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Mirror Symmetry and Elliptic Curves[*]

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Abstract

I review how recent results in quantum field theory confirm two general predictions of the mirror symmetry program in the special case of elliptic curves: (1) counting functions of holomorphic curves on a Calabi-Yau space (Gromov-Witten invariants) are 'quasi-modular forms' for the mirror family; (2) they can be computed by a summation over trivalent Feynman graphs.

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1. Introduction

As discussed in detail by Kontsevich in this volume [1] the moduli space $\mathcal{M}_g$ of algebraic curves of genus g has an interesting generalisation to a moduli space $\mathcal{M}_g(X,d)$ of pairs (C,f) with C a curve and $f : C \to X$ a degree d map into a variety X. Tautological cohomology classes in the stable compactification $\overline{\mathcal{M}}_g(X,d)$ are known as Gromov-Witten invariants. They appeared in Gromov's fundamental work on pseudo-holomorphic curves in symplectic geometry [2] and Witten's equally fundamental study of topological sigma models [3]. In the special case of genus zero, Gromov-Witten invariants are directly related to the quantum cohomology of X [4] and the symplectic Floer cohomology of the loop space LX [5].

The space $\mathcal{M}_g(X,d)$ is also the primary object of study in the mirror symmetry program [6]. Mirror symmetry is concerned with counting the number of holomorphic curves on Calabi-Yau manifolds, *i.e.* compact Kähler manifolds X with trivial canonical bundle K_X. One tries to define and calculate the generating functions

$$F_g(t) = \sum_d N_{g,d}\, q^d, \qquad q = e^{2\pi i t}, \tag{1.1}$$

where $N_{g,d}$ is the appropriately defined 'number' of genus g, degree d curves on X. It can for example be given by the (orbifold) Euler character of $\mathcal{M}_g(X,d)$.

[In the above we assumed for convenience that $H^2(X)$ is one-dimensional and generated by the Kähler form ω; otherwise, the degree is actually a multi-degree and F_g a multi-variable function. The above definiton should also be slightly modified in the case $g = 0$ or 1, since these curves are not stable. For rational curves $C \cong \mathbb{P}^1$ we pick three hypersurfaces $H_0, H_1, H_\infty \subset X$, Poincaré dual to ω, and consider maps $x : \mathbb{P}^1 \to X$ such that $x(z) \in H_z$ for $z = 0, 1, \infty$. This defines the third derivative F_0''' of F_0. In case of an elliptic curve we pick a point P and demand $x(P) \in H_0$, which then gives F_1'. In this note we will however be mainly concerned with the case $g > 1$.]

The generating functions $F_g(t)$ are more or less by definition topological or, more precisely, *symplectic* manifold invariants of X. They do not depend on the complex structure of X, *i.e.* on the particular point in the moduli space $\mathcal{M}_X$ of manifolds of type X, but there is the obvious dependence on the parameter $t \in H^2(X)$, that labels the symplectic class. The mirror conjecture states that the functions $F_g(t)$ for a Calabi-Yau manifold have an alternative interpretations as *complex* manifold invariants of a family of 'mirror' Calabi-Yau manifolds Y_t, where t is interpreted as a suitable coordinate on $\mathcal{M}_Y$, the moduli space of manifolds of type Y.

Until recently all calculations were concerned with genus zero, where mirror symmetry is suppose to relate the function $F_0(t)$, that computes (part of) the quantum cohomology of X, to variation of Hodge structures for the family Y_t. Actually, the precise formulation of the mirror symmetry conjecture for higher genus, *i.e.* the interpretation of the objects $F_g(t)$ in terms of the geometry of the mirror family Y_t, was not even clear. This has changed remarkably with the beautiful work of Bershadsky, Cecotti, Ooguri and Vafa [7]. They have indicated the nature of the objects associated to Y that are conjecturally equivalent to the invariants F_g associated to X, at least for the case of Calabi-Yau three-folds. This leads to two interesting predictions:

First, $F_g(t)$ should be a meromorphic object that can be obtained as the limit

$$F_g(t) = \lim_{\bar{t}\to\infty} F_g^*(t,\bar{t}) \tag{1.2}$$

of a non-holomorphic section F_g^* of the line bundle $L^{\otimes(2g-2)}$ over $\mathcal{M}_Y$. Here L is the bundle of holomorphic 3-forms with fiber $H^0(K_Y)$. Sections of powers of this line bundle can be considered as generalized modular forms. The limiting objects F_g will have anomalous transformation properties, and will be named quasi-modular forms. So, roughly we have

