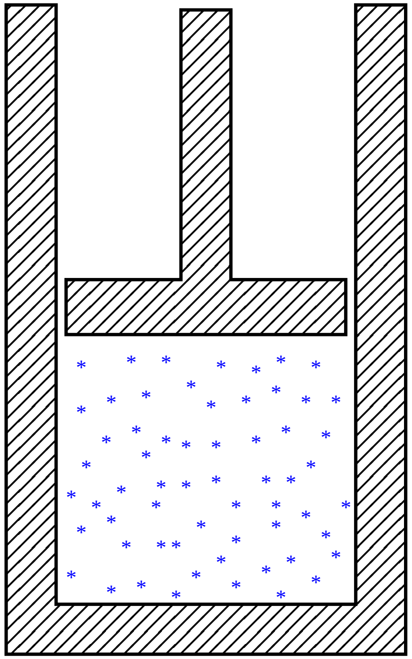
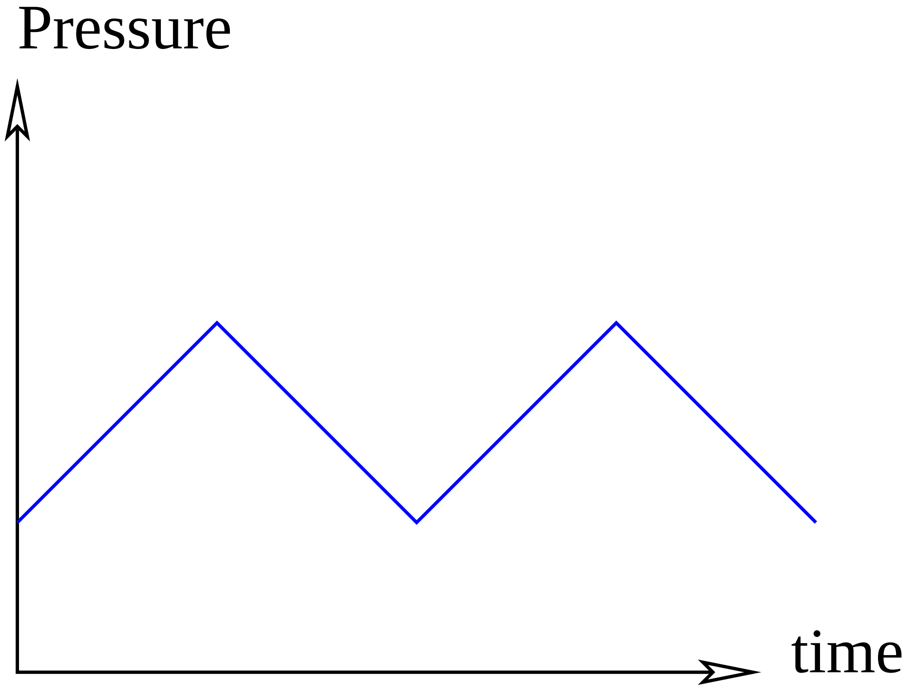
T^μ_μ spectral function and bulk viscosity
Where we can calculate it

Guy D. Moore, Omid Saremi

- Review of bulk viscosity, spectral function
- Perturbative regime: kinetic theory
  - ∗ High frequency: rising cut
  - ∗ Low frequency: peak
- Near the critical point: universal scaling
  - ∗ Dynamical universality classes: QCD vs. liquid-gas
  - ∗ Critical slowing down and Bulk viscosity
- Summary and conclusions

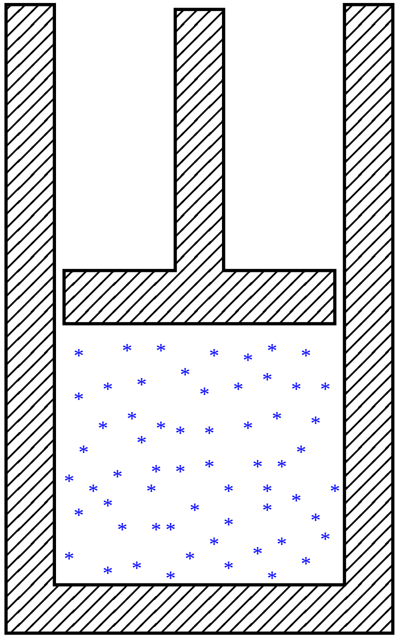
Raise and lower a piston: compress and decompress gas

