Oefententamen

1. (a) Show that

$$\int_0^1 p(x)dx = \tfrac{1}{2}(p(0)+p(1)) + \tfrac{1}{12}(p'(0)-p'(1))$$

for any $q \in \mathcal{P}_3$.

(b) Show that for a constant $q \in \mathbb{R}$, for any sufficiently smooth f,

$$\int_0^1 f(x)dx - \{\tfrac{1}{2}[f(0)+f(1)] + \tfrac{1}{12}[f'(0)-f'(1)]\} = qf^{(4)}(\xi)$$

for some $\xi \in [0,1]$, and determine q.

(c) Show that for $c < d$ and sufficiently smooth g,

$$\int_c^d g(x)dx - \{\tfrac{1}{2}(d-c)[g(c)+g(d)] + \tfrac{1}{12}(d-c)^2[g'(c)-g'(d)]\}$$
$$= q(d-c)^5 g^{(4)}(\eta) \text{ voor e.o.a. } \eta \in [c,d],$$

(Hint: use $r : [0,1] \to [c,d] : y \mapsto (d-c)y + c$)

(d) Let $a < b$, $m \in \mathbb{N}$, and define $x_i := a + ih$ with $h := \frac{b-a}{m}$. Show that

$$\int_a^b f(x)dx - \big\{h\big[\tfrac{1}{2}f(a) + f(x_1) + \cdots + f(x_{m-1}) + \tfrac{1}{2}f(b)\big] + \tfrac{1}{12}h^2[f'(b)-f'(a)]\big\}$$
$$= qh^4(b-a)f^{(4)}(\gamma) \text{ for some } \gamma \in [a,b].$$

2. We consider $C[a, b]$ with scalar product $\langle f, g\rangle := \int_a^b w(x)f(x)g(x)dx$, where w is a positive, continuous, and integrable weight function on (a, b).

   (a) Show that $0 \neq p_{n+1} \in \mathcal{P}_{n+1}$ with $p_{n+1} \perp \mathcal{P}_n$ has $n+1$ different zeros $x_0, \dots, x_n$ on (a, b).

   (b) Let $w_0, \dots w_n$ be weights such that $\int_a^b w(x)f(x)dx = \sum_{i=0}^n w_i f(x_i)$ for all $f \in \mathcal{P}_n$. Show that $Q(f) := \sum_{i=0}^n w_i f(x_i)$ is even exact for all $f \in \mathcal{P}_{2n+1}$ (Hint: Write f as $qp_{n+1} + r$ for certain polynomials q en r).

   (c) By using a suitable interpolation polynomial, show that for $f \in C^{2n+2}[a, b]$,

$$\int_a^b w(x)f(x)dx - \sum_{i=0}^n w_i f(x_i) = \frac{f^{2n+2}(\xi)}{(2n+2)!}\int_a^b w(x)\prod_{i=0}^n (x - x_i)^2 dx,$$

   for some $\xi \in (a, b)$.

3. We consider $\begin{cases} y'(x) = f(x, y(x)), \\ y(x_0) = y_0, \end{cases}$ where f is continuous and Lipschitz continuous with respect to the second variable.

   Given $X_M > x_0$ and $N \in \mathbb{N}$, let $h := \frac{X_M - x_0}{N}$, and $x_i := x_0 + ih$ $(i = 0, \dots, N)$. We consider two numerical approximation methods:

$$y^F_{n+1} := y^F_n + hf(x_n, y^F_n) \quad \text{(Forward Euler)}$$
$$y^B_{n+1} := y^B_n + hf(x_{n+1}, y^B_{n+1}) \quad \text{(Backward Euler)}$$

   where $y^F_0 = y^B_0 := y_0$. We define

$$e^F_n := y(x_n) - y^F_n, \quad T^F_n := \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n))$$
$$e^B_n := y(x_n) - y^B_n, \quad T^B_n := \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_{n+1}, y(x_{n+1}))$$

   (a) It is known that $|T^F_n| \leq \frac{1}{2}h \max_{x\in[x_0,X_M]} |y''(x)|$ for $n \in \{0, \dots, N-1\}$. Prove the same estimate for T^B_n.

   From here on, we consider f to be of the form $f(x, y) = -\lambda y + g(x)$ for some continuous function g and constant $\lambda \geq 0$.

   (b) Determine σ^F, σ^B such that

$$e^F_{n+1} = \sigma^F e^F_n + hT^F_n, \quad e^B_{n+1} = \sigma^B(e^B_n + hT^B_n).$$

   (c) Show that $|e^B_n| \leq (X_M - x_0) \max_{0\leq i\leq N-1} |T^B_i|$. Also show that $|e^F_n| \leq (X_M - x_0) \max_{0\leq i\leq N-1} |T^F_i|$ when $h\lambda$ is sufficiently small. What can you expect from Forward Euler when the last condition is violated?