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Topological Strings in $d < 1$

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Abstract

We calculate correlation functions in minimal topological field theories. These twisted version of $N = 2$ minimal models have recently been proposed to describe $d < 1$ matrix models, once coupled to topological gravity. In our calculation we make use of the Landau-Ginzburg formulation of the $N = 2$ models, and we find a direct relation between the Landau-Ginzburg superpotential and the KdV differential operator. Using this correspondence we show that the minimal topological models are in perfect agreement with the matrix models as solved in terms of the KdV hierarchy. This proves the equivalence at tree-level of topological and ordinary string theory in $d < 1$.

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1. Introduction

The recent progress in the study of low dimensional string theories using matrix models [1] has led to a number of remarkable discoveries. Not only has it revealed a new connection between two-dimensional gravity and generalized KdV hierarchies [2]-[4], in addition it was found that both subjects are intimitely linked with 2-d topological field theory [5]-[8]. So far these relationships have been only partially clarified, and to make further progress a better understanding seems necessary. In this paper we will examine and try to elucidate the connection of 2-d topological field theory with the multi-matrix models and KdV equations.

Soon after Witten's observation [5] that the one matrix model can be identified with the topological theory of 2-d gravity, it was recognized [7] that the multi-matrix model corresponds to topological gravity coupled to some matter system, although precisely which system was unknown. Recently, however, it was proposed by Li [9] that the appropriate topological theories are the so-called twisted versions of the $N = 2$ minimal superconformal models, introduced in [10]. In [9] Li supported this conjecture with convincing arguments, most of which were based on specific properties of 2-d topological gravity. However, already at the level of the pure topological matter theory, the correspondence with the multi-matrix model has quite non-trivial consequences. In particular, when combined with Douglas' work [4], it provides a definite prediction for all the amplitudes in the topological models. One of our aims is to verify this prediction.

The topological properties of a $N = 2$ minimal superconformal model are most evident via its identification with the renormalization group fixed point of a Landau-Ginzburg field theory [11]-[13]. In particular, this allows one to characterize the ring of chiral primary fields in terms of the LG superpotential $W(x)$. Since in the topological counterparts of the $N = 2$ models one is essentially left with only the chiral ring, it is natural to expect that in this case the superpotential $W(x)$ will play an even more prominent role. As one of our main results we will show that the form of the superpotential is indeed sufficient to determine all physical correlation functions in the topological model. Furthermore, in comparing these results with the amplitudes obtained in the multi-matrix models, we will find an intriguing connection between the superpotential W and the differential operator Q of the KdV hierarchy.

The organization of this paper is as follows. In section 2 we review some of the properties of topological conformal field theory. We describe its symmetry algebra, which is given by a twisted version of the $N=2$ algebra [15, 10], and the set of physical operators. Here we borrow extensively from existing results in the $N=2$ literature, in particular [13]. In section 3 we consider the correlation functions in these models and show that all information about them can be stored into a finite dimensional associative ring, whose structure coefficients depend on the physical couplings. We proof some specific properties of these topological correlation function using techniques from conformal field theory. Section 4 is devoted to the analysis of the twisted $N=2$ minimal models. After deriving some selection rules, we will develop an explicit method for calculating all physical amplitudes in these models, which makes use of the Landau-Ginzburg formulation of the $N=2$ discrete series. The proof that these amplitudes are identical to those of the matrix models is given in section 4.4. Section 5 contains some concluding remarks.

2. Topological Conformal Field Theory

Topological field theories were introduced by Witten some years ago [14, 15]. These theories possess a nilpotent symmetry Q satisfying

$$Q^2 = 0, \tag{2.1}$$

and the physical observables are the cohomology classes of the operator Q as it acts in the full Hilbert space. For that reason these models are also refered to as cohomological field theories. Since the stress-energy tensor $T_{\alpha\beta}$ is Q-exact

$$T_{\alpha\beta} = \{Q, G_{\alpha\beta}\}, \tag{2.2}$$

correlation functions of the observables are independent of the two-dimensional metric $g_{\alpha\beta}$. The topological theories therefore have a much larger symmetry than conformal theories. However, even in the category of topological quantum field theory there is a counterpart of conformal invariance, since we can distinguish the special class of models in which $T_{\alpha\beta}$ is traceless even before restricting to the Q-cohomology. These are what we will call topological conformal field theories (TCFT's).

In the same way that the conformal field theories correspond to the critical points in the space of ordinary quantum field theories, the topological CFT's are the critical points in the space of topological QFT's. One can consider perturbations of a TCFT by turning on the couplings of the appropriate physical fields. This maintains the topological symmetry but in general destroys the conformal invariance, so one obtains a parameter family of more general 'massive' topological theories. In this space, the conformal models are distinguished by the fact that all their correlation functions scale as a function of the couplings.

2.1. The symmetry algebra of topological CFT.

The combined presence of conformal invariance and the topological symmetry implies that the generator Q can be written as a sum

$$Q = Q_L + Q_R \tag{2.3}$$

of left- and right-moving charges. We will restrict our discussion to the left-moving sector. The charge Q_L is expressed as a contour integral of a holomorphic current $Q(z)$

$$Q_L = \oint Q(z). \tag{2.4}$$

The presence of the nilpotent Q-symmetry tells us that in addition we must have two other holomorphic fields: a fermionic spin-2 field $G(z)$ and a bosonic $U(1)$-current $J(z)$ which are the Q-partners of the stress-tensor $T(z)$ and and the fermionic current $Q(z)$

$$\begin{aligned} T(z) &= \{Q, G(z)\}, \\ Q(z) &= [Q, J(z)]. \end{aligned} \tag{2.5}$$

All these fields have an integer moded Laurent expansion. Without introducing any other fields there is a unique consistent commutator algebra for these models. It reads

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}, & [J_m, J_n] &= d\, m\delta_{n+m,0}, \\ [L_m, G_n] &= (m-n)G_{m+n}, & [J_m, G_n] &= -G_{m+n}, \\ [L_m, Q_n] &= -nQ_{m+n}, & [J_m, Q_n] &= Q_{m+n}, \end{aligned}$$

$$
\begin{aligned}
\{G_m, Q_n\} &= L_{m+n} + nJ_{m+n} + \tfrac{d}{2}m(m+1)\delta_{n+m,0}, \\
[L_m, J_n] &= -nJ_{m+n} + \tfrac{d}{2}m(m+1)\delta_{m+n,0}.
\end{aligned}
$$

In fact, this algebra (2.6) can be obtained from the $N=2$ superconformal algebra by modifying the $N=2$ stress-tensor as follows

$$
T(z) \to T(z) + \frac{1}{2}\partial J(z). \tag{2.6}
$$

As a result of this modification the super-currents $G^{\pm}(z)$ of the $N = 2$ algebra acquire conformal spin $\frac{3}{2} \pm \frac{1}{2}$, and become identified with $G(z)$ and $Q(z)$. It has been noted previously [15, 5, 10] that by twisting a $N=2$ SCFT in this way one gets a topological field theory. However, we have just argued that *any* two-dimensional conformally invariant topological field theory is of this kind*. This observation will enable us to take over much of the $N=2$ technology that has been developed in the literature [13] to topological CFT's. Before we can do so, one has to understand in more detail what the effect of the modification (2.6) is.

Let us discuss some of the features of the symmetry algebra (2.6). Note that there is no central charge in the Virasoro algebra, *i.e.* $c = 0$. However, we do have a central extension d showing up in some of the other relations.† In particular the last commutator tells us that the $U(1)$-current has an anomalous conservation law: there is a background charge equal to d. This has important consequences for the correlation functions and the Hilbert space of the theory.

2.2. Hilbert space and physical operators.

The Hilbert space $\mathcal{H}$ is divided into sectors corresponding to different $U(1)$-charge q

$$
J_0|\phi\rangle = q|\phi\rangle. \tag{2.7}
$$

The theory has a unique (right) vacuum state $|0\rangle$ with $q = 0$ corresponding to the identity operator. The presence of the background charge d implies that the left

*As an aside we note that the algebra (2.6) is present in any conformal invariant theory with a BRST symmetry, so in particular in the bosonic string where $G = b$, $J = bc$ and $d = -3$.