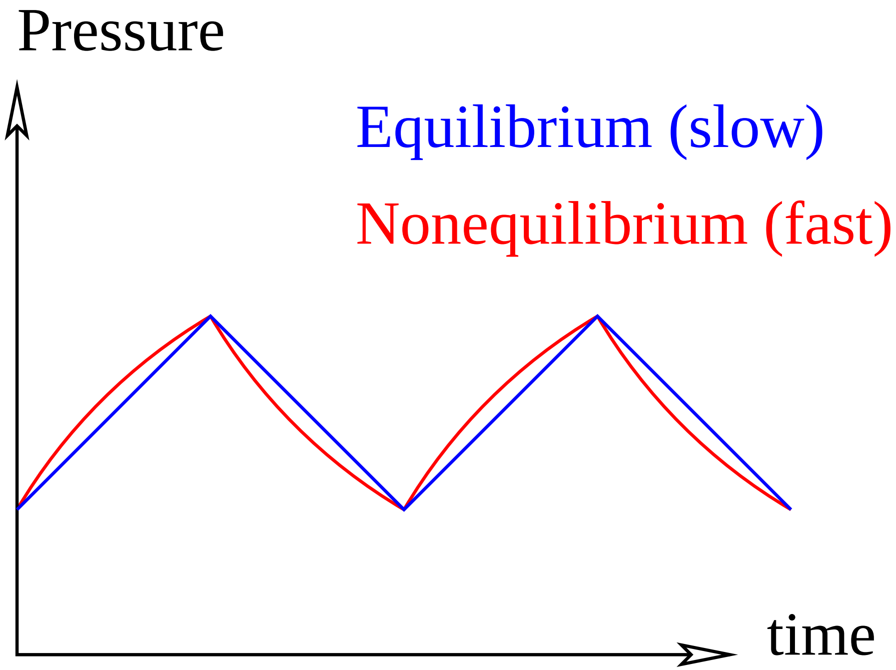




Pressure rises and falls as you compress and decompress.

Compress faster: pressure deviates from equilibrium version



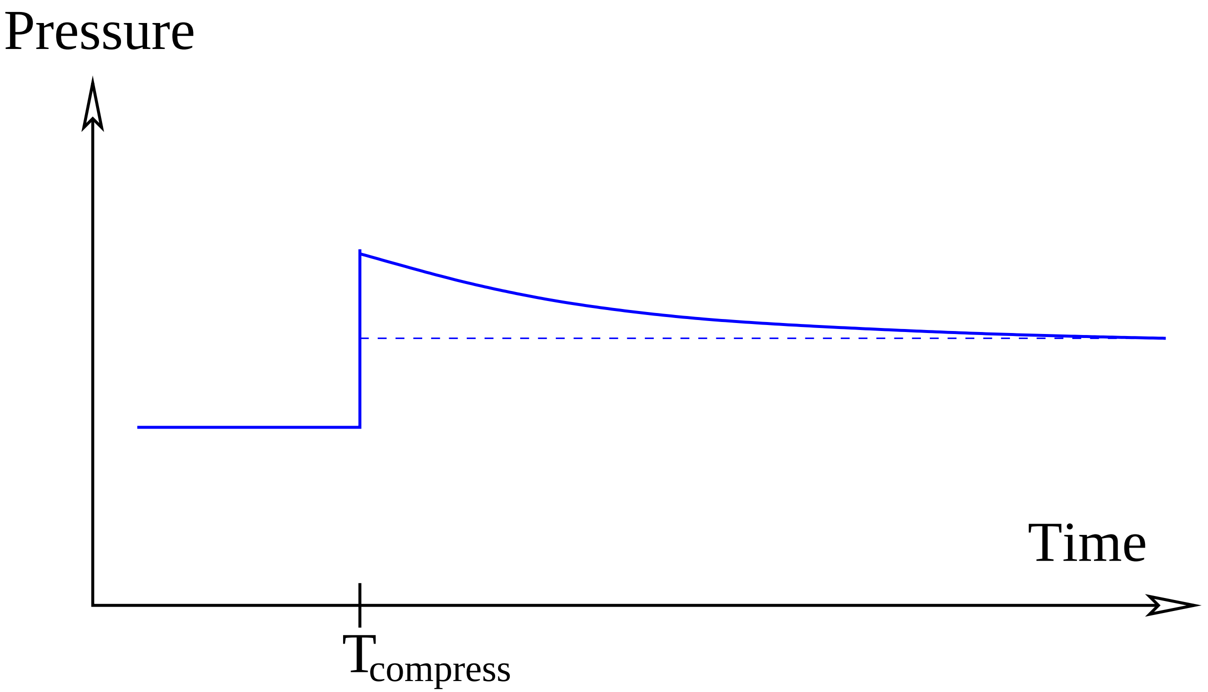


Compression: pressure higher

Decompression: pressure lower Second Law of Thermodynamics

Difference is characterized by Bulk Viscosity

Consider small, sudden compression:



If operator $\mathcal{O}_1$ causes compression, $\mathcal{O}_2$ measures P:

- $\lim_{t\to 0}\langle \mathcal{O}_1(0)\mathcal{O}_2(t)\rangle$ gives height of discontinuity
- $\int_0^\infty dt\,\langle \mathcal{O}_1(0)\mathcal{O}_2(t)\rangle$ gives area under difference curve. Scale by $\Delta V/V$: that defines bulk visc ζ.