Claim I: The counting functions $F_g(t)$ on X are quasi-modular forms for the mirror family Y_t. (1.3)

Second, on the mirror manifold we should count 'constant' maps $f : C \to Y$. For genus zero, these maps just send $\mathbb{P}^1$ to a point $y \in Y$. However, for genus $g > 0$ there appear non-trivial 'constant' maps when the curve degenerates completely into thrice-punctured $\mathbb{P}^1$'s and each rational component is mapped to a different point in Y. Such a completely degenerated curve is combinatorically described by a trivalent graph, where the vertices correspond to $\mathbb{P}^1$'s. This reduces the calculation to a sum over (Feynman) graphs with vertices labelled by points in Y:

Claim II: The counting functions F_g^* and F_g can be computed with trivalent Feynman graphs on Y. (1.4)

It is useful to combine the functions F_g^* into the so-called partition function

$$Z^* = \exp \sum_{g=1}^{\infty} \lambda^{2g-2} F_g^* \tag{1.5}$$

(where λ is a section of L^{-1}). This partition function has then a physical interpretation as the path-integral for the Kodaira-Spencer quantum field $\varphi \in \Omega^1(\bigwedge^2 T_Y)$,

$$Z^* = \int [d\varphi] e^{-S(\varphi)}, \tag{1.6}$$

with a cubic action

$$S(\varphi) = \int_Y \left(\tfrac{1}{2}\partial\varphi \wedge \overline{\partial}\varphi + \tfrac{\lambda}{6}\partial\varphi \wedge \partial\varphi \wedge \partial\varphi\right). \tag{1.7}$$

Here the holomorphic 3-form is used to 'integrate' a section of $\Omega^3(\bigwedge^3 T_Y)$. According to standard arguments of perturbative quantum field theory, the objects F_g^*, and therefore also the derived quantities F_g, should be computable by evaluating cubic Feynman diagrams. We expect an expression of the form

$$F_g = \sum_{\Gamma \in G_g} \frac{I_\Gamma}{\#\mathrm{Aut}\,\Gamma}, \tag{1.8}$$

where G_g is the set of closed connected trivalent graphs with Euler number $1-g$. I_Γ is the weight of the diagram Γ and is computed using geometric objects on Y.

Finally, the failure of holomorphicity of the physical objects F_g^* is related to effects due to the compactification of $\mathcal{M}_g(X)$ and is expressed in the so-called holomorphic anomaly equation [7]. It reads symbolically (for $g > 1$)

$$\frac{\partial F_g^*}{\partial \bar{t}} \propto \sum_{h=0}^{g} \frac{\partial F_h^*}{\partial t}\frac{\partial F_{g-h}^*}{\partial t} + \frac{\partial^2 F_{g-1}^*}{\partial t^2}. \tag{1.9}$$

Equivalently, the partition function Z^* satisfies a linear differential equation of the form

$$\frac{\partial Z^*}{\partial \bar{t}} \propto \lambda^2 \frac{\partial^2 Z^*}{\partial t^2}. \tag{1.10}$$

The precise details of all these formulas are rather intimidating and are to a large extend not computable, in the sense that the weight function I_Γ is not explicitely known and that the definition is plagued with the usual divergencies and indeterminacies of a non-renormalizable quantum field theory. Although these problems are likely to be overcome in the future, it would be an overstatement to say that the mirror symmetry conjecture leads at this moment to directly calculable predictions for the functions F_g for all genera in the case of general Calabi-Yau space X, even if the mirror family Y is known. (Although some beautiful formulas were obtained in special examples in low genus [7].)

However, here we will be concerned with a model that is computable and rigorously defined. In fact, our aim will be to show how the above two claims are concretely realized in the simplest example of the mirror symmetry program, the torus or elliptic curve — the unique one-dimensional Calabi-Yau space. In our analysis we will make use of the renewed interest in counting covers of Riemann surfaces (not necessarily of genus one) inspired by the fundamental work of Gross and Taylor on two-dimensional $U(N)$ Yang-Mills theory in the large N expansion [8]. In this remarkable development the classical 19th century work of Hurwitz, Schur *et al.* on the combinatorics of Riemann surfaces has been rediscovered and expanded. It allows us to compute the objects F_g and F_g^* and verify their conjectured properties. It also introduces some interesting modular objects.