†We have denoted the central extension by d because in specific examples it equals the complex dimension of the target space.

vacuum state $\langle\infty|$ that is dual to $|0\rangle$ carries $U(1)$-charge $q = d$. For each local operator ϕ we introduce the states

$$
\begin{aligned}
|\phi\rangle &= \phi(0)|0\rangle, \\
\langle\phi^*| &= \langle\infty|\phi^*(\infty).
\end{aligned} \tag{2.8}
$$

We now assume the the theory is unitary in the sense that the norm defined by

$$
\| \, |\phi\rangle \, \|^2 \equiv \langle\phi^*|\phi\rangle \geq 0 \tag{2.9}
$$

is positive definite. This inner product coincides with the usual one in the corresponding $N = 2$ theory.

We are now in the position to take over the arguments of [13] to derive various results for the physical states of the theory. The physical states $|\phi_i\rangle$ in the Hilbert space are defined by the condition that they are Q-closed and equivalent up to Q-exact states

$$
\begin{aligned}
Q|\phi_i\rangle &= 0, \\
|\phi_i\rangle &\sim |\phi_i\rangle + Q|\lambda\rangle.
\end{aligned} \tag{2.10}
$$

We now make use of the fact that with respect to (2.9) the hermitean conjugate of Q is $Q^* = G_0$ to choose a representative for each physical state that satisfies

$$
G_0|\phi_i\rangle = 0. \tag{2.11}
$$

The fact that such a 'Hodge-representative' exists was shown in the case of the $N = 2$ CFT's by Lerche, Vafa, and Warner [13], and their proof goes through in this situation. In the $N\!=\!2$ context the conditions (2.10) and (2.11) precisely single out the so-called 'chiral primary fields.' For these fields it is known that the $U(1)$-charge q takes its values in the interval

$$
0 \leq q \leq d. \tag{2.12}
$$

Furthermore, each chiral primary field ϕ_i with charge q has a unique 'Poincaré dual' field $\phi_i^{dual} = \eta_{ij}\phi_j$ with charge $d - q$, with

$$
\langle\phi_i\phi_j\rangle = \eta_{ij}. \tag{2.13}
$$

All this also holds in the topological CFT. An alternative characterization of the physical states $|\phi_i\rangle$, which is equivalent to (2.10) and (2.11), is that they satisfy

$$L_0|\phi_i\rangle = 0. \tag{2.14}$$

In usual CFT's the vacuum $|0\rangle$ is the only state that is annihilated by L_0, but apparently this axiom does not hold for topological CFT's. Here we find that the physical states, of which there are more than one (except for $d = 0$), can be made to satisfy (2.14) and are then also annihilated by the positive modes of the stress-tensor and the other currents. A useful identity, which we will need later on, is that the states $|\phi_i\rangle$, when acted upon with L_{-1}, give Q-exact states

$$L_{-1}|\phi_i\rangle = Q(G_{-1}|\phi_i\rangle). \tag{2.15}$$

We further note that a physical field will have left- and right-moving $U(1)$-charges q, $\overline{q}$. Since before twisting the conformal dimensions are given by $h = q/2$, the spin and according the statistics of the physical fields is determined by $s = (q - \overline{q})/2$. This can be either integer or half-integer. Twisting will not alter the statistics of the operators, so we can have both world-sheet bosons and fermions in our topological field theory, even though after twisting $h = \overline{h} = 0$. In order to keep our formulas tractible we will assume subsequently that all fields are commuting. This will also turn out to be the situation for the minimal models we consider in section 4, which satisfy $q = \overline{q}$.

The relation between the symmetry algebra (2.6) and the $N=2$ superconformal algebra suggests that topological CFT's also have super-field formulations. Indeed, we can introduce an odd coordinate θ which is related to usual complex coordinate z by the Q-symmetry

$$\delta_Q z = \theta, \quad , \quad \delta_Q \theta = 0. \tag{2.16}$$

The group of transformations generated by $T(z)$ and $G(z)$ is a simple generalization of the conformal group, and is represented on (z, θ) as

$$\begin{aligned} z &\rightarrow f(z), \\ \theta &\rightarrow \partial f(z)\,\theta + \hat{f}(z). \end{aligned} \tag{2.17}$$

Note that θ transforms as a (-1)-differential, and not as a spinor. The $U(1)$-current $J(z)$ is associated with the global rotations of θ, *i.e.* $\theta \rightarrow e^{i\alpha}\theta$. To the physical state

$|\phi_i\rangle$ one can associate not just a scalar field $\phi_i(z,\overline{z})$ but a complete 'super-field' $\Phi_i(z,\overline{z},\theta,\overline{\theta})$ with the expansion

$$\Phi_i = \phi_i^{(0)} + \theta\,\phi_i^{(1)} + \overline{\theta}\,\overline{\phi}_i^{(1)} + \theta\overline{\theta}\,\phi_i^{(2)}, \tag{2.18}$$

where the first component is just $\phi_i^{(0)} \equiv \phi_i$. The other components $(\phi_i^{(1)}, \overline{\phi}_i^{(1)})$ and $\phi_i^{(2)}$ build up a 1- and 2-form respectively. Their corresponding states in the Hilbert space $\mathcal{H}$ are obtained from $|\phi_i\rangle$ by acting with G_{-1} and $\overline{G}_{-1}$. For example $\phi_i^{(2)}$ corresponds to the state $G_{-1}\overline{G}_{-1}|\phi_i\rangle$. Now from (2.15) we learn that the different components of the superfield are related by the descent equations, which read [15]

$$\begin{aligned} d\phi^{(0)} &= \{Q, \phi^{(1)}\}, \\ d\phi^{(1)} &= \{Q, \phi^{(2)}\}. \end{aligned} \tag{2.19}$$

In fact, together with (2.16) these equations express simply the fact that the super-field Φ is Q-invariant. Furthermore, the above equations show that the line resp. surface integral of $\phi^{(1)}$ and $\phi^{(2)}$ commute with Q. Hence, for each physical operator ϕ in TCFT we can associate three types of observables

$$\begin{aligned} \Phi(P) &= \phi^{(0)}, \\ \Phi(C) &= \oint_C \phi^{(1)}, \\ \Phi(\Sigma) &= \int_\Sigma \phi^{(2)}. \end{aligned} \tag{2.20}$$

where P is a point and C is a contour on the surface Σ. In the next section we will study the correlation functions of these observables.

3. Correlation Functions

We will now discuss some general properties of the correlation functions of the various physical operators in a two-dimensional conformal-invariant topological field theory. We consider only the zero and two-form operators $\phi_i^{(0)}$ and $\phi_i^{(2)}$.

3.1. Factorization and the chiral ring.

Let us start with the correlators of the zero-form operators. The first important observation is that these correlation functions are independent of the positions of the operator insertions, *i.e.*

$$\frac{\partial}{\partial z_i}\Big\langle \phi_{i_1}^{(0)}(z_1,\overline{z}_1)\ldots\phi_{i_s}^{(0)}(z_s,\overline{z}_s)\Big\rangle = 0. \tag{3.1}$$

Namely, when we differentiate with respect to one of the positions, then, according to the descent-equation (2.19), we can write the result as a Q-variation, which vanishes because all the other insertions are Q-invariant. This implies that the correlation function

$$\langle \phi_{i_1}\ldots\phi_{i_s}\rangle \tag{3.2}$$

(where we write $\phi_i \equiv \phi_i^{(0)}$) is simply a constant. A related property is that the correlation function does not depend on the choice of representative of a particular Q-cohomology class, since operators of the form $\{Q,\lambda\}$ decouple.

In general only some of the correlators (3.2) are non-vanishing. Conservation of the $U(1)$-current $J(z)$ tells us that, in order to have a non-zero correlation function on a genus g surface, the $U(1)$-charges must satisfy

$$\sum_i q_i = d(1-g). \tag{3.3}$$

Hence we see that because all $q_i \geq 0$ there are only non-trivial correlators of this type on the sphere. Of course, in genereral there will also be other selection rules besides $U(1)$-charge conservation, but their form will be different for each topological CFT.