Bulk viscosity:

$$\int dt(P - P_{\rm eq}) = -\zeta \Delta V/V = -\zeta \int dt\, \vec{\nabla}\cdot\vec{v}$$

or

$$P - P_{\rm eq} = -\zeta\, \vec{\nabla}\cdot\vec{v}$$

Related to correlator of pressure operator $\mathcal{O}_2 = P = \frac{1}{3}T^i_i$ and gen. of expansions $\mathcal{O}_1 = \mathcal{O}_2$. Usual arguments:

$$\zeta = \frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\int_{-\infty}^{\infty} dt e^{i\omega t}\int d^3x \left\langle \frac{1}{9}\Big[\, T^i_i(x,t)\,,\, T^j_j(0,0)\,\Big]\right\rangle\ .$$

And small t response described by ω integral.

Is it T^i_i? Or T^μ_μ?

It doesn't matter! $\left[\mathcal{O}_1^\dagger\,,\,\mathcal{O}_1\right] = \left[\mathcal{O}_1^\dagger + c\,,\,\mathcal{O}_1 + c\right]$.
T^0_0 acts almost like constant[a] as energy is conserved.

Useful choices:

- T^i_i: intuitively clear
- T^μ_μ: sum rules and exact results
- $T^i_i - \langle T^i_i \rangle \simeq T^i_i + 3c_\mathrm{s}^2 T^0_0$: allows KMS

$$\int dt e^{i\omega t} \left\langle \mathcal{O}^\dagger(t)\mathcal{O}(0) \right\rangle = \frac{e^{\omega/T}}{e^{\omega/T}-1} \int dt e^{i\omega t} \left\langle \left[\mathcal{O}(t)\,,\,\mathcal{O}(0) \right] \right\rangle$$

without need to subtract disconnected part

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[a] almost–T^{00} shift adds t-independent contrib: $\delta(\omega)$, const in $G_\mathrm{E}(\tau)$.

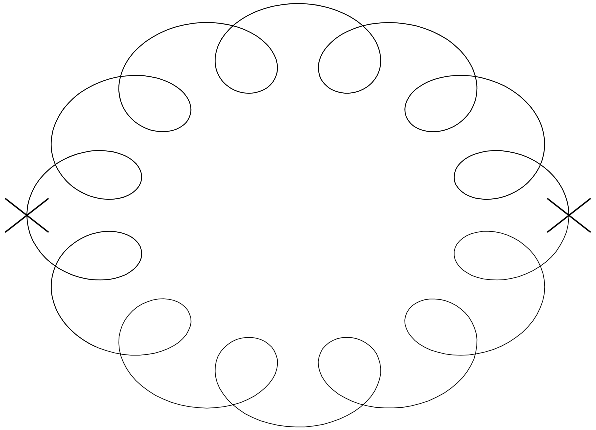
Perturbative regime

Normalize so $S = \int d^4x \frac{1}{2g^2} \mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu}$.

Do pure glue for simplicity. Conformal anomaly:

$$T^{\mu}_{\mu} = \frac{\beta}{g^4} \mathrm{Tr}\, G_{\mu\nu}G^{\mu\nu} \,, \qquad \beta \equiv \frac{\mu^2 d}{d\mu^2} g^2 \sim g^4 \,.$$

Evaluate Wightman correlator of $(\beta/g^4)G^2 - (1-3c_{\mathrm{s}}^2)T^0_0$.



Leading diagram.

Note $(1-3c_{\mathrm{s}}^2) \sim g^4$ is small;

$(1-3c_{\mathrm{s}}^2)T^0_0$ is g^2 suppressed.

Leading perturbative result

$$G^{>}(Q)=\frac{2\beta^2 d_{\rm A}}{9g^4}\int\frac{d^4P d^4R}{(2\pi)^8}G^{>}_{\mu\alpha}(P)G^{>}_{\nu\beta}(R)(2\pi)^4\delta^4(Q{-}P{-}R)$$
$$\times(g^{\mu\nu}P{\cdot}R-P^\mu R^\nu)(g^{\alpha\beta}P{\cdot}R-P^\alpha R^\beta)$$

Cut propagator

$$G^{>}_{\mu\nu}(P)=\left[n_b(p^0){+}1\right]2\pi\delta(P^2{+}m_\infty^2)\sum_\lambda \epsilon_\mu(\lambda)\epsilon^*_\nu(\lambda)\,,$$

Two contributions:

- Cut (continuum): both lines positive frequency
- Pole ($\delta(\omega)$): one pos, one neg frequency

“Pole” contribution

Pole in G_{R}, δ in spectral weight.

