Fourier Analysis on the Line and the Circle from a Distribution Theory Point of View

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1 Introduction

In this paper we give a short introduction to Fourier analysis of tempered distributions. An attractive aspect of this theory is that Fourier analysis on the circle turns out to be a special case of Fourier analysis on the line. Using the Fourier transform on the line we will show that any periodic distribution can be expanded in a Fourier series. This approach gives the additional insight that the Fourier transform of such a distribution is a train of delta distributions whose coefficients are precisely its Fourier coefficients.

The theory can be generalized to arbitrary dimension but in order to make the presentation as concise and clear as possible we restrict ourselves to the one dimensional case. Furthermore we will only consider tempered distributions and leave the general theory of distributions aside since tempered distributions are most important in Fourier analysis.

In the second chapter we introduce tempered distributions in the framework of locally convex topological vector spaces. To make the notion of a tempered distribution more transparent the questions of continuity and convergence will be examined carefully for locally convex topological vector spaces. The space of tempered distributions will be defined as the dual of a suitable space of functions. We also give some examples and show how the basic operations of functions can be generalized to tempered distributions.

The third chapter gives a brief overview of the basic Fourier analysis on the real line in so far as it is needed to set up Fourier analysis on the circle. Proofs are omitted. After that we apply the theory to set up Fourier analysis on the circle. This is done by considering tempered distributions on the circle as periodic tempered distributions on the line. We also discuss the Poisson summation formula.

2 Locally convex topological vector spaces

In this section we introduce the most important spaces of functions for Fourier analysis. Since these spaces are all locally convex topological vector spaces it is natural to use this framework. We will consider only complex vector spaces.

Definition 1. *Let $\mathcal{V}$ be a $\mathbb{C}$-vector space, and $N : \mathcal{V} \to [0, \infty)$ a function. N is called a semi-norm if N satisfies the triangle inequality and for any scalar λ, and $v \in \mathcal{V}$ we have $N(\lambda v) = |\lambda| N(v)$.*
Let $\mathcal{N}$ be a collection of semi-norms on $\mathcal{V}$. The pair $(\mathcal{V}, \mathcal{N})$ is called a locally convex topological vector space if for any $v \in \mathcal{V}$ we have $v = 0$ if and only if $\forall N \in \mathcal{N} \quad N(v) = 0$.
The topology of $(\mathcal{V}, \mathcal{N})$ is defined as follows: Let $\mathcal{B}$ be the set of unions of finite intersections of sets $B(N, \epsilon) = \{v \in \mathcal{V} | N(v) < \epsilon\}$, where $N \in \mathcal{N}$ and $\epsilon < 0$. A subset $U \subset \mathcal{V}$ is open if for every $u \in U$ there is a $B \in \mathcal{B}$ such that $B + u \subset U$.

We shall see that the topology we just defined is compatible with the vector space structure. The topology can also be characterized as the smallest topology that makes the semi-norms continuous.

Easy examples of locally convex topological vector spaces are supplied by the normed linear spaces where the norm is the only semi-norm. For example the space L^1 of Lebesgue integrable functions on $\mathbb{R}$.

The continuity of linear maps between locally convex topological vector spaces can be checked in terms of semi-norm estimates. This is the content of the next lemma.

Lemma 1. *Let $T : (\mathcal{V}, \mathcal{N}) \to (\mathcal{W}, \mathcal{M})$ be a linear map of locally convex topological vector spaces. T is continuous if and only if for every $M \in \mathcal{M}$ there is a $C > 0$ and there are $N_1, .., N_k \in \mathcal{N}$ such that for every $v \in \mathcal{V}$:*