Correlation functions of the ϕ_i's can be simply determined using factorization in terms of the three-point function

$$c_{ijk} = \langle \phi_i\phi_j\phi_k\rangle. \tag{3.4}$$

We insert a sum over a complete set of states in each intermediate channel. Inductively one shows that at each vertex at least two of the three states are Q-closed, and hence only the physical states contribute. Thus we can write the insertion as

$$\mathbf{1} = \sum_{i,j} |\phi_i\rangle\eta^{ij}\langle\phi_j| \tag{3.5}$$

where $\eta_{ij} = \langle \phi_i \phi_j \rangle$ is the metric defined by the two-point function. For example, the factorization of the four-point function is given by

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle = \sum_m \langle \phi_i \phi_j \phi^m \rangle \langle \phi_m \phi_k \phi_l \rangle = \sum_m c_{ij}{}^m c_{mkl}, \tag{3.6}$$

where we have used η_{ij} to raise the index. Consistency of the factorization of this four-point function in the s-channel and t-channel implies the condition

$$\sum_m c_{ij}{}^m c_{mkl} = \sum_m c_{ik}{}^m c_{mjl}. \tag{3.7}$$

This property of the c_{ijk} can be reformulated as the statement that the operator algebra of the fields ϕ_i, given by

$$\phi_i \phi_j = \sum_k c_{ij}{}^k \phi_k, \tag{3.8}$$

is associative. This algebra is known in the $N = 2$ context as the chiral primary ring, and has been discussed extensively in [13, 16]. In the conformal topological models this ring has very special properties, namely, it respects the $U(1)$ counting and furthermore it is always nilpotent.

3.2. Perturbed correlation functions.

Let us now consider the more general correlation function of the type*

$$\langle \phi_{i_1} \ldots \phi_{i_s} \int \phi_{j_1} \ldots \int \phi_{j_r} \rangle. \tag{3.9}$$

The selection rule in this case reads

$$\sum_i q_i + \sum_j (q_j - 1) = d(1 - g). \tag{3.10}$$

Note that these correlation functions can exist at higher genus. Again we can learn a lot about these correlation functions by considering their factorization properties.

*In the following all integrated operators will be understood to be two-forms, and we will in these cases drop the superscript of $\phi^{(2)}$.

Let us illustrate this with the example of the insertion of one integrated operator $\int \phi_n$ into a multi-point function of zero-form fields. It is clear that if the number of zero-form operators is bigger than three (or $g > 0$), we can still apply the same procedure of inserting (3.5) in each channel and factorize it into lower amplitudes. The only difference now is that when the factorization is done at a dividing cycle there will be two terms corresponding to the contributions of the integral $\int \phi_n$ over the two components of the surface. So for example, the four-point point function in the presence of $\int \phi_n$ can be factorized as

$$\langle \phi_i \phi_j \phi_k \phi_l \int \phi_n \rangle = \sum_m (c_{ij}{}^m{}_{,n}\, c_{mkl} + c_{ij}{}^m c_{mkl,n}) \tag{3.11}$$

where we introduced the notation

$$c_{ijk,n} = \langle \phi_i \phi_j \phi_k \int \phi_n \rangle. \tag{3.12}$$

In order to derive (3.11) we have assumed that at the dividing node of the surface there is no extra contribution due to the integral $\int \phi_n$. This is equivalent to the statement that the expression $\eta_{ij,n} = \langle \phi_i \phi_j \int \phi_n \rangle$ vanishes. We will indeed show in a moment that this property holds.

In a similar way as in (3.11), all amplitudes with one integrated operator can be decomposed into the fundamental amplitudes c_{ijk} and $c_{ijk,n}$. Notice that (3.12) cannot be factorized any further, since it is given by the integral of a volume-form over $\mathcal{M}_{0,4}$.

Given the three-point functions c_{ijk}, the consistency of the above factorization procedure imposes a strong restriction on the four-point functions $c_{ijk,n}$, namely that the right-hand side of (3.11) is symmetric in the labels i, j, k and l. The form of this consistency condition shows that the insertion of $\int \phi_n$ acts like a derivation. Indeed, we can introduce coupling constants t_n for each of the operators $\int \phi_n$ and consider the perturbed three-point functions

$$c_{ijk}(t) = \Big\langle \phi_i \phi_j \phi_k \exp\Big(\sum_n t_n \int \phi_n\Big)\Big\rangle. \tag{3.13}$$

We now have relations of the form

$$c_{ijk,m} = \partial_m c_{ijk}\big|_{t=0} \tag{3.14}$$

with $\partial_m = \partial/\partial t_m$. With this definition the consistency of the factorization formula (3.11) can now be recognized as the first derivative with respect to t_n (at all $t_i = 0$) of the associativity condition (3.7) of the perturbed three-point amplitudes $c_{ijk}(t)$.

We could now repeat the same kind of reasoning for correlations functions with more than one insertions of integrated operators. They can again be decomposed into fundamental amplitudes of the form $\langle \phi_i \phi_j \phi_k \int \phi_n \ldots \int \phi_s \rangle$, which in turn can be written as multiple derivatives of the perturbed three-point function $c_{ijk}(t)$. One then finds that the consistency of the factorization procedure implies that for all values of t_n the coefficients $c_{ijk}(t)$ define an associative ring. We will refer to it as the *perturbed chiral ring.*

We can understand that the functions (3.13) define an associative ring from the fact that they represent the three-point functions of the perturbed topological theory. Although this perturbed model is no longer conformal, one can still show [17] that its three-point amplitudes satisfy (3.7). It is interesting to note that, in general, the perturbed chiral ring is no longer nilpotent. We now want to derive two important additional properties.

(*i*) The two-point function in the perturbed theory is independent of the couplings

$$c_{ij0}(t) = \eta_{ij}. \tag{3.15}$$

(*ii*) The coefficients satisfy the integrability condition

$$\partial_m c_{ijk}(t) = \partial_k c_{ijm}(t). \tag{3.16}$$

This second property guarantees that one can integrate $c_{ijk} = \partial_i g_{jk}$. Since the three-point functions themselves are symmetric, we can in fact integrate twice more†. Therefore we arrive at the important conclusion that there exists a unique function $F(t)$ such that

$$c_{ijk}(t) = \partial_i \partial_j \partial_k F(t), \tag{3.17}$$

†Here we assumed that the space of coupling constants is an affine space and has no cohomology. Although in all generality this need not be true, it will be a correct assumption in the case of the $d < 1$ models that we consider in the next section.

and $F(t) = \frac{1}{6}c_{ijk}t^i t^j t^k + O(t^4)$. We will sometimes refer to this function as the generating functional, since with an appropriate definition it can be written as

$$F(t) = \Big\langle \exp\Big(\sum_n t_n \int \phi_n\Big)\Big\rangle. \tag{3.18}$$

In topological string theory F will correspond to the (tree level) free energy.

The proof of (i) and (ii) follows from the Ward identity of the spin-2 field $G(z)$. On the sphere the Ward identity takes the form

$$\begin{aligned}\Big\langle \oint \xi(w)G(w)\phi_{i_1}(z_1)\dots\phi_{i_s}(z_s)\Big\rangle = \\ \sum_{n=1}^{s}\xi(z_n)\Big\langle\phi_{i_1}(z_1)\dots\phi^{(1)}_{in}(z_n)\dots\phi_{i_s}(z_s)\Big\rangle = 0,\end{aligned} \tag{3.19}$$

where $\xi(w)$ is a globally defined holomorphic vector field, necessarily of the form

$$\xi(w) = aw^2 + bw + c.$$

One can choose a suitable vector field that vanishes at the positions of two of the operators, say, at z_1 and z_2. In the case of a 3-point function this choice leads to the result

$$\Big\langle\phi_i(z_1)\phi_j(z_2)\phi^{(2)}_k(z_3)\Big\rangle = 0. \tag{3.20}$$

Here we have applied the Ward identity for both left- and right-movers. This establishes that the first derivative of c_{ij0} vanishes, and that indeed to first order $c_{ij0}(t) = \eta_{ij}$. However, we now note that

$$\oint \xi(w)G(w)\phi^{(2)}_n(z) = 0, \tag{3.21}$$

since the left-hand side corresponds to the state $G^2_{-1}|\phi_n\rangle = 0$. Therefore we see that the Ward identity (3.19) remains unchanged under insertion of two-form operators. This is sufficient to proof property (i) for arbitrary values of the couplings t_n.