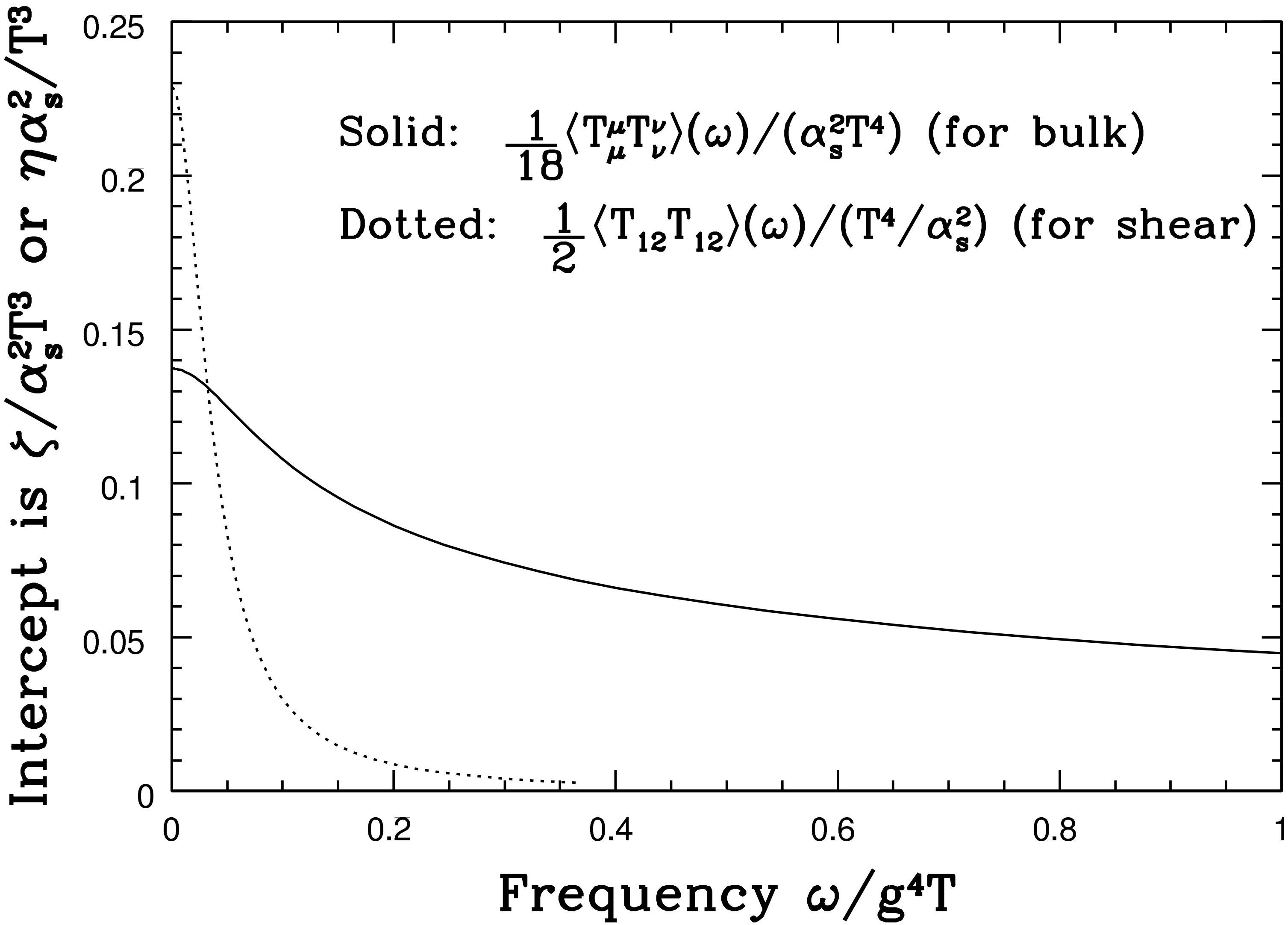
Area of delta function:

$$G^{>}_{\mathrm{pole}}(\omega,0) \;=\; \delta(\omega)\frac{2}{9}\,2d_{\mathrm{A}}\frac{1}{4\pi}\int_0^\infty n(p)(1+n(p)) \times\left[\left(\frac{1}{3}-c_{\mathrm{s}}^2\right)p^2+\frac{\beta m_\infty^2}{g^2}\right]^2 dp\,.$$

