conditions

Summary

The unrenormalized 2PI effective action is gauge invariant

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

The renormalized 2PI effective action is a priori not gauge invariant

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}+\delta\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=\left(\bar Z_2-Z_2-(\bar Z_1-Z_1)\frac{\bar Z_2}{Z_1}\right)\delta_\alpha\int_x\bar\psi(x)i\partial\!\!\!/_x\psi(x)$$

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WT identities + Renormalization conditions

$$\left(\bar{Z}_2-Z_2-(\bar{Z}_1-Z_1)\frac{\bar{Z}_2}{Z_1}\right)=0$$

It is then possible to define a gauge-invariant renormalization scheme.

Renormalization conditions

13 counterterms need 13 renormalization conditions!

$$\begin{aligned}
\delta\Gamma_{2\mathrm{PI}} = {} & \frac{\delta Z_3}{2}\int_x A_\mu(x)\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)A_\nu(x)\\
& + \delta Z_2\int_x \bar\psi(x)\,i\not\partial_x\psi(x) - m\,\delta Z_0\int_x \bar\psi(x)\psi(x)\\
& - e\,\delta Z_1\int_x \bar\psi(x)\not A(x)\psi(x)\\
& + \frac{\delta\bar Z_3}{2}\int_x \Big[\Big(g^{\mu\nu}\partial_x^2-\partial_x^\mu\partial_x^\nu\Big)G_{\mu\nu}(x,y)\Big]_{x=y}\\
& + \frac{\delta\bar Z_L}{2}\int_x \Big[\partial_x^\mu\partial_x^\nu G_{\mu\nu}(x,y)\Big]_{x=y} + \frac{\delta\bar M^2}{2}\int_x G^\mu_\mu(x,x)\\
& + \frac{\delta\bar g_1}{8}\int_x G^\mu_\mu(x,x)G^\nu_\nu(x,x) + \frac{\delta\bar g_2}{4}\int_x G^{\mu\nu}(x,x)\,G_{\mu\nu}(x,x)\\
& - \delta\bar Z_2\int_x \mathrm{tr}\,[i\not\partial_x D(x,y)]_{x=y} + m\,\delta\bar Z_0\int_x \mathrm{tr}D(x,x)\\
& + e\,\delta\bar Z_1\int_x \mathrm{tr}\,[\not A(x)D(x,x)] - e\,\delta\tilde Z_1\int_x\big[\bar\psi(x)K(x,x)+\bar K(x,x)\psi(x)\big]
\end{aligned}$$

Renormalization conditions

Green counterterms (δZ_1 here) fix the parameters of the theory

$$e\,\gamma^{\mu}_{\bar{\alpha}\alpha} = \delta Z_1 e\,\gamma^{\mu}_{\bar{\alpha}\alpha} + \left.\frac{\delta^3\Gamma}{\delta A_\mu \delta\psi_\alpha \delta\bar{\psi}_{\bar{\alpha}}}\right|_{\star}$$

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Blue counterterms ($\delta\bar Z_1$ and $\delta\tilde Z_1$ here) restore unicity of vertices at the renormalization point

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Blue counterterms ($\delta\bar{Z}_1$ and $\delta\tilde{Z}_1$ here) restore unicity of vertices at the renormalization point

$$\frac{i\delta\bar{\mathcal{G}}^{-1}_{\bar{\alpha}\alpha}}{\delta A_\mu}\neq\frac{i\delta\bar{\mathcal{G}}^{-1}_{\bar{\alpha}\mu}}{\delta\psi_\alpha}\neq\frac{\delta^3\Gamma}{\delta A_\mu\delta\psi_\alpha\delta\bar{\psi}_{\bar{\alpha}}}$$

Renormalization conditions

Green counterterms (δZ_1 here) fix the parameters of the theory

$$e\,\gamma^{\mu}_{\bar{\alpha}\alpha}=\delta Z_1 e\,\gamma^{\mu}_{\bar{\alpha}\alpha}+\left.\frac{\delta^3\Gamma}{\delta A_\mu\delta\psi_\alpha\delta\bar{\psi}_{\bar{\alpha}}}\right|_{\star}$$

Blue counterterms ($\delta\bar{Z}_1$ and $\delta\tilde{Z}_1$ here) restore unicity of vertices at the renormalization point

$$\delta\bar{Z}_1 e\,\gamma^{\mu}_{\bar{\alpha}\alpha}+\left.\frac{i\delta\bar{\mathcal{G}}^{-1}_{\bar{\alpha}\alpha}}{\delta A_\mu}\right|_{\star}=\delta\tilde{Z}_1 e\,\gamma^{\mu}_{\bar{\alpha}\alpha}+\left.\frac{i\delta\bar{\mathcal{G}}^{-1}_{\bar{\alpha}\mu}}{\delta\psi_\alpha}\right|_{\star}=\delta Z_1\,e\,\gamma^{\mu}_{\bar{\alpha}\alpha}+\left.\frac{\delta^3\Gamma}{\delta A_\mu\delta\psi_\alpha\delta\bar{\psi}_{\bar{\alpha}}}\right|_{\star}$$