$$M(Tv) \leq C \max_{j=1..k} N_j(v)$$

Proof. T is continuous if and only if T is continuous in 0 because of the definition of the topologies on $\mathcal{V}$ and $\mathcal{W}$. Now if T is continuous in 0 and $M \in \mathcal{M}$ then there is a neighborhood U of 0 in $\mathcal{V}$ such that $T(U) \subset B(M, 1)$. Since U is the union of finite intersections of sets $B(N, \epsilon)$ there are $N_1, .., N_k \in \mathcal{N}$ and $\epsilon_1, .., \epsilon_k > 0$ such that $\bigcap_{j=1}^k B(N_j, \epsilon_j) \subset U$. This means that $\forall j : N_j(x) < \epsilon_j \Rightarrow M(Tx) < 1$. Define $N(v) = \max_{i=1..k} N_i(v)$ and $\epsilon = \min_{i=1..k} \epsilon_i$ and $C = \frac{2}{\epsilon}$ and let $v \in \mathcal{V}$. When we replace v in the previous equation by tv, where $t > 0$, we get: $\forall j : tN_j(v) < \epsilon_j \Rightarrow tM(Tv) < 1$. If $N(v) = 0$ then $tM(Tv) < 1$ holds for any t so $M(Tv) = 0$ and certainly $M(Tv) \leq CN(v)$. If $N(v) \neq 0$ then we choose $t = \frac{\epsilon}{2N(v)} > 0$. This gives the equation

$$\forall j : \epsilon N_j(v) < \epsilon_j 2N(v) \Rightarrow M(Tv) < CN(v)$$

Since the statement $\epsilon N_j(v) < \epsilon_j 2N(v)$ is always true, we are done.

On the other hand suppose that for $M \in \mathcal{M}$ there is a $C > 0$ and there are $N_1, .., N_k \in \mathcal{N}$ such that for every $v \in \mathcal{V}$: $M(Tv) \leq C \max_{j=1..k} N_j(v)$.

Let G be a neighborhood of 0 in $\mathcal{W}$ then there is a finite intersection $A = \bigcap_{r=1}^m B(M_r, \epsilon_r) \subset G$. For every r we have a $C_r > 0$ and $N_{r,1}, .., N_{r,k_r} \in \mathcal{N}$ such that for every $v \in \mathcal{V}$: $M_r(Tv) \leq C_r \max_{j=1..k_r} N_{r,j}(v)$. Define an open neighborhood U of $0 \in \mathcal{V}$ by $U = \bigcap_{r=1}^m \bigcap_{j=1}^{k_r} B(N_{r,j}, \frac{\epsilon_r}{C_r})$, then $T(U) \subset A \subset G$ so T is continuous in 0. Indeed for any $r = 1..m$ and $a \in A$ we have $M_r(Ta) \leq C_r \max_{j=1..k_r} N_{r,j}(a) < C_r \frac{\epsilon_r}{C_r} < \epsilon_r$. $\square$

As a first application of lemma 1 we prove that the vector space structure of a locally convex topological vector space $(\mathcal{V}, \mathcal{N})$ is compatible with the topology defined above. More precisely we will show that vector addition $A : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and the scalar multiplication $S : \mathbb{C} \times \mathcal{V} \to \mathcal{V}$ are continuous. To apply the lemma we have to give the cartesian products a locally convex topological vector space structure: The product $(\mathcal{V}, \mathcal{N}) \times (\mathcal{W}, \mathcal{M})$ becomes a locally convex topological vector space if we define the semi-norms as follows: If $N \in \mathcal{N}$ and $M \in \mathcal{M}$ then $(v, w) \mapsto \max(N(v), M(w))$ is a semi-norm on the product.

As A is linear we can now apply lemma 1: Let $N \in \mathcal{N}$. For the semi-norm $\max(N, N)$ on the product we have $N(A(v, v')) = N(v + v') \leq N(v) + N(v') \leq 2 \max(N, N)(v, v')$ so A is continuous according to lemma 1. The map S is bilinear and it suffices to show that it is continuous in both arguments: If we fix $\lambda \in \mathbb{C}$ then $N(S(\lambda, v)) = |\lambda| N(v)$, so for the semi-norm N we get the same semi-norm N and the constant $C = |\lambda|$ showing that $S(\lambda, .)$ is continuous. If we fix v then the same equation gives the continuity in the other variable with constant $C = N(v)$.

Convergence in locally convex topological vector spaces can also be expressed in terms of semi-norms:

Lemma 2. *Let (u_n) be a sequence in a locally convex topological vector space $(\mathcal{V}, \mathcal{N})$. Then (u_n) converges to $u \in \mathcal{V}$ if and only if $\forall N \in \mathcal{N}$ $\lim_{n \to \infty} N(u_n - u) = 0$.*

Proof. Without loss of generality we can assume that $u = 0$. Choose $\epsilon > 0$ and let $N \in \mathcal{N}$. The semi-norm N is continuous (this follows from lemma 1), so there is a neighborhood U of 0 such that $u \in U \Rightarrow N(u) < \epsilon$. Since (u_n) converges to 0 there is an m such that $n \geq m \Rightarrow u_n \in U$. Therefore we have $n \geq m \Rightarrow N(u_n) < \epsilon$.