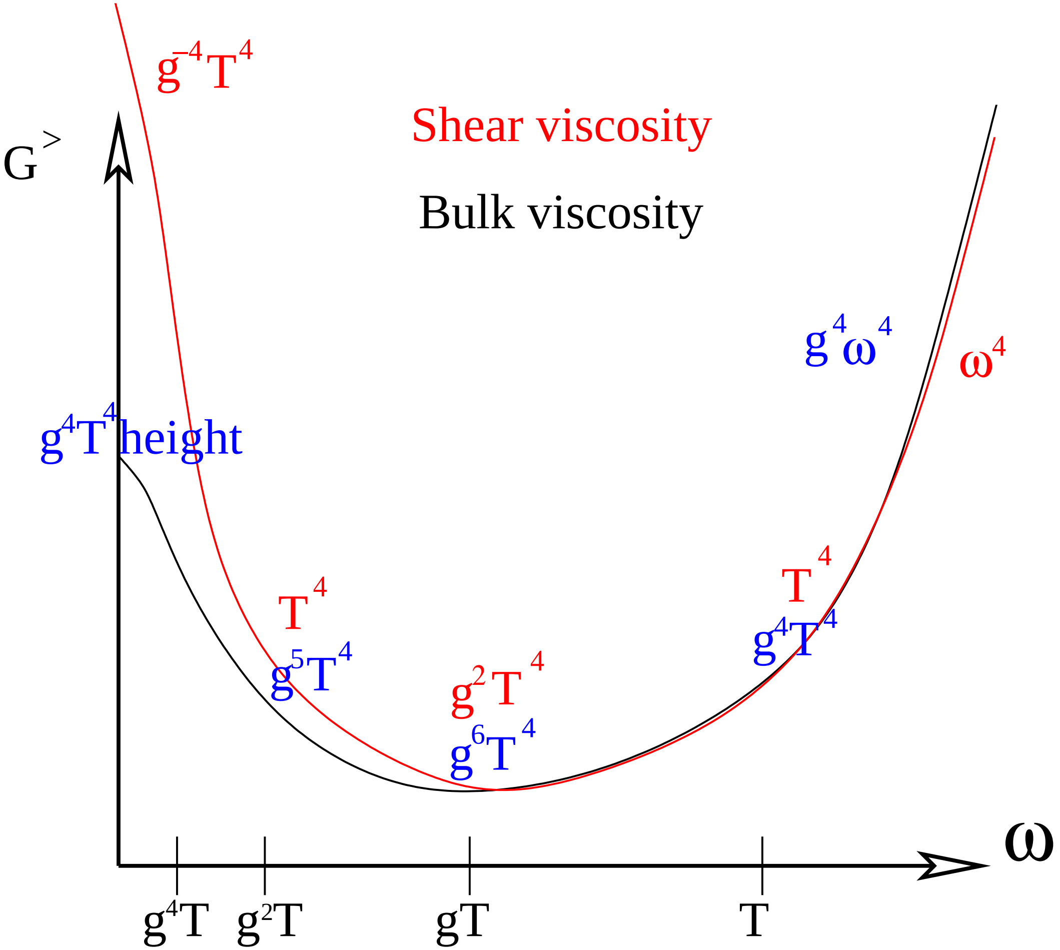
IR singular: Order g^7T^4.

Detailed shape: need kinetic theory at finite ω.

Shape of low frequency peak

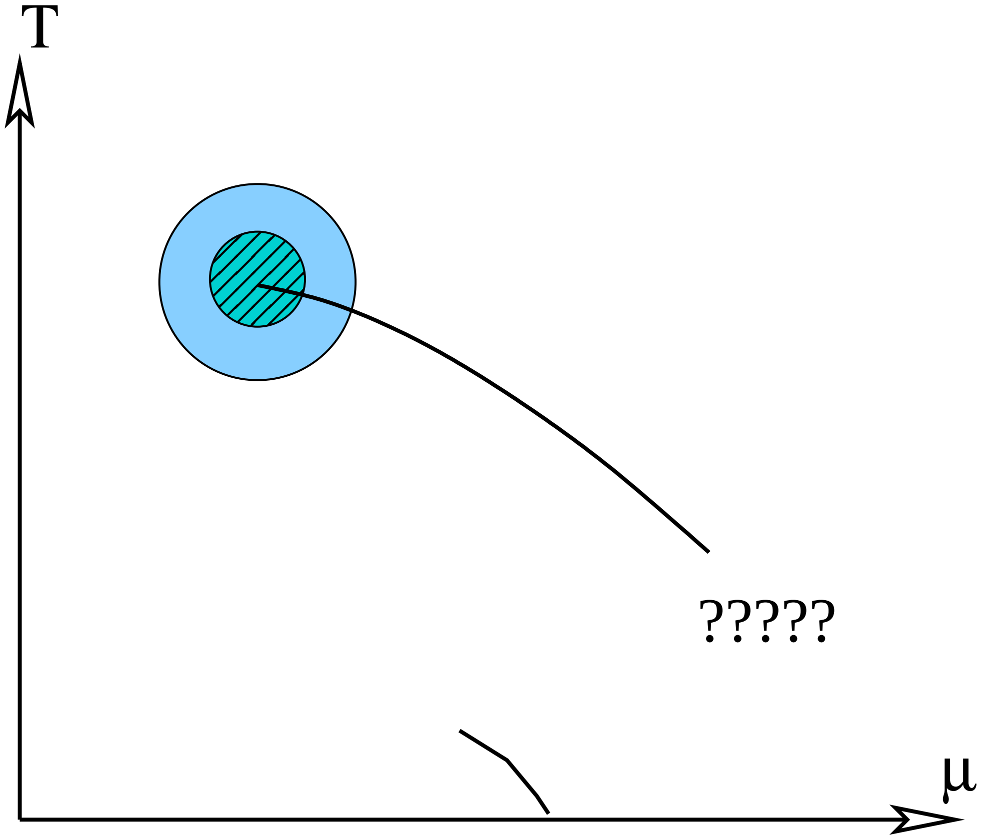


Summary:


$g^{-4}T^4$
Shear viscosity
Bulk viscosity
$G^>$
$g^4\omega^4$
ω^4
g^4T^4 height
T^4
g^5T^4
g^2T^4
g^6T^4
T^4
g^4T^4
ω
g^4T
g^2T
gT
T

Another analytically tractible case

Critical region near second-order transition point:



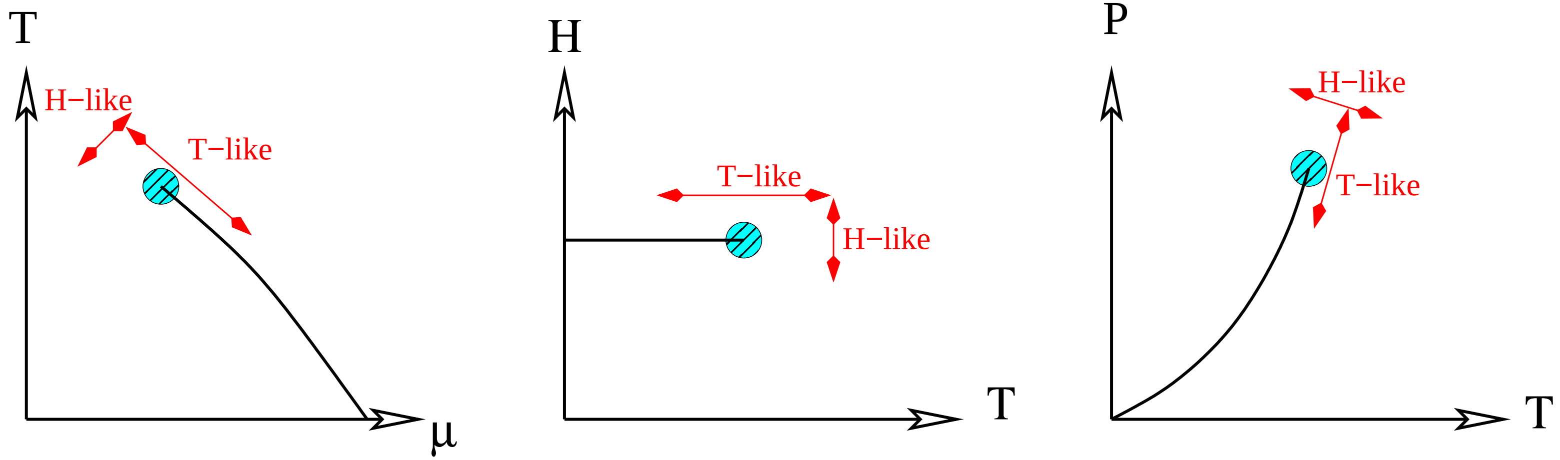
Possible to compute parametric behaviors analytically

Static universality

“Chiral” phase transition not true symmetry breaking.

Order parameter $\boldsymbol{\psi} = \langle\bar{\psi}\psi\rangle$ same universality as Ising

T, μ map linearly into H, T (Ising) or T, P (liquid-gas)



Corr. length $\xi \sim (T - T_c)^{-\nu} = t^{-\nu}$ $[\nu = 0.630]$

Heat capacity $C_v \sim t^{-\alpha}$ $[\alpha = .110]$ but dP/dt nonsingular

Dynamics

Two sets of degrees of freedom:

- Short-distance: $\Delta P/\Delta E \sim 1$. Rapid equilibration
- Long-range [ψ fluct]: $\Delta P/\Delta E = 0$. Dominate C_v.

Long range equilibrate diffusively and slowly:

$$\langle \boldsymbol{\psi}(k,t)\boldsymbol{\psi}(-k,0)\rangle \sim \chi(k)\,\exp(-t/\tau)\,, \qquad \tau \sim k^{-z}$$

with z dynamic critical exponent.

Sudden compression:

- First, short-dist DOF adjust, $\Delta P \sim \Delta E$
- Later, ψ adjusts, ΔP relaxes back in time $\propto \xi^{-z}$

Dynamic universality: value of z

Long scale dynamics essentially hydrodynamic

Hohenberg Halperin Rev Mod Phys 49 p435 (1977)

Depend on what quantities are conserved (ψ is not)

Conserved: $T^{0\mu} = (\epsilon, \vec{P})$ and ρ_B

Liquid-gas system: $\epsilon, \vec{P}, \rho$ conserved.

Same dynamic universality as Liquid-Gas

Son Stephanov hep-ph/0401052

Dynamics analyzed to death by CM physics people: $z \simeq 3$.

Even bulk viscosity analyzed Onuki, Phys Rev E 55 p403 (1997)

Previous argument: $\Delta V/V$ leads to instantaneous $\Delta P \sim \Delta E \sim T^4 \Delta V/V$.

But ΔP relaxes to $\simeq 0$ in time $\tau \sim \xi^z$.

Hence, expect $\zeta \sim T^3(\xi T)^z \sim T^3 t^{-z\nu} \simeq T^3 t^{-2}$.

Subtlety: $\boldsymbol{\psi}$ relaxes in stages from $k \sim T$ to $k \sim \xi^{-1}$.