In case of a four-point function the Ward identity gives us

$$\Big\langle\phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\phi^{(2)}_m(z_4)\Big\rangle = \left|\frac{\xi(z_3)}{\xi(z_4)}\right|^2 \Big\langle\phi_i(z_1)\phi_j(z_2)\phi^{(2)}_k(z_3)\phi_m(z_4)\Big\rangle \tag{3.22}$$

with $\xi(w) = (w - z_1)(w - z_2)$. To see that this implies the symmetry of the quantity $c_{ijk,m}$ we can use the *conformal* Ward identities to write both sides of the equation in terms of the anharmonic ratio

$$\zeta = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)} \tag{3.23}$$

One finds

$$\left\langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\phi_m^{(2)}(z_4) \right\rangle = \left| \frac{\partial \zeta}{\partial z_4} \right|^2 G_{ijk,m}(\zeta) \tag{3.24}$$

with $G_{ijk,m}$ some function. Using (3.22) and the identity

$$\xi(z_3)\frac{\partial \zeta}{\partial z_3} + \xi(z_4)\frac{\partial \zeta}{\partial z_4} = 0 \tag{3.25}$$

we infer the symmetry $G_{ijk,m} = G_{ijm,k}$. Then, since we can write

$$c_{ijk,m} = \int d^2\zeta \, G_{ijk,m}(\zeta) \tag{3.26}$$

this establishes $\partial_m c_{ijk} = \partial_k c_{ijm}$ for $t = 0$. Finally, again with the aid of (3.21) we conclude that the above arguments are still valid at non-zero values of the couplings which proves the integrability condition (*ii*).

4. Topological minimal models

We will now consider what can be properly called topological minimal models – the twisted versions of the $N = 2$ discrete series. These models were introduced in a recent paper by Eguchi and Yang [10], following suggestions of Witten [15, 5]. Subsequently it was proposed by Li [9] that, when coupled to topological gravity, these minimal models in fact correspond to the matrix chain models. Douglas' solution of these matrix models [4] provides us with definite answers for all correlation functions, in particular for the operators that would correspond to the topological matter fields. The goal of this section is to actually calculate these n-point functions in the topological model, and verify that they agree with the matrix model results.

4.1. Twisted minimal $N = 2$ superconformal models.

Let us begin with recalling some facts about minimal $N = 2$ SCFT. The classification of the unitary $N = 2$ superconformal field theories with central charge $d = c/3 < 1$ follows the familiar ADE pattern. The models occur at the values

$$d = \frac{k}{k+2} \tag{4.1}$$

($k = 1, 2, \ldots$) and the different modular invariant partition functions are in one-to-one correspondence with simply-laced Lie groups G with dual Coxeter number h related to k by $h = k + 2$. The chiral primary fields ϕ_i are labeled by integers $0 \leq i \leq k$ for the A_{k+1} model. For the D and E invariants we have one chiral primary ϕ_i for each exponent $i+1$ of the Lie group G. Recall that the exponents of a group equal the orders of the Casimirs minus one. The number of chiral primary fields equals the rank of G. The field ϕ_i has $U(1)$-charge given by

$$q_i = \frac{i}{k+2} \tag{4.2}$$

A convenient representation of the $N = 2$ minimal models is that in terms of $\mathbf{Z}_k$ parafermion fields ψ^l_m together with a free boson φ [18]. Here the boson φ is obtained by bosonizing the $U(1)$-current of the $N = 2$ algebra, $J = i\sqrt{\frac{k}{k+2}}\partial\varphi$. In the minimal topological theory the boson φ is coupled to a background charge $\frac{k}{k+2}$. Primary fields in the $N = 2$ models are products of primary parafermion operators and bosonic vertex operators, *i.e.* they are of the form

$$\phi^l_m = \psi^l_m e^{im\varphi/\alpha}, \tag{4.3}$$

where $\alpha = \sqrt{k(k+2)}$. (For convenience of notation, we suppress all anti-holomorphic labels.) The parafermion operators ψ^l_m are labeled by integers l, m satisfying $0 \leq l \leq k$, $-k+1 \leq m \leq k$, $l - m = 0 \pmod 2$, and are subject to the identification

$$\psi^l_m = \psi^{k-l}_{m+l} \tag{4.4}$$

with m taken modulo $2k$. The conformal dimension of ψ^l_m is given by $\frac{l(l+2)}{4(k+2)} - \frac{m^2}{4k}$. The chiral primary operators in the $N = 2$ models correspond to the special primary fields ϕ^l_m with $l = m$, $l + 1$ an exponent of G

$$\phi^{(0)}_l = \psi^l_l e^{il\varphi/\alpha}. \tag{4.5}$$

In the minimal topological theory these operators have zero conformal dimension: they are the zero-form versions of the physical fields in the model. The supercurrent $G(z)$ has the parafermionic representation $G = \psi^0_{-2} e^{-i(k+2)\phi/\alpha}$ and from this we find that the two-form version $\phi^{(2)}_l = G_{-1}\overline{G}_{-1}\phi^{(0)}_l$ of the field ϕ_l is given by

$$\phi^{(2)}_l = \psi^l_{l-2} e^{i(l-k-2)\varphi/\alpha}. \tag{4.6}$$

This field has $U(1)$-charge $q_l - 1$ and, after twisting, conformal weight $(1, 1)$. The main virtue of this parafermion representation of the $N = 2$ models is that it naturally divides the selection rules of the theory into $U(1)$-charge conservation and those encoded in the parafermion fusion rules, which are basically those of the $SU(2)$ level k current algebra. We will consider both selection rules in the next subsection.

4.2. Some selection rules

Our final aim in this section is to determine the general s-point functions in minimal topological conformal field theories

$$\langle \phi_{l_1} \dots \phi_{l_r} \int \phi_{l_{r+1}} \dots \int \phi_{l_s} \rangle_g \tag{4.7}$$

Let us first consider the selection rules satisfied by these correlators. The most obvious one is of course $U(1)$-charge conservation. Applying the general rule (3.10) we see that at genus g (4.7) is non-vanishing only if

$$\sum_{n=1}^{s} l_n = (1-g)k + (s-r)(k+2). \tag{4.8}$$

In particular, on the sphere we have that

$$\langle \phi_i \phi_j \rangle = \eta_{ij}, \qquad \eta_{ij} \equiv \delta_{i+j,k}. \tag{4.9}$$

Hence the operator ϕ_i is conjugate to ϕ_{k-i}.

As explained in section 3, we can use the factorization expansion, obtained by inserting a complete set of states in intermediate channels, to decompose a general amplitude (4.7) into a sum of products of lower, fundamental amplitudes of the form

$$\langle\phi_{l_1}\phi_{l_2}\phi_{l_3}\int\phi_{l_4}\ldots\int\phi_{l_s}\rangle \tag{4.10}$$

at genus zero. These correlation functions are fundamental because they are the integrals of a volume form on the moduli space of the punctured sphere, since $\dim\mathcal{M}_{0,s} = 2(s-3)$, and for this reason cannot be factorized into lower amplitudes. As we have shown in section 3, the fundamental amplitudes (4.10) do not depend on which three operators ϕ_l are represented as zero-forms, that is, they are symmetric in all labels $l_1,\ldots,l_s$.