On the other hand suppose that U is an open neighborhood of 0, then there are $N_1, ..N_k \in \mathcal{N}$ and $\epsilon_1, .., \epsilon_k > 0$ such that $\bigcap_{j=1}^k B(N_j, \epsilon_j) \subset U$. If we choose m such that $N_j(u_n) < \epsilon_j$ for all j and all $n \geq m$, then $u_n \in U$ for $n \geq m$. $\square$

The notions of convergence and continuity are connected by the following lemma:

Lemma 3. *Let $T : (\mathcal{V}, \mathcal{N}) \to (\mathcal{W}, \mathcal{M})$ be a linear map of locally convex topological vector spaces and let (u_n) be a sequence that converges to u. If T is continuous then the sequence $(T(u_n))$ converges to $T(u)$.*

Proof. Let $M \in \mathcal{M}$ and choose $\epsilon > 0$. According to lemma 1 there are semi-norms $N_1, .., N_k \in \mathcal{N}$ and a $C > 0$ such that $M(T(u_n) - T(u)) = M(T(u_n - u)) \leq C \max_{j=1..k} N_j(u_n - u)$. By lemma 2 we can choose a number m such that for $n \geq m$ we have $\max_{j=1..k} N_j(u_n - u) < \frac{\epsilon}{C}$. It follows that $n \geq m \Rightarrow M(T(u_n) - T(u)) < \epsilon$, so $T(u_n)$ converges to $T(u)$. □

Although convenient in the given proof, the linearity of T is not necessary: one can prove lemma 3 for general continuous maps T. However the converse of the lemma is not true for general locally convex topological vector spaces, it is true when $\mathcal{V}$ is countable, see [1].

An important example of a locally convex topological vector space is the Schwartz[1] space $\mathcal{S}$. This space is especially useful in Fourier analysis.

Definition 2. *Let $\mathcal{S}$ be the space of C^∞ functions $f : \mathbb{R} \to \mathbb{C}$ such that for all $a, b \in \mathbb{N}$ we have $\sup_{x\in\mathbb{R}} |x^a f^{(b)}(x)| < \infty$. For $m \in \mathbb{N}$ we define semi-norms N_m on $\mathcal{S}$ by*

$$N_m(s) = \sup_{0\leq a,b\leq m, x\in\mathbb{R}} |x^a s^{(b)}(x)|$$

It is clear that $\mathcal{S}$ is a vector space and that the maps N_m are semi-norms. It remains to check that $\mathcal{S}$ becomes a locally convex topological vector space when equipped with these semi-norms. Now if $s = 0$ then obviously $N_m(s) = 0$. On the other hand suppose that $N_0(s) = 0$ then $\sup_{x\in\mathbb{R}}(|s(x)|) = 0$, so $s = 0$. The semi-norms on the Schwartz space have an important additional property: if $m < n$ then $N_m < N_n$. This means that lemma 1 simplifies to the statement that a map $T : \mathcal{S} \to (\mathcal{W}, \mathcal{M})$ is continuous if and only if for every M there is a constant C and a natural number n such that $M(Tv) \leq CN_n(v)$.

Examples of elements of $\mathcal{S}$ are the C^∞ functions with compact support and the Gaussian function $g(x) = e^{-\pi x^2}$.

In many parts of mathematical analysis there is the need to generalize the concept of a function. A generalization especially suitable for Fourier analysis is provided by passing to the dual of the Schwartz space. We will explain this concept for general locally convex topological vector spaces.

Definition 3. *The dual of a locally convex topological vector space $(\mathcal{V}, \mathcal{N})$ is the pair $(\mathcal{V}^*, \mathcal{N}^*)$, where $\mathcal{V}^*$ is the vector space of continuous linear forms on $\mathcal{V}$ and $\mathcal{N}^*$ is the collection of semi-norms $N_v : u \mapsto |u(v)|$ where $v \in \mathcal{V}$.*

A bilinear form on a locally convex topological vector space $(\mathcal{V}, \mathcal{N})$ is a map $< ., . >: V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear in both arguments and is non-degenerate and continuous.