Renormalization conditions

Green counterterms (δZ_1 here) fix the parameters of the theory

$$e\,\gamma^{\mu}_{\bar\alpha\alpha} = \delta Z_1 e\,\gamma^{\mu}_{\bar\alpha\alpha} + \left.\frac{\delta^3\Gamma}{\delta A_\mu \delta\psi_\alpha \delta\bar\psi_{\bar\alpha}}\right|_\star$$

Blue counterterms ($\delta\bar Z_1$ and $\delta\tilde Z_1$ here) restore unicity of vertices at the renormalization point

$$\delta\bar Z_1 e\,\gamma^{\mu}_{\bar\alpha\alpha} + \left.\frac{i\delta\bar{\mathcal{G}}^{-1}_{\bar\alpha\alpha}}{\delta A_\mu}\right|_\star = \delta\tilde Z_1 e\,\gamma^{\mu}_{\bar\alpha\alpha} + \left.\frac{i\delta\bar{\mathcal{G}}^{-1}_{\bar\alpha\mu}}{\delta\psi_\alpha}\right|_\star = \delta Z_1\, e\,\gamma^{\mu}_{\bar\alpha\alpha} + \left.\frac{\delta^3\Gamma}{\delta A_\mu \delta\psi_\alpha \delta\bar\psi_{\bar\alpha}}\right|_\star$$

Red counterterms restore the transversality of γ-vertices at the renormalization point

$$k^\mu\left[i\bar G^{-1}_{\mu\nu} + \delta\bar M^2 g_{\mu\nu} + \delta\bar Z_L k_\mu k_\nu\right]_\star = k^\mu \left.\frac{\delta^2\Gamma}{\delta A_\mu \delta A_\nu}\right|_\star = k^\mu i G^{-1}_{0,\mu\nu}$$

Summary

Summary

Approximations of the 2PI effective action for QED are gauge invariant:

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

The corresponding – 2PI and 2PI-resummed – vertices are renormalizable:

- Doubled counterterms: $(\delta Z_3,\delta\bar{Z}_3)$, $(\delta Z_2,\delta\bar{Z}_2)$, $(\delta Z_1,\delta\bar{Z}_1)$, $(\delta Z_0,\delta\bar{Z}_0)$.
- Additional counterterms: $\delta\bar{g}_1$, $\delta\bar{g}_2$, $\delta\bar{Z}_L$ and $\delta\bar{M}^2$.

The renormalization procedure is consistent:

- All these new features are allowed by the symmetry:

$$\delta_\alpha\Big(\Gamma_{\rm 2PI}+\delta\Gamma_{\rm 2PI}-S_{\rm gf}[A]\Big)=0$$

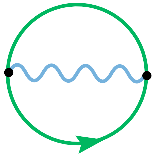
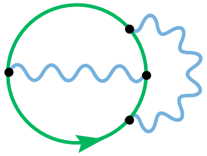
- No new parameters in the game.

$\Rightarrow$ $\boxed{\text{Systematic application of 2PI techniques to abelian gauge theories}}$ $\Leftarrow$

Gauge-fixing dependence

Example: Consider the pressure of the system

$$\mathcal{P}_{2\mathrm{PI}} = -\mathrm{Tr}\left[\ln \bar{D}^{-1}(\xi) + D_0^{-1}\,\bar{D}(\xi)\right] + \frac{1}{2}\mathrm{Tr}\left[\ln \bar{G}^{-1}(\xi) + G_0^{-1}(\xi)\,\bar{G}(\xi)\right]$$