Thus we may restrict our attention to the correlation functions of the form (4.10). Conservation of $U(1)$-charge tells us that

$$\sum_{n=1}^{s} l_n = k + (s-3)(k+2). \tag{4.11}$$

In addition, as we will now show, there is a second important selection rule which states that the amplitude (4.10) vanishes unless

$$s \leq 3 + l_n, \quad \text{for all } n. \tag{4.12}$$

That is, whenever the operator ϕ_l occurs, the correlation function can be at most of order $l+3$. Notice that one special case of this rule is that the identity $\phi_0 = 1$ has only two and three-point functions, as was already shown in the previous section. The above selection rule will be helpful in finding the explicit solution of the models.

There are a number of ways to prove (4.12). Here we present a derivation which uses the above parafermion representation*. From equations (4.5) and (4.6) we find that the correlator in (4.10) contains as a factor the parafermionic correlation function

$$\langle\psi^{l_1}_{l_1}\,\psi^{l_2}_{l_2}\,\psi^{l_3}_{l_3}\,\psi^{l_4}_{l_4-2}\ldots\psi^{l_s}_{l_s-2}\rangle. \tag{4.13}$$

Parafermions are subject to selection rules both in the l and m quantum number [18]. We have that, modulo the identification (4.4), the m-charge is conserved modulo $2k$

*An independent derivation using $N=2$ null-state equations has been given by N. Warner [19].

and the l's satisfy the affine $SU(2)$ level k fusion rules. It is not difficult to see that the m-charge conservation is guaranteed by the $U(1)$ selection rule (4.11). So we can focus on the labels l_n. Next we use the identification (4.4) to replace all but two of the l_n's into $\bar{l}_n = k - l_n$, and put (4.13) in the following form

$$\langle \psi^{l_1} \psi^{l_2} \psi^{\bar{l}_3} \dots \psi^{\bar{l}_s} \rangle, \qquad \bar{l}_n = k - l_n, \tag{4.14}$$

where we suppressed the m labels. The $U(1)$-charge counting (4.11) now reads

$$l_1 + l_2 = \sum_{n=3}^{s} \bar{l}_n + 2(s-3), \tag{4.15}$$

and in addition we know that the labels $(l_1, l_2, \bar{l}_3, \dots, \bar{l}_s)$ have to satisfy the fusion rules of $SU(2)_k$. This in particular means that it must be possible to produce, say, l_1 by fusing $l_2, \bar{l}_3, \dots, \bar{l}_s$. This condition implies the inequality

$$l_1 \leq l_2 + \sum_{n=3}^{s} \bar{l}_n, \tag{4.16}$$

since the isospin produced from fusing is bounded by the sum of the contributing isospins. Combining (4.15) with (4.16) we deduce the selection rule $s - 3 - l_2 \leq 0$. Since we could have taken any operator to be ϕ_{l_2}, this establishes condition (4.12).

4.3. Construction of the correlation functions

We will now determine the correlation functions in the $d < 1$ topological models, using the several properties we have established so far. All information about these correlators is encoded in the partition function $F(t)$, or equivalently in the structure coefficients $c_{ijl}(t)$ of the perturbed chiral ring, as a function of the coupling constants t_n of the physical fields. These coupling constants t_n provide a parametrization of the space of topological models, which are described as deformations of the twisted minimal $N = 2$ models. The structure constants $c_{ijl}(t)$ can be represented as the expectation value in the conformal point

$$c_{ijl}(t) = \Big\langle \phi_i \, \phi_j \, \phi_l \, \exp\Big(\sum_{n=0}^{k} t_n \int \phi_n \Big) \Big\rangle. \tag{4.17}$$

First let us summarize what we know so far. It was shown in sections 3 and 4.2 that (i) the two-point function $c_{0ij} = \eta_{ij}$ is independent of the couplings t_n; (ii) the four-point amplitude $a_{mijl} = \partial_m c_{ijl}$ is symmetric in all indices; (iii) at $t_n = 0$ the ring respects the $U(1)$-charge conservation of the twisted $N = 2$ model; (iv) $c_{ijl}(t)$ is a polynomial of finite order p in the t_n's, where p equals the minimum of i, j, and l. Whereas the first three properties are true for general topological models, the last property is characteristic only of the models at $d < 1$ and follows from the selection rule (4.12). Because of all these constraints the construction of the correlation functions will be in fact an overdetermined problem.

A very convenient way of encoding the information of the chiral ring of the minimal $N = 2$ models is in terms the corresponding Landau-Ginzburg potentials. The LG description of the A_n minimal models uses one single LG field x and in that of the other models there are two fields x_i. The action of a general $N = 2$ LG model reads

$$S = \int d^2z d^4\theta \, K(x_i, \overline{x}_i) + \left[\int d^2z d^2\theta \, W(x_i) + c.c. \right] \tag{4.18}$$

with $K(x_i, \overline{x}_i)$ the kinetic term and $W(x_i)$ the so-called superpotential. The fields x_i are chiral $N = 2$ superfields and their dynamics is entirely determined by the LG superpotential $W(x_i)$ at the fixed point of the renormalization group. In particular, the chiral ring $\mathcal{R}$ is isomorphic to the ring of polynomials in the LG fields x_i, modulo the equation of motion $\partial_{x_i} W = 0$,

$$\mathcal{R} = \frac{\mathbf{C}[x_i]}{\partial W(x_i)}. \tag{4.19}$$

The potential $W(x_i)$ for the separate cases is given by

$$\begin{aligned} A_n &: \; x^{n+1}, \\ D_n &: \; x^{n-1} + xy^2, \\ E_6 &: \; x^3 + y^4, \\ E_7 &: \; x^3 + xy^3, \\ E_8 &: \; x^3 + y^5. \end{aligned} \tag{4.20}$$

The chiral primary fields ϕ_l are given by the monomials in x and y, and the $U(1)$-charges of x and y are such that the superpotential W has charge equal to one.

Our strategy for obtaining the structure constants $c_{ijl}(t)$ of the general models will be to construct a *perturbed* potential $W(x_i, t)$ that encodes all information of the *perturbed* chiral ring. Let us explain this in detail for the simplest case, the A_{k+1} model. In this case the LG potential will be a function of one variable x. Before turning on the couplings t_n the potential is[†]

$$W(x) = \frac{x^{k+2}}{k+2} \tag{4.21}$$

and the chiral primaries are identified with $\phi_i = x^i$, $i = 0, \ldots, k$, and generate the ring

$$\phi_i \phi_j = \begin{cases} \phi_{i+j}, & i+j \leq k, \\ 0, & i+j > k. \end{cases} \tag{4.22}$$

We will now write the perturbed potential as

$$W(x,t) = \frac{x^{k+2}}{k+2} - \sum_{i=0}^{k} g_i(t) x^i. \tag{4.23}$$

The functions $g_i(t)$ are *a priori* arbitrary functions of the coupling constants t_j. The only thing we know is that to first order they are given by $g_i(t) = t_i$, since the first order deformations of W are in one-to-one correspondence with the chiral primary fields. This is not true in higher order because the operators ϕ_i will no longer be described by simple powers x^i, but acquire a non-trivial dependence on the couplings t_n. Therefore, in general we have to define the operators as

$$\phi_i(x;t) = -\partial_i W(x,t), \tag{4.24}$$

with $\partial_i = \partial/\partial t_i$. The polynomial $\phi_i(x,t)$ will be of weight $q_i = i/(k+2)$, where x has weight $1/(k+2)$ and t_j weight $1 - q_j$. We have $\phi_i = x^i + O(x^{i-2})$. The chiral ring will now be given by the polynomial ring of the ϕ_i's modulo $W' \equiv \partial_x W$

$$\phi_i(x,t)\, \phi_j(x,t) = \sum_l c_{ij}{}^l(t)\, \phi_l(x,t) \qquad (\text{mod } W'). \tag{4.25}$$

Here the coefficients $c_{ij}{}^l(t)$ are to be identified with the perturbed three-point functions (4.17) via $c_{ijl} = \sum_m \eta_{lm} c_{ij}{}^m = c_{ij}{}^{k-l}$. Thus we find that all information of the

[†]The prefactor $\frac{1}{k+2}$ is introduced for later convenience.

amplitudes in the topological theory is encoded in the potential W as a function of x and the t_n's.