Although the notion of a sesquilinear form is more standard for complex linear spaces, the above type of bilinear forms are used in distribution theory. It should be noted that such bilinear forms do not give rise to norms in the usual way.

The category of locally convex topological vector spaces is closed under taking duals as will be clear from the following considerations. If u and u' are continuous linear forms, then then $u + \lambda u'$ is continuous as well, because the semi-norm estimates for u and u' imply a semi-norm estimate for $u + \lambda u'$: $|(u + \lambda u')(v)| \leq |u(v)| + |\lambda||u'(v)|$. Therefore $\mathcal{V}^*$ is a vector space. Furthermore

[1]After Laurent Schwartz (1915-2002) one of the pioneers in distribution theory.

the properties of the absolute value show that the N_v are semi-norms and if $u = 0$ then $N_v(u) = |u(v)| = 0$ for all $v \in \mathcal{V}$. Finally if $N_v(u) = 0$ for all $v \in \mathcal{V}$ then $|u(v)| = 0$ for all v, so $u = 0$.

The dual of the Schwartz space is called the space of tempered distributions and this is the space on which we will do Fourier analysis in the next chapter.

Definition 4. *The dual $\mathcal{S}^*$ of the Schwartz space $\mathcal{S}$ is called the space of tempered distributions on $\mathbb{R}$.*

The refinement of lemma 1 for the Schwartz space is very convenient for checking whether a linear form u on $\mathcal{S}$ is a tempered distribution: $u \in \mathcal{S}^*$ if and only if for every $s \in \mathcal{S}$ there is a $C > 0$ and a natural number n such that $|u(s)| \leq CN_n(s)$.

Tempered distributions can be viewed as generalizations of functions by the following general construction:

Lemma 4. *If $< ., . >$ is a bilinear form on $(\mathcal{V}, \mathcal{N})$, then the map $\Phi : v \mapsto < v, . >$ is a continuous linear injection of $(\mathcal{V}, \mathcal{N})$ into its dual.*

Proof. The assumed continuity of $< v, . >$ implies that $< v, . > \in \mathcal{V}^*$. The non-degeneracy of the form shows that the map Φ is injective. To show that Φ is continuous we use lemma 1. For any $v' \in \mathcal{V}$ we need to find a constant $C > 0$ and a finite set of semi-norms $N_1, .., N_k$ such that $N_{v'}(\Phi(v)) \leq C \max_{j=1..k} N_j(v)$ for all v. Since $N_{v'}(\Phi(v)) = | < v, v' > |$ this is equivalent to the continuity of $< ., v' >$. □

On the Schwartz space $\mathcal{S}$ we will always use the symmetric bilinear form $< s, t >= \int st$. It non-degenerate because $< s, \bar{s} > = \|s\|^2_{L^2}$. Because of the symmetry we only have to check the continuity in the second variable. This is shown by the following semi-norm estimate:

$$| < v, s > | \leq \int |v||s| \leq N_0(v) \int (x^2 + 1)|s| \frac{1}{x^2 + 1} \leq N_2(s) N_0(v) \int \frac{2}{x^2 + 1}$$

With the same bilinear form we can also inject larger function spaces, such as L^1, continuously into $\mathcal{S}^*$. More precisely we consider the map $\Phi : L^1 \to \mathcal{S}^*$ on defined by $\Phi(f)(s) = < f, s >$. The continuity of Φ is shown by the following estimate: $| < f, s > | \leq \int |f||s| \leq \|f\|_{L^1} N_0(s)$. For L^p with $1 \leq p \leq \infty$ a similar proof works, see [1].

A simple example of a tempered distribution that does not come from a function is the delta distribution $\delta_x : s \mapsto s(x)$ for any fixed real number x. To check that this is indeed a tempered distribution the estimate $|\delta_x(s)| = |s(x)| \leq N_0(s)$ suffices. Since $\mathcal{S}^*$ is a vector space finite linear combinations of delta distributions are tempered distributions too. We can also consider infinite sums of delta functions and these will be important in the next chapter:

Lemma 5. *Let (a_n) be a doubly infinite sequence of complex numbers and define a map $\Delta_{(a_n)}$ by $s \mapsto \sum_{n\in\mathbb{Z}} a_n \delta_n(s)$. Then $\Delta_{(a_n)} = \sum_{n\in\mathbb{Z}} a_n \delta_n$ is a tempered distribution if and only if there is an $M > 0$ such that $|a_n| = \mathcal{O}(n^M)$ as $n \to \pm\infty$. If $\Delta_{(a_n)}$ is a tempered distribution then the sequence of tempered distributions $\Delta_k = \sum_{n=-k}^{k} a_n \delta_n$ converges to $\Delta_{(a_n)}$ in $\mathcal{S}^*$.*

Proof. Suppose there is a $C > 0$ and an $M > 0$ such that $|a_n| \leq C(1+|n|)^M$ for all n and let $s \in \mathcal{S}$. By lemma 1 and the remarks following the definition of $\mathcal{S}$ we can use the following estimate to show that $\Delta_{(a_n)}$ is a tempered distribution:

$$|\Delta_{(a_n)}(s)| \leq \sum_{n\in\mathbb{Z}} |a_n||s(n)| \leq \sum_{n\in\mathbb{Z}} C(1+|n|)^M|s(n)|$$

$$\leq C\sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|)^2}(1+|n|)^{M+2}|s(n)| \leq 2C\left(\sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|)^2}\right) N_{M+2}(s)$$

The same estimate for applied to $|(\Delta_{(a_n)} - \Delta_k)(s)|$ show that $N_s(\Delta_{(a_n)} - \Delta_k)$ converges to 0, because $\lim_{k\to\infty} \sum_{|n|\geq k} \frac{1}{(1+|n|)^2} = 0$.

On the other hand define s_m to be a C^∞ function supported inside $[m - \frac{1}{2}, m + \frac{1}{2}]$ such that $s_m(m) = 1$. Then $\Delta_{(a_n)}(s_m) = \sum_{n\in\mathbb{Z}} a_n s_m(n) = a_n$. If $\Delta_{(a_n)}$ is a tempered distribution then we also have a $C > 0$ and a N such satisfying $|a_m| = |\Delta_{(a_n)}(s_m)| < CN_k(s_m)$. Since s_m is supported inside $[m - \frac{1}{2}, m + \frac{1}{2}]$ we can estimate

$$N_k(s_m) = \sup_{r,j\leq k, x\in\mathbb{R}} |s_m^{(j)}(x)x^r| \leq \sup_{j\leq k, x\in\mathbb{R}} |s_m^{(j)}(x)(|m|+1)^k|$$

Now $\sup_{j\leq k,x\in\mathbb{R}} |s_m^{(j)}(x)| = \sup_{j\leq k,x\in\mathbb{R}} |s_0^{(j)}(x)|$ so this is a constant independent of m. Therefore $a_m = \mathcal{O}(n^k)$. □

Our next task is to extend the usual operations on functions such as differentiation and translation to tempered distributions. This construction works for any locally convex topological vector space with a bilinear form as above:

Lemma 6. *Let $(\mathcal{V}, \mathcal{N})$ be a locally convex topological vector space with a bilinear form $< .,. >$ and let $T : (\mathcal{V}, \mathcal{N}) \to (\mathcal{V}, \mathcal{N})$ be a continuous linear map. If T has a transpose, i.e. a continuous linear map T' such that for $v, w \in \mathcal{V}$ we have $< T'v, w > = < v, Tw >$, then T' unique. Moreover the map $(T')^* : \mathcal{S}^* \to \mathcal{S}^*$ is continuous and linear as well and it extends T to $\mathcal{S}^*$ in the sense that $(T')^*(< v, . >) = < Tv, . >$.*

Proof. If it exists the transpose of T is unique because if $< T''v, w > = < v, Tw > = < T'v, w >$ for all w then we get $< T''v - T'v, w > = 0$ for all w. The non-degeneracy of the bilinear form implies that $T''v = T'v$. If T is continuous and linear then T^* is linear and also continuous. The last statement can be checked using lemma 1: Let $v \in \mathcal{V}$ then we have $N_v(T^*u) = |T^*u(v)| = |u(Tv)| = N_{Tv}(u)$ for all $u \in \mathcal{V}^*$. Now suppose that T has a transpose T' and let $v, w \in \mathcal{V}$, then $(T')^*(< v, . >)(w) = < v, T'w > = < Tv, w >$. □

Some examples for the case $\mathcal{V} = \mathcal{S}$ are given below:
(1) Define the translation maps T_a by $s(x) \mapsto s(x - a)$ for any real number a. Let $m \in \mathbb{N}$ and $s \in \mathcal{S}$. It is obvious that $T_a(s) \in \mathcal{S}$ and that $N_m(T_a(s)) = N_m(s(x - a)) = N_m(s)$ so T_a is continuous by lemma 1.

