IAC2009: Gamma-Ray Bursts — Problem set week 1
Solutions

Problem 1: Isotropy of the bright stars

The bright stars are roughly isotropic in the sky. How far away can they be?

We know that the stars locally have a disk-like distribution in space, with the Sun close to the midplane. If the stars we see were on average more than one disk thickness away, we would clearly see that they prefer a belt in the sky around the Galactic Plane. Isotropy therefore means most must be less than one disk scale height away. The disk thickness varies from 70–100 pc for young stars to 300–400 pc for old stars, so the limit is in this range. Actually the majority of bright stars we see are a few times more massive than the Sun, so the smaller scale height is more appropriate.

Problem 2: Supernova rate and star formation

The supernova rate in a galaxy is one per century. What is (a) The birth rate of stars above 10 $\mathrm{M}_\odot$? (b) The total star formation rate in $\mathrm{M}_\odot$/yr. (c) The half-life of the molecular gas reservoir?

Let $N(>M)$ be the birth rate of stars above mass M; usually we approximate $N(>M) = K(M/M_\odot)^{-x}$, where common values of x are 1.35 (Salpeter) and 1.7 (Miller-Scalo). Let's take $x = 1.5$. The differential distribution function of mass is then $-\mathrm{d}N/\mathrm{d}M$ (why the minus sign?), or

$$-\frac{\mathrm{d}N}{\mathrm{d}M} = x\frac{K}{M_\odot}\left(\frac{M}{M_\odot}\right)^{-x-1} .$$

The total star formation rate is then found by integrating the differential distribution times the mass over all masses above some minimum, resulting in

$$SFR = \frac{x}{x-1} K M_\odot \left(\frac{M_{\rm min}}{M_\odot}\right)^{-x+1} .$$

Since $10\,M_\odot$ is the minimum mass for a star to give a supernova, the birth rate of stars above that limit should equal the supernova rate, i.e., one star per century. From the normalisation to the supernova rate of 0.01/yr, we get $K \simeq 0.3$/yr, which then gives $SFR \simeq 3M_\odot$/yr, for a reasonable value of $M_{\rm min} = 0.1M_\odot$.

A typical galaxy like ours has a few times $10^9 M_\odot$ of cold atomic and molecular gas, so at this SFR its half life is about 1 Gyr.

Problem 3: Supernova remnant

A supernova with explosion energy 10^{51} erg has expanded to 10 pc size in normal ISM. (a) What is its total mass? (b) What are its temperature and expansion speed? (c) How old is it?

Let us assume an average ISM density for the SN environment, i.e., 1 atom per cc. The total mass of ISM swept up is then simply the volume of the SNR times the density, which comes to $100\,M_\odot$ for the numbers we have assumed. This is much more than the typical 5–10 $M_\odot$ of stellar envelope that would be ejected, so we will neglect the latter.

To get the temperature and expansion speed, we take an advance on results to be derived later for SN shocks, namely that the SNR evolves adiabatically at first, and that its energy is roughly equally divided between thermal energy and bulk kinetic energy. The thermal energy is then $E_{th} = 3NkT/2$, where N is the number of particles swept up. Setting this equal to half the explosion energy, we get $T \simeq 10^7$ K (or half that if one is more precise and notes that because the gas is fully ionised, we have 2 particles for each proton mass). Similarly, setting $E_k = Mv_{exp}^2/2$, we find $v_{exp} \simeq 700$ km/s.

To get the age of the remnant, we can simply estimate it as $t = R/v_{exp}$, which for the numbers we find above gives t =14 kyr. If we assume that indeed the kinetic energy of expansion is constant, we can derive a more precise expansion law: $E_k \propto Mv^2 = cst$, from which we see, since $v = dR/dt$, that $R^3(dR/dt)^2 = cst$. That we can write as an easy differential equation, $R^{3/2}(dR/dt) = cst$, which solves to $d(R^{5/2})/dt = cst$, or $R^{5/2} = k.t$, and thus $R(t) \propto t^{2/5}$. By diffentiation, we can then see that in terms of the current size and expansion speed, the true age is $t = (2/5)R/v$, which in the present case is about 5,600 yr.