$+$  $+$  $+ \ldots$

At a given order of approximation, there is a residual gauge-fixing dependence

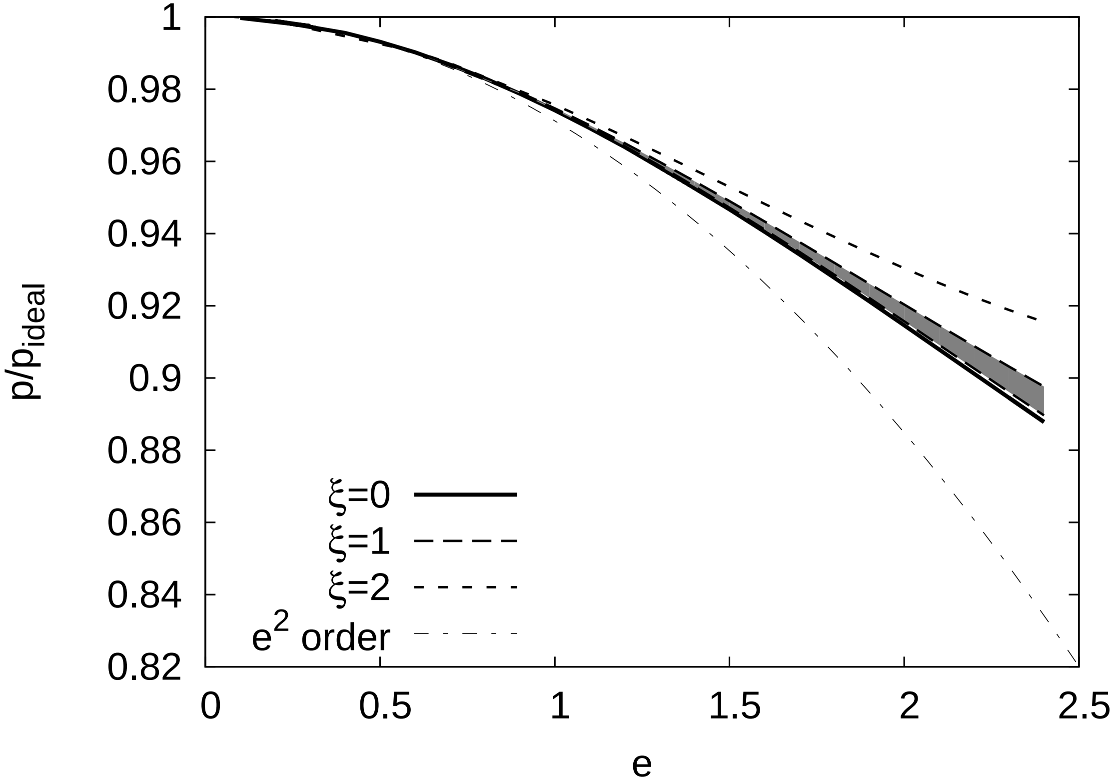
At order e^2, $\quad \dfrac{d}{d\xi}\mathcal{P}_{2\mathrm{PI}} = \mathcal{O}(e^4)$

At order e^4, $\quad \dfrac{d}{d\xi}\mathcal{P}_{2\mathrm{PI}} = \mathcal{O}(e^6)$, and so on

Convergence $\Rightarrow$ The residual gauge-fixing dependence is an asset rather than a problem.

Two-loop result (1/2)

In the range of converge $\xi \in [0,2]$, the ξ-dependence is not dramatically big: comparable to the μ-dependence in the range $\mu \in [\pi T, 4\pi T]$

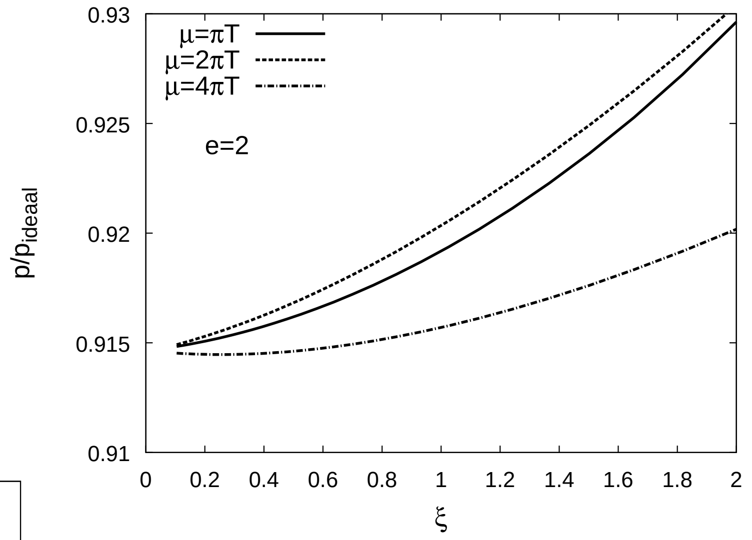


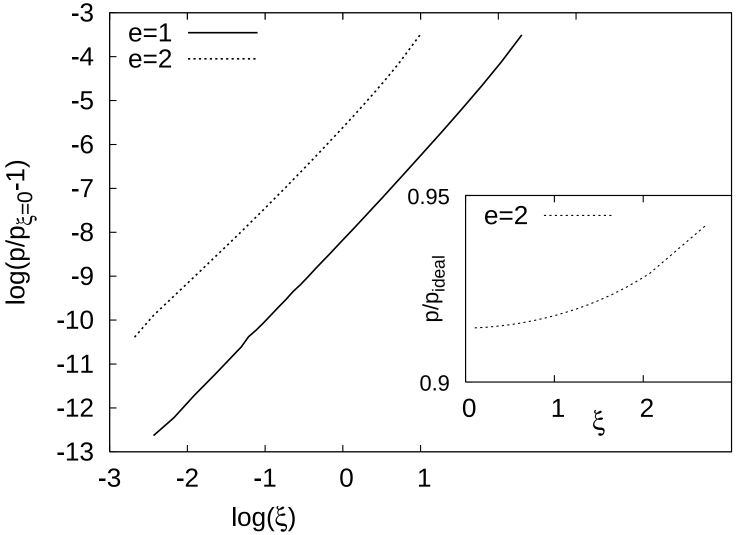
UR & Sz. Borsányi, PLB 661 (2008)

Two-loop result (2/2)

Minimum sensitivity obtained for $\xi = 0$ (Landau gauge):

μ dependence minimal for $\xi = 0$





$$\mathcal{P}_{\rm 2PI}(\xi) - \mathcal{P}_{\rm 2PI}(0) \sim \alpha(e,\mu)\,\xi^2$$