(2) The next example is the differentiation operator $D : s \mapsto s'$. Again the definition of the norms on the Schwartz space make it clear that $D(s) \in \mathcal{S}$ and that $N_m(Ds) \leq N_m(s)$, so D is continuous.

(3) Define the set $C_P^\infty = \{f \in C^\infty | \ \forall n \ \exists M : f^{(n)}(x) = \mathcal{O}(x^M)\}$. Let $f \in C_P^\infty$ and define the map $M_f : s \mapsto fs$. We have $fs \in \mathcal{S}$, moreover if $\max_{j=1..m} |f^{(j)}(x)| < C(1+|x|)^M$ then

$$N_m(fs) = \sup_{j,k \leq m, x \in \mathbb{R}} |(fs)^{(j)}(x)x^k| \leq \sup_{j,k \leq m, x \in \mathbb{R}} \sum_{i=1}^{j} \binom{j}{i} |f^{(i)}(x)s^{(j)}(x)x^k|$$

$$\leq \sup_{j,k \leq m, x \in \mathbb{R}} \sum_{i=1}^{j} \binom{j}{i} |C(1+|x|)^M s^{(j)}(x)x^k| \leq C' N_{M+m}(s)$$

where C' is some constant. This shows that $M_f(s)$ is in $\mathcal{S}$ and by lemma 1 that M_f is continuous.

(4) The last example is the map $R : s(x) \mapsto s(-x)$. It is obviously continuous.

All the maps in the examples are linear and all maps except D are their own transpose with respect to the bilinear form $< .,. >$ on $\mathcal{S}$ defined earlier. It follows from partial integration that $D' = -D$. By lemma 6 all these maps can now be extended to $\mathcal{S}^*$. In what follows we will usually identify the the above function spaces with their injective images in $\mathcal{S}^*$ and likewise we will identify the map T with $(T')^*$ whenever this is possible.

We can now state the general structure theorem for tempered distributions:

Theorem 1. *Any tempered distribution is a finite order derivative of some continuous function of polynomial growth.*

Proof. The proof can be found in section 8.3 of [2]. □

We close this chapter with a definition of the support of a tempered distribution.

Definition 5. *Let $B \in \mathbb{R}$ be an open set and let u be a tempered distribution. We say that $u = 0$ on B if $u(s) = 0$ for every $s \in \mathcal{S}$ with $\mathrm{supp}(s) \in B$. We also say that two tempered distributions u and v are equal on B if $u - v = 0$ on B.*

The support of a tempered distribution u is defined to be the complement of the set $\{x \in \mathbb{R} | u = 0 \text{ on } B_\epsilon(x)\}$, where $B_\epsilon(x)$ is the open ϵ-ball around x.

3 Fourier Analysis on the Line and the circle

In the first subsection we recall the basic results on Fourier analysis on $\mathbb{R}$. We start out with the Schwartz space and extend the theory to the space of tempered distributions. The second paragraph applies the theory to periodic tempered distributions and shows how they can all be expanded into a Fourier series. Finally we discuss the Poisson summation formula.

3.1 Fourier Analysis on $\mathbb{R}$

We start to work on the Schwartz space $\mathcal{S}$. For $s \in \mathcal{S}$ define the Fourier transform F by $F(s)(x) = \int s(y)e^{-2\pi ixy}\mathrm{d}y$. The basic features of F on $\mathcal{S}$ are summarized in theorem 2:

Theorem 2. *a) The Fourier transform is a linear homeomorphism the locally convex topological vector space $\mathcal{S}$ to itself.*
b) The inverse of F is given by $F^{-1} = FR$, where R is defined as in the previous chapter.
c) Moreover we have $FT_a = M_{e^{-2\pi iax}}F$ and $T_aF = FM_{e^{2\pi iax}}$.
d) $FD = M_{2\pi ix}F$ and $DF = FM_{-2\pi ix}$.
e) $F' = F$, i.e. F is its own transpose with respect to the bilinear form $< .,. >$ on $\mathcal{S}$.