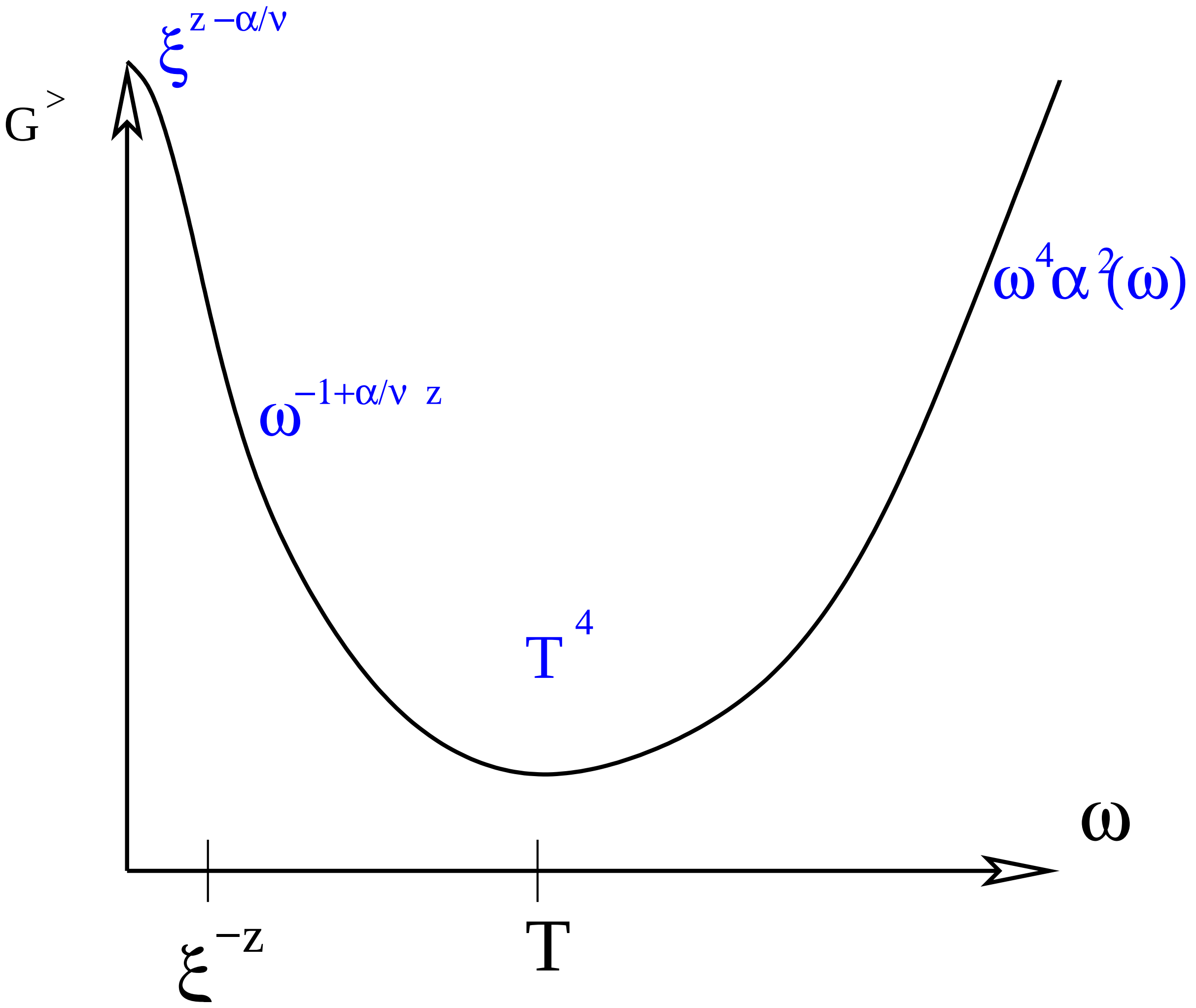
At time τ, modes with $k > \tau^{-1/z}$ are equilibrated.

These carry $C_v \sim k^{-\alpha/\nu}$. So $\Delta P \sim \Delta E k^{\alpha/\nu}$.

Integrate this behavior for all intermediate times....

$$\zeta \sim T^3(\xi T)^{z-\alpha/\nu} \sim T^3 t^{-z\nu+\alpha}$$

Slow dynamics: another low ω peak!