In order to make this more explicit, let us introduce an inner product on the polynomial ring $\mathcal{R} = \mathbf{C}[x]/W'(x)$. For two polynomials ϕ and χ in $\mathcal{R}$ we define their inner product

$$\langle\phi|\chi\rangle = \mathrm{res}\left(\frac{\phi\,\chi}{W'}\right) \equiv \oint \frac{dx}{2\pi i}\frac{\phi(x)\,\chi(x)}{W'(x)}. \tag{4.26}$$

One easily shows that this indeed defines an inner product on $\mathcal{R}$, *i.e.* it is independent of choice of representatives; terms proportional to W' clearly don't contribute. It follows from $\mathrm{res}(\phi_i/W') = \delta_{ik}$ that the polynomials ϕ_i have inner product

$$\langle\phi_i|\phi_j\rangle = \sum_l c_{ij}{}^l\,\mathrm{res}\left(\frac{\phi_l}{W'}\right) = c_{ij0}. \tag{4.27}$$

Now we use the non-trivial input that the coefficients c_{ijl} are completely symmetric, since they represent three-point functions. This implies $c_{ij0} = c_{0ij} \equiv \eta_{ij}$, and we find

$$\langle\phi_i|\phi_j\rangle = \eta_{ij}. \tag{4.28}$$

This allows us to express the $c_{ijl}(t)$ as

$$c_{ijl}(t) = \mathrm{res}\left(\frac{\phi_i\phi_j\phi_l}{W'}\right). \tag{4.29}$$

It is important to note, that, for given $W(x,t)$, the orthogonality condition (4.28) together with the leading behaviour $\phi_i \sim x^i$ uniquely determines the polynomials ϕ_i. The fact that we also have to satisfy relation (4.24) can in principle be used to solve the potential $W(x,t)$, as well as the polynomials $\phi_i(x,t)$. We will use this angle of attack in the next subsection when we discuss the relation with the matrix model. Here, however, we will proceed along a slightly different route, using the selection rules that we derived in the previous section.

A useful way to think about the representation of the chiral ring in terms of polynomials in x is that it provides the *diagonalization* of the coefficients $c_{ij}{}^l$. In other words, we may interpret the equation

$$\phi_1 = x \tag{4.30}$$

as the formal identification of x with (one of) the *eigenvalues* of $\phi_1 = c_{1i}{}^j$. An important conclusion we can draw from this interpretation is that the equation $W'(x) = 0$ has to coincide with the characteristic equation determining the eigenvalues of $c_{1i}{}^j$ [16]

$$W'(x) = \det{(x\delta_i^j - c_{1i}{}^j)}. \tag{4.31}$$

Thus, if we succeed in determining the structure coefficients $c_{1i}{}^j(t)$ of $\phi_1 = x$, we have obtained the superpotential $W(x,t)$ and, via (4.24), the polynomials ϕ_i. We know already that, because of the selection rule (4.12) of the previous subsection, all coefficients c_{1ij} are at most linear in the coupling constants; correlation functions featuring the field ϕ_1 are at most of order 4. This result, combined with $U(1)$-charge conservation, implies that the product of $\phi_1 = x$ with some other field ϕ_i is of the form

$$x\phi_i = \phi_{i+1} + \sum_{j=0}^{i-1} a_i{}^j t_{j-i+k+1}\phi_j, \tag{4.32}$$

where $i = 0, \ldots, k$ and $\phi_{k+1} = 0$. The coefficients a_{ij} $(\equiv a_i^{k-j})$ are identified with the four-point amplitudes

$$a_{ij} = \langle \phi_1\phi_i\phi_j\phi_{2k+1-i-j} \rangle, \tag{4.33}$$

where the position of one of the four operators is integrated over. All that remains is to compute the amplitudes a_{ij}. The crucial requirement which uniquely determines them is consistency with the associativity of the perturbed ring. To this end let us consider the situation in which all couplings t_n vanish, except for t_1. The chiral ring then becomes

$$\phi_i\phi_j = \begin{cases} \phi_{i+j}, & i+j \leq k, \\ t_1 a_{ij}\phi_{i+j-k-1}, & i+j > k. \end{cases} \tag{4.34}$$

By direct inspection we see that this algebra is associative if and only if all a_{ij} are equal. We may set[‡]

$$a_{ij} = \begin{cases} 0, & i+j \leq k, \\ 1, & i+j > k. \end{cases} \tag{4.35}$$

Inserting this into (4.32) we obtain a recursion relation from which we can explicitly

[‡]We will return to the issue of normalization in a moment.

solve the $\phi_i(x)$

$$\phi_i(x,t) = (-1)^i \det \begin{pmatrix} -x & 1 & 0 & \cdots & 0 \\ t_k & -x & 1 & \ddots & \vdots \\ t_{k-1} & t_k & \ddots & & 0 \\ \vdots & & \ddots & & 1 \\ t_{k-i+2} & \cdots & t_{k-1} & t_k & -x \end{pmatrix}. \tag{4.36}$$

Furthermore, we find that

$$W'(x,t) = \phi_{k+1}(x,t), \tag{4.37}$$

where $\phi_{k+1}(x,t)$ is defined by putting $i = k+1$ in (4.36). We can now integrate W' and $\partial_i W = -\phi_i$ to obtain $W(x,t)$. Together with equation (4.29) this gives the complete solution of the A_{k+1} topological minimal models. We leave it to the reader to verify that the above result for the $c_{ijl}(t)$ indeed satisfies the properties $(i) - (iv)$ described at the beginning of this subsection. Their associativity is guaranteed by construction.

Let us return to the issue of normalization. We have seen that in (4.35) we could as well have put $a_{ij} = \mu$, with μ arbitrary. In this way our expressions for the polynomials $\phi_i(x,t)$ and structure coefficients $c_{ijl}(t)$ will be modified by a rescaling of all coupling constants $t_i \to \mu t_i$. This implies that all correlation functions rescale with a factor μ^n, where n is the number of two-form operators. However, we always have the freedom to change the normalization of the operators and to rescale the absolute expectation values on a genus g surface by a factor λ^{2g-2}. Here λ is known as the string coupling constant. If we rescale $\phi_i \to \phi_i \mu^{-q_i}$ and put $\lambda = \mu^{-d/2}$, the factor μ^n is canceled due to the anomalous $U(1)$ conservation law (3.10). So we see that the ambiguity in the normalization that we encountered in our solution of the minimal model is directly related to the string coupling constant λ.

4.4. Correspondence with matrix models

There is an alternative way to express the result of the previous subsection, which not only makes clear that the c_{ijl} can be integrated with respect to the couplings t_n, but also directly proves that the amplitudes of the topological models are identical to the ones found in the matrix models. We introduce the quantity $L(x)$ by

$$\frac{1}{k+2}L^{k+2}(x) = W(x). \tag{4.38}$$

Of course, L is no longer a polynomial but has an infinite Laurent expansion

$$L(x) = x + \sum_{j=1}^{\infty} b_j x^{-j}. \tag{4.39}$$

We now claim that the polynomials ϕ_i defined by (4.36) can be expressed in terms of this fractional power of the superpotential as

$$\phi_i(x) = \left[L^i \partial_x L\right]_+ , \tag{4.40}$$

where the subscript $+$ indicates a truncation to non-negative powers of x. It is clear that $\phi_i \sim x^i$, so in order to establish (4.40) it suffices to verify that the inner product is given by $\langle\phi_i|\phi_j\rangle = \eta_{ij}$. Indeed, we can calculate

$$\begin{aligned} \langle\phi_i|\phi_j\rangle = \mathrm{res}\left(\frac{\phi_i\phi_j}{W'}\right) &= \mathrm{res}(L^{i-k-1}\phi_j) \\ &= \mathrm{res}(L^{i+j-k-1}\partial_x L) = \eta_{ij}. \end{aligned} \tag{4.41}$$

Here we used the identity

$$L^{k+1}\partial_x L = W'. \tag{4.42}$$

The dependence of the potential W on the couplings t_i can now be determined by requiring that we still have $\phi_i = -\partial_i W = -L^{k+1}\partial_i L$. This gives the relation

$$-\frac{1}{i+1}\partial_j \mathrm{res}(L^{i+1}) = \mathrm{res}(L^{i-k-1}\phi_j) = \eta_{ij}. \tag{4.43}$$

By integrating once we thus find that the couplings t_i are determined by

$$t_{k-i} = -\frac{1}{i+1}\mathrm{res}(L^{i+1}). \tag{4.44}$$

It is a straightforward calculation to verify that the ϕ_i's defined above satisfy the recursion relation (4.32) with $a_{ij} = 1$.