Proof. See [2] 8.2 for the continuity and [4] for the rest. □

The beauty of setting up the theory in this way is that it is now very easy to extend this theorem to the dual of $\mathcal{S}$, the tempered distributions. Note that we used the same notations for the extensions of the maps F, R, T and D to $\mathcal{S}^*$.

Theorem 3. *a) The Fourier transform is a linear homeomorphism $F : \mathcal{S}^* \to \mathcal{S}^*$.*
b) The inverse of F is given by $F^{-1} = FR$, where R is defined as in the previous chapter.
c) Moreover we have $FT_a = M_{e^{-2\pi iax}}F$ and $T_aF = FM_{e^{2\pi iax}}$.
d) $FD = M_{2\pi ix}F$ and $DF = FM_{-2\pi ix}$.

Proof. First of all F is a continuous linear homeomorphism on $\mathcal{S}$ with inverse FR. Therefore the extension $(F')^*$ of F to $\mathcal{S}^*$ is continuous and linear too by lemma 5. The same holds for $((FR)')^* = (F')^*(R')^*$ and the relation $FRF = I$ on $\mathcal{S}$ extends to show that $(F')^*(R')^*(F')^* = ((FRF)')^* = I$. Parts c and d follow in the same way by applying transpose and $*$ to the corresponding relations in $\mathcal{S}$. □

Some simple examples of Fourier transforms are those of the delta distributions δ_a, $a \in \mathbb{R}$. Let $s \in \mathcal{S}$ and observe that $F(\delta_a)(s) = \delta_a(F(s)) = F(s)(a) = \int s(y)e^{-2\pi iay}\mathrm{d}y = e^{-2\pi iax}(s)$. Therefore $F(\delta_a) = e^{-2\pi iax}$. By linearity we also know the Fourier transform of a finite linear combination of delta distributions. Even more generally we can compute the Fourier transform of the delta trains from lemma 5.

For a doubly infinite sequence (a_n) such that there is an $M > 0$ such that $|a_n| = \mathcal{O}(n^M)$ as $n \to \pm\infty$. We will now compute the Fourier transform of the tempered distribution $\Delta_{(a_n)} = \sum_{n\in\mathbb{Z}} a_n\delta_n$. In lemma 5 we proved that the symmetric partial sums $\Delta_k = \sum_{n=-k}^{k} a_n\delta_n$ converge to $\Delta_{(a_n)}$ in $\mathcal{S}^*$. By the sequential continuity of F (lemma 3) we see that the distributions $F(\Delta_k) = \sum_{n=-k}^{k} a_n e^{-2\pi inx}$ converge to $F(\Delta_{(a_n)})$. This means that $F(\Delta_{(a_n)}) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi inx}$.

3.2 Fourier Analysis on the Circle

In this section we will apply the theory to tempered distributions on the circle. By a tempered distribution on the circle we mean a 1-periodic tempered distribution. These are the elements $u \in \mathcal{S}^*$ such that $T_1 u = u$. The main theorem is the following:

Theorem 4. *For any 1-periodic tempered distribution u we have $Fu = \Delta_{(a_n)}$ and $u = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ with convergence in $\mathcal{S}^*$. The doubly infinite sequence (a_n) does not grow too rapidly: there is an $M > 0$ such that $|a_n| = \mathcal{O}(n^M)$ as $n \to \pm\infty$.*

Proof. Let u be a 1-periodic tempered distribution, that is $T_1 u = u$. Taking the Fourier transform of this relation we get $Fu = FT_1 u = e^{-2\pi i x} Fu$ and hence $(e^{-2\pi i x} - 1)Fu = 0$. This shows that the support of Fu is $\mathbb{Z}$. To see this take an $s \in \mathcal{S}$ with $\mathrm{supp}(s) \subset \mathbb{R} - \mathbb{Z}$ then we can construct a C_P^∞ function w by $w(x) = \frac{1}{(e^{-2\pi i x}-1)}$ on $\mathrm{supp}(s)$ and by extending w in any bounded and smooth fashion to all of $\mathbb{R}$. Observe that $Fu(s) = Fu(sw(e^{-2\pi i x}-1)) = w(e^{-2\pi i x}-1)\cdot Fu(s) = 0$.