Implications for Euclidean correlators

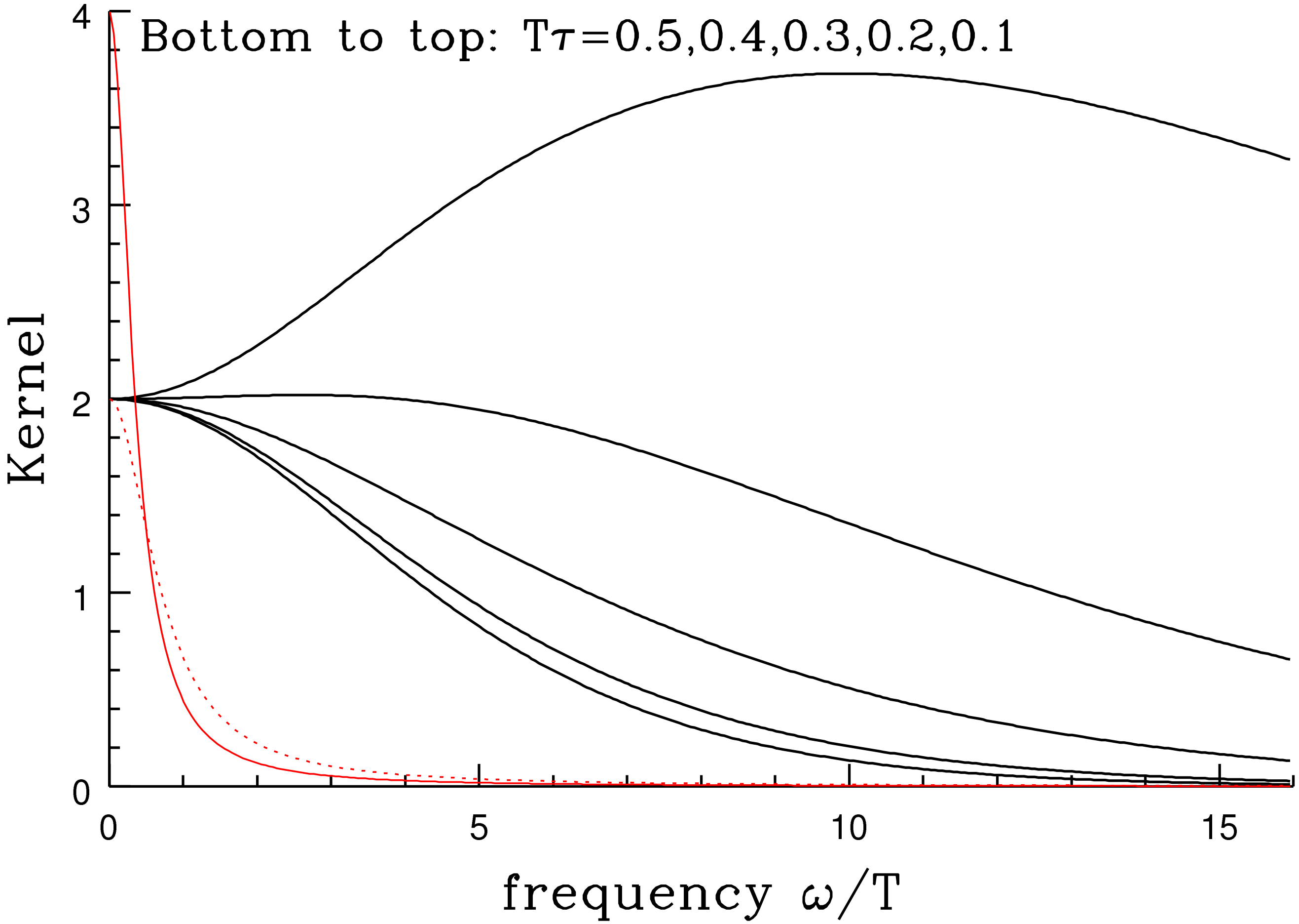
Integral relation between $G_\mathrm{E}(\tau)$ and ρ:

$$
\begin{aligned}
G_\mathrm{E}(\tau) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\left\langle \left[\mathcal{O}_1 \,, \mathcal{O}_1 \right] \right\rangle (\omega)}{\omega} K(\omega,\tau)\,, \\
K(\omega,\tau) &= \frac{\omega \cosh[\omega(\tau - \beta/2)]}{\sinh(\beta\omega/2)}\,.
\end{aligned}
$$

Knowing ρ, can compute G_E.

Then, ask how we could “guess” that ρ from G_E.

The function $K(\omega, \tau)$



Peak near zero gives common contrib to $G_E(\tau)$, all τ

Implications for spectral function

Sharp peak: contribution to $G_{\mathrm{E}}(\tau)$

- Almost independent of shape of peak
- Almost independent of value of τ

All $G_{\mathrm{E}}(\tau)$ raised by common amount: Area under peak.

- Determined by static universality
- Does not diverge as $T \to T_c$

Shape of peak essential to finding $\zeta =$height.
Very hard to determine from $G_{\mathrm{E}}(\tau)$.

Conclusions

- Slow equilibration $\Rightarrow$ Peak in spectral function
- Slow equil. at weak coupling: "Wide shouldered" peak
- Slow equilibration near critical point:
  nearly ω^{-1} shaped peak, $\zeta \sim \xi^{z-\alpha/\nu}$
- Euclidean Green function cannot find shape of peak

If crossover is rapid (near-critical): big ζ

In this case, Euclid. methods CANNOT say much directly

we CAN say something useful from universality+static info

Comment on Kharzeev and Tuchin

Their sum rule:

$$G_{\rm E}(\omega' = 0) = \int d\omega \frac{\rho}{\omega} = \frac{T^5 \partial}{\partial T} \left(\frac{\epsilon - 3P}{T^4} \right)$$

$g \ll 1$: RHS$\sim g^6 T^4$. LHS$\sim g^7 T^4$[peak]$+g^4 T^4$[cut]

Similar sum rule for $T_{xy} T_{xy}$ correlator

$$G_{\rm E, T_{xy} T_{xy}}(\omega' = 0) = \int d\omega \frac{\rho}{\omega} = P$$

Making same assumption about *shape* of peak as they make gives $\eta \sim T^3/g$ not T^3/g^4.

Also, convergence/validity questions with Kramers-Kronig