With the expression of the polynomials as derivatives of fractional powers of the potential it now becomes possible to explicitly integrate the c_{ijk} and obtain the generating function $F(t)$ (3.18). Hereto we have to use several times the useful relations (4.42) and

$$\partial_j \phi_i = \partial_x \left[-\phi_j L^{i-k-1} \right]_+ . \tag{4.45}$$

The calculation now proceeds as follows, where we hope the reader is able to reconstruct the intermediate steps

$$\begin{aligned} c_{ijl} = \mathrm{res}\left(\frac{\phi_i \phi_j \phi_l}{W'} \right) &= \mathrm{res}\left(\phi_i \phi_j L^{l-k-1} \right) - \mathrm{res}\left(\frac{\phi_i \phi_j}{W'} \left[L^l \partial_x L \right]_- \right) \\ &= -\frac{1}{l+1} \mathrm{res}\left(\phi_i \partial_j L^{l+1} \right) + \frac{1}{l+1} \mathrm{res}\left(\partial_x \left[L^{i-k-1} \phi_j \right]_+ L^{l+1} \right) \\ &= -\frac{1}{l+1} \partial_j \mathrm{res}\left(\phi_i L^{l+1} \right) \\ &= \frac{1}{(l+1)(k+l+3)} \partial_i \partial_j \mathrm{res}(L^{k+l+3}). \end{aligned} \tag{4.46}$$

So, we find for the first derivatives of $F(t)$ the result

$$\partial_i F = \frac{\mathrm{res}(L^{k+i+3})}{(i+1)(k+i+3)}. \tag{4.47}$$

In topological string theory this will correspond to the one-point functions, $\langle \phi_i \rangle = \partial_i F$, as a function of the coupling constants. Unfortunately, we are not able to perform the last integration explicitly. We can however check the integrability condition.

Let us proceed to show that the above results are identical to the ones obtained in the matrix chain models. In fact, since we already formulated our answers in terms of residues of fractional powers of the superpotential, the correspondence with the KdV hierarchy that appears in the matrix model is easily established. The solution of the $k+1$ matrix model is expressed in terms of the (generalized) KdV operator

$$Q = D^{k+2} - (k+2) \sum_{i=0}^{k} u_i(t) D^i, \tag{4.48}$$

where $D = \lambda \partial / \partial t_0$, and the coefficients $u_i(t)$ are functions which are eventually determined through the so-called string equation [2, 3, 4]. Here λ is the string coupling constant. In the matrix model, which should correspond to a minimal topological field theory coupled to topological gravity, there is an infinite set of operators. We do not only have the primary fields ϕ_i but also their 'gravitational descendants' $\sigma_n(\phi_i)$, $n > 0$ [5]. Correlation functions are in general computed with the aid of the fractional power $\hat{L} = Q^{1/k+2}$. In particular the two-point function of the so-called puncture operator $P \equiv \phi_0$ with any other field can, with a particular normalization, be expressed as [7, 20]

$$\langle P\sigma_n(\phi_i)\rangle = c_{n,i}\mathrm{res}(\hat{L}^{n(k+2)+i+1}) \tag{4.49}$$

with

$$c_{n,i} = (-1)^{n+1}/(i+1)(k+2+i+1)\cdots(n(k+2)+i+1). \tag{4.50}$$

Here the residue is defined as the coefficient of D^{-1} in the Laurent expansion of the pseudo-differential operator. Since we will be interested in correlation functions on the sphere, we can discard any derivatives of the u_i, and the residues are polynomials in these coefficients. In particular, we have

$$\langle P\phi_i\rangle = -\frac{1}{i+1}\mathrm{res}(\hat{L}^{i+1}). \tag{4.51}$$

In the special case that only the couplings t_i to the 'primary' fields ϕ_i are non-zero the string equation tells us that [7]

$$\langle P\phi_i\rangle = \eta_{ij}t_j. \tag{4.52}$$

These last two equations determine the functions $u_i(t)$ in (4.48). In this case it also not difficult to calculate the free energy of the matrix model. Namely, since we only couple to the primary fields ϕ_i, the so-called puncture equation [7, 8] tells us that $\langle P\sigma_1(\phi_i)\rangle = \langle\phi_i\rangle$. (This relation is no longer true if we introduce non-zero couplings to the descendant fields.) We now conclude from (4.49)

$$\langle\phi_i\rangle = \frac{\mathrm{res}(\hat{L}^{k+i+3})}{(i+1)(k+i+3)}. \tag{4.53}$$

If we compare the above equations with (4.44) and (4.47), we see that the equivalence of the topological and matrix model can now be formulated in one remarkable statement. *After we substitute $D \to x$ and $u_i \to g_i$ the KdV operator Q and the LG superpotential W are identical.* With this identification we have shown that the free energy of the matrix model equals the generating function of the topological model, at least for the correlation functions of the 'primary' operators.

We can extend the correspondence between W and Q to the polynomials ϕ_i and the KdV flows. Recall that the general KdV flow is written as

$$\partial_i Q = \left[\hat{L}_+^{i+1}, Q\right]. \tag{4.54}$$

We have seen that Q is of a special form, if only the primary couplings are non-zero. In particular, the coupling t_0 only appears linearly in Q and not at all in $\hat{L}_+^{i+1}$ if $i \leq k$. With this observation we can rewrite the above equation as

$$\partial_i Q = \left[\hat{L}_+^{i+1}, t_0\right]. \tag{4.55}$$

With the identifications $Q \to W$, $\hat{L} \to L$, $D \to x$, and $t_0 \to \partial_x$ we find

$$\phi_i = -\partial_i W = \partial_x L_+^{i+1}, \tag{4.56}$$

which corresponds to our definition (4.40).

4.5. D_n MODELS

Now that we have solved the models in the A-series, it is not very difficult to obtain the correlation function of the minimal TCFT of D-type. Recall that the D_n models can be seen as orbifolds of the minimal models labeled by A_{2n-3}; both have Coxeter number $h = k+2 = 2n-2$. The spectrum of chiral primary fields is given by the even fields ϕ_{2i}, $i = 0, \ldots, n-2$, together with one twist field $\phi_* = \phi_{n-2}$. The $U(1)$-charges are respectively $\frac{2i}{k+2} = \frac{i}{n-1}$ and $\frac{n-2}{2n-2}$.

We introduce again coupling constants t_{2i} and t_* to the two-form versions of these operators, and note that under the $\mathbf{Z}_2$ symmetry of the model the fields ϕ_{2i} (and consequently also the coupling constants t_{2i}) are even, while ϕ_* and t_* are odd. The superpotential is now a two variable polynomial and is, before coupling to the fields, given by

$$W(x,y) = \frac{x^{n-1}}{2n-2} + \tfrac{1}{2}xy^2. \tag{4.57}$$

In general it will be a $\mathbf{Z}_2$-even function of $U(1)$-charge one. Giving these restrictions the most general form of the perturbed potential will read in a suitable normalization

$$W(x,y) = \frac{x^{n-1}}{2n-2} + \tfrac{1}{2}xy^2 - \sum_{i=0}^{n-2} g_{2i}(t)x^i - t_* y. \tag{4.58}$$

Here the functions $g_{2i}(t)$ only depend on the even couplings t_{2i} and are in fact identical to those of the corresponding A_{2n-3} model, since for $t_* = 0$ we should reproduce the (even) correlation functions of the A-model. (Note that it is indeed consistent to put all couplings $t_{2i+1} = 0$ in that model.) We further remark that $U(1)$-charge counting and $\mathbf{Z}_2$ behavior of $W(x,y)$ imply that the odd coupling t_* can only appear linearly, as indicated.