Now observe that we can write $(e^{-2\pi i x}-1)\frac{1}{g(x)} = (x-n)$ with $\frac{1}{g(x)} \in C_P^\infty$ on $(n-1, n+1)$. Multiplying with Fu we get $(Fu)\cdot(x-n) = (Fu)\cdot(e^{-2\pi i x}-1)\cdot\frac{1}{g(x)} = 0 \cdot \frac{1}{g(x)} = 0$ on $(n-1, n+1)$. Take an $s \in \mathcal{S}$ with support inside $(n-1, n+1)$ then Taylor's theorem says that we can write $s(x) = s(n) + (x-n)t(x)$ with $t(x) \in \mathcal{S}$. Then $Fu(s) = (Fu)(s(n)) + (Fu)((x-n)t(x)) = (Fu)(1)\cdot s(n) + (Fu\cdot(x-n))(t) = a_n\delta_n(s) + 0$. Therefore $Fu = a_n\delta_n$ on $(n-1, n+1)$ for some $a_n \in \mathbb{C}$.

Let μ_n be a smooth bump-function with $\mu_n = 1$ on $[n - \frac{1}{2}, n + \frac{1}{2}]$ that is supported inside $(n-1, n+1)$. For any $s \in \mathcal{S}$ we have $\mathrm{supp}(s\mu_n) \subset (n-1, n+1)$ and so $Fu(s\mu_n) = a_n\delta_n(s\mu_n) = a_n s(n) = a_n\delta_n(s)$. Furthermore $Fu(s) = Fu(\sum_{n\in\mathbb{Z}} s\mu_n)$ because the difference is $Fu(s(1-\sum_{n\in\mathbb{Z}}\mu_n)) = 0$ since $s(1-\sum_{n\in\mathbb{Z}}\mu_n)$ is supported inside $\mathbb{R}-\mathbb{Z}$. Therefore $Fu(s) = Fu(\sum_{n\in\mathbb{Z}} s\mu_n) = \sum_{n\in\mathbb{Z}} Fu(s\mu_n) = \sum_{n\in\mathbb{Z}} a_n\delta_n(s)$. This means that $Fu = \sum_{n\in\mathbb{Z}} a_n\delta_n = \Delta_{(a_n)}$. Since Fu is a tempered distribution, lemma 5 gives the growth condition on (a_n). Applying FR to both sides we get $u = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ with convergence in $\mathcal{S}^*$. See the last example in the previous paragraph. □

As an application of the theorem or rather of its proof we prove Poisson's summation formula.

Theorem 5. *Let $u = \sum_{n\in\mathbb{Z}} \delta_n$, then $Fu = u$.*
Therefore $\sum_{n\in\mathbb{Z}} s(n) = \sum_{n\in\mathbb{Z}} Fs(n)$ for all $s \in \mathcal{S}$.

Proof. Since u is a 1-periodic tempered distribution we can copy the proof of the previous theorem and find that $Fu = a_n\delta_n$ on $(n-1, n+1)$. On the other hand the last example in the previous paragraph shows that $Fu = \sum_{n\in\mathbb{Z}} e^{2\pi i n x}$, so Fu is 1-periodic too. This implies that $a_n = a$ for all n and some constant a, so $Fu = au$. To show that the constant is 1 we consider the Gaussian $g(x) = e^{-\pi x^2}$. Since $Fg = g$ we find $Fu(g) = u(Fg) = u(g)$ and hence $a = 1$. □

It should be noted that the summation formula is valid for a much wider class and is a special case of more general trace formulas, see [3] chapter 30.

4 Conclusion

We have investigated the definition of tempered distributions in terms of locally convex topological vector spaces. This somewhat general approach has the advantage that it clarifies the origins of the various constructions are used in distribution theory. Especially the use of semi-norm estimates becomes clear.

In the third chapter we showed how distributions can be used to set up Fourier analysis quickly and elegantly. After the main theorems are proved in the Schwartz class they are readily extended to the dual space of tempered distributions. Distribution theory also sheds light on the relation between Fourier series and the Fourier transform. The Fourier transform of a periodic function is a train of delta distributions whose coefficients are exactly the coefficients of the Fourier series. The ideas that lead to this theorem can be specialized to give quick proof the beautiful Poisson summation formula. In conclusion it is fair to say that distribution theory makes Fourier analysis more elegant and that it will probably do the same in other parts of analysis.

References

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