We will now show that this solution indeed satisfies all conditions that we derived up to now. To this end we first notice that the variable y only appears quadratically and so can be integrated out. This will produce an inverse power of x in the potential

$$W(x) = \frac{x^{n-1}}{2n-2} - \sum_{i=0}^{n-2} g_{2i}(t)x^i - \tfrac{1}{2}t_*^2 x^{-1}. \tag{4.59}$$

Next we will substitute $x = z^2$. Recall that z was the fundamental field in the corresponding A-model. In fact, we have

$$W(z) = W_0(z) - \tfrac{1}{2}t_*^2 z^{-2}, \tag{4.60}$$

with $W_0(z)$ the superpotential corresponding to A_{2n-3}, as defined in the previous subsection, with all couplings t_{2i+1} set to zero. We again introduce the fractional powers of W through

$$\frac{1}{2n-2}L^{2n-2}(z) = W(z). \tag{4.61}$$

We notice that in this case all powers of L contain inverse powers of z. The fields are defined by

$$\phi_{2i} = \left[L^{2i}\partial_z L\right]_+, \qquad \phi_* = t_* z^{-2}, \tag{4.62}$$

which implies the relation

$$t_{2i} = -\frac{1}{2i+1}\mathrm{res}(L^{2i+1}). \tag{4.63}$$

It is easy to verify that indeed in these expressions for ϕ_{2i} and t_{2i} the odd coupling t_* cannot appear, so that we just as well could have worked with the potential W_0. But, although the operators resemble the operators of the A-model, the chiral ring is different, since the equation of motion $\partial_z W$ has been modified.

To show that the structure coefficients of this chiral ring are still integrable we can perform the same manipulations as in the previous discussion, at least in the case of the even couplings. We find

$$\partial_{2i}F = \frac{\mathrm{res}(L^{2i+2n-1})}{(2i+1)(2i+2n-1)}. \tag{4.64}$$

To obtain the derivative of F with respect to t_* we calculate

$$\begin{aligned}\partial_*\partial_{2i}F = -\mathrm{res}\left(\frac{L^{2i+2n-2}}{2i+1}\partial_* L\right) = \mathrm{res}\left(\frac{L^{2i+1}}{2i+1}t_* z^{-2}\right) = \\ \mathrm{res}\left(t_* z^{-1}L^{2i}\partial_z L\right) = -\partial_{2i}\left[\mathrm{res}\left(t_* z^{-1}W\right)\right].\end{aligned} \tag{4.65}$$

We now see that

$$\partial_* F = -t_*\,\mathrm{res}\left(z^{-1}W\right) = t_* g_0(t). \tag{4.66}$$

This completes the solution of the D_n-model. The full partition function is

$$F(t) = F_0(t) + \tfrac{1}{2}g_0(t)t_*^2 \tag{4.67}$$

with $F_0(t)$ the generating function of the A_{2n-3}-model.

A very interesting point is that we can again translate the function L into a pseudo-differential operator. This is *exactly* the operator Drinfeld and Sokolov find for the generalized KdV hierarchies of type D_n [21]. In [20] this D_n hierarchy was proposed to describe the correlation functions of the matrix model based on the same Lie group. We have established here that this is indeed the correct answer for the D_n minimal topological models.

Let us finally discuss the exceptional models E_6, E_7, E_8. First it is clear that, although we do not have some elegant scheme, the chiral rings for these models can be uniquely determined by imposing all our constraints, since the problem is overdetermined. For instance, for the E_6 model we have found the superpotential

$$\begin{aligned}W(x,y) = \tfrac{1}{3}x^3 + \tfrac{1}{4}y^4 - t_{10}\,xy^2 - t_7 xy - (t_6 - \tfrac{1}{2}t_{10}^3)y^2 \\ -(t_4 - t_6 t_{10} + \tfrac{1}{12}t_{10}^4)x - (t_3 - t_7 t_{10}^2)y \\ -(t_0 - \tfrac{1}{2}t_6^2 + \tfrac{1}{6}t_6 t_{10}^3 - \tfrac{1}{2}t_4 t_{10}^2 - \tfrac{1}{2}t_7^2 t_{10}).\end{aligned} \tag{4.68}$$

Similar concrete expressions can be given for E_7 and E_8.

Both the E_6 and E_8 model can be obtained as tensor products of A-series minimal models. However, we note that correlation functions of tensor product models cannot be simply determined once we know the correlators of the respective factors. This is essentially due to the following phenomenon. Any physical zero-form operator in the tensor product is necessarily of the form

$$\Phi^{(0)} = \phi_1^{(0)}\phi_2^{(0)}, \tag{4.69}$$

where the $\phi_i^{(0)}$ are physical operators in the two factors. The two-form version reads

$$\Phi^{(2)} = \phi_1^{(2)}\phi_2^{(0)} + \phi_1^{(1)} \wedge \phi_2^{(1)} + \phi_1^{(0)}\phi_2^{(2)}, \tag{4.70}$$

and contains one-form components that we have not yet considered. We leave the analysis of these one-form operators and tensor products in general for future study.

5. Conclusions

Topological string theories in $d < 1$ are described by the coupling of minimal topological CFT's to two-dimensional topological gravity. In this work we have presented a complete solution of the correlation functions of the minimal TCFT's. The knowledge of these correlators is sufficient to determine the tree-level amplitudes of all operators $\sigma_n(\phi_i)$ in the corresponding topological string theory by means of the set of recursion relations derived in [5]. The existence of the same recursion relations in multi-matrix models [7] ensures that topological and ordinary string theory in $d < 1$ are identical, at least at tree-level.

The extension of these results to arbitrary genus requires a careful reconsideration of the factorization properties of the topological correlation functions in the presence of gravity. As seen in the $d = 0$ theory the interactions of the gravitational fields σ_n are governed completely by a contact term algebra, which is isomorphic to the Virasoro algebra [8]. It has been proposed [22, 23] that for $0 < d < 1$ this algebra will be generalized to a W-algebra, in fact the W-algebra based on the same simply-laced

Lie group G that labels the minimal model. The generators of the W-algebra are again labeled by the exponents of G and are in one-to-one correspondence with the primary fields ϕ_i. Considerable evidence for the presence of such a W-structure has been presented in [9]. Unfortunately, to obtain a complete and rigorous derivation for arbitrary G seems to be a formidable task.

Even at genus zero the W-constraints have important consequences for the correlation functions of the primary fields ϕ_i. If the couplings to all gravitational descendants are put to zero, the W-constraint associated with ϕ_i attains the form

$$\partial_i F(t) = W_{i+2}(t) \tag{5.1}$$

with $W_{i+2}(t)$ a polynomial in the coupling constants t_n, at most of order $i+2$. This immediately implies the selection rule that correlation functions with ϕ_i can contain not more than $i+2$ other fields. This is precisely the relation we derived in section 4.2, and our derivation can be seen as substantial, independent evidence for the presence of W-constraints in topological string theory in $d < 1$.

Let us finish with a few speculations and generalizations. The most striking result we found was the direct correspondence at tree-level between the LG superpotentials and the (algebraic truncation of the) KdV differential operators. An important question is whether the $N = 2$ Landau-Ginzburg frame-work can be extended to the quantum string theory, and can encompass the full KdV hierarchy of differential equations. In this way one might perhaps arrive at a target space interpretation of $d < 1$ topological string theory as some generalization of supersymmetric quantum mechanics with superpotential $W(x)$.

Many of our results can be generalized for TCFT's with $d \geq 1$. In a large class of models the chiral ring can still be characterized by a Landau-Ginzburg potential. In principle, given the constraints we derived, it is still possible to construct the perturbed superpotential and derive in this way the correlation functions. Of course, there are new technical problems, *e.g.* the existence of marginal operators with $q = \overline{q} = 1$. In these cases the $c_{ijl}(t)$ are no longer polynomial functions. It would be interesting to study the string theories based on these models, in particular for $d = 1$, and see to what extend the relation with integrable hierarchies is still present.

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