Finally a warning: There is hardly any new mathematics in this note. Most of the material is already in the literature. However, the matter is usually not presented from the point of view of mirror symmetry, and it might be useful for (algebraic) geometers in its present form.

2. The mirror of an elliptic curve

We should stress that mirror symmetry for elliptic curves is a simple, but certainly not a vacuous statement. Recall that a Calabi-Yau space X has two kinds of moduli. First of all, we have the moduli space $\mathcal{M}_X$ of inequivalent complex structures. In the case of elliptic curves this is the familiar space $\mathcal{M}_1 = \mathbb{H}/PSL(2,\mathbb{Z})$. That is, we represent the elliptic curve E as $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ and identify τ in the upper-half plane $\mathbb{H}$ by

$$\tau \rightarrow \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}). \tag{2.1}$$

We write $\tau = \tau_1 + i\tau_2$ with imaginary part $\tau_2 > 0$. The second modulus of a Calabi-Yau space is the (complexified) Kähler class $[\omega] \in H^2(X,\mathbb{C})$. In our case we choose to parametrize ω with $t \in \mathbb{H}$ as

$$\omega = -\frac{\pi t}{\tau_2} dz \wedge d\bar{z}, \qquad \int_E \omega = 2\pi i t. \tag{2.2}$$

Again, we write $t = t_1 + it_2$, now with $t_2 > 0$, the area of the surface. The one-dimensional Calabi-Yau space we so obtain is denoted as $E_{\tau,t}$. Mirror symmetry for elliptic curves is simply the interchange of τ and t [9]

$$X = E_{\tau,t} \iff Y = E_{t,\tau}. \tag{2.3}$$

This is already a remarkable formula, since it interchanges two variables with an alltogether different interpretation. It implies that the modular group $PSL(2,\mathbb{Z})$

acts naturally on the Kähler modulus t and thus on the counting functions $F_g(t)$. The transformation $t \to t+1$ is rather obvious, since in the definition of F_g we only use the exponent $q = e^{2\pi it}$. But the interpretation of the second generator of the modular group, $t \to -1/t$, is much less evident. It interchanges large area with small area, and is the most well-known example of a so-called duality transformation in string theory [10]. In fact, the corresponding quantum field theory is defined by a *four*-dimensional lattice in $\mathbb{C}^2$ with a metric of signature $(2,2)$ spanned by the vectors $(1,1)$, (τ,τ), $(t,\bar{t})$, $(t\tau,\bar{t}\tau)$. It has an automorphism group $O(2,2,\mathbb{Z})$ that contains the $\mathbb{Z}_2$ mirror symmetry (2.3), see [9].

We now come to the precise definition of the counting functions $F_g(t)$ for elliptic curves, see also [11]. First of all let us consider the case $g > 1$. A holomorphic map of degree d from a genus g curve C_g to an elliptic curve E is simply a d-fold, connected cover of E. This reduces the problem to combinatorics of S_d, the symmetric group on d elements. Let $X_{g,d}$ be the set of simple branched (topological) covers of genus g and degree d. Simple means here that all branch points have branching number one. The precise definition of the set $X_{g,d}$ in terms of representations of the fundamental group is as follows. Choose b unordered points $P_1,\ldots,P_b \in E$ and let π_1^b be the fundamental group of the b-punctured curve $E - \{P_1,\ldots,P_b\}$. We now have

$$X_{g,d} = \mathrm{Hom}'(\pi_1^b, S_d)/S_d, \tag{2.4}$$

where the prime indicates that: (i) the holonomy around all punctures P_i lies in the conjugacy class of single transpositions in S_d; (ii) the resulting cover is connected. The group S_d acts on the homomorphisms by conjugation. The number of branch points b is determined by the Riemann-Hurwitz theorem,

$$b = 2g-2. \tag{2.5}$$

We see that the number of branch points does not depend on the degree d of the map. This is a general feature of Calabi-Yau spaces: by the Riemann-Roch theorem the (virtual) dimension of $\mathcal{M}_g(X,d)$ is independent of the degree of the map $C_g \to X$. The set $X_{g,d}$ can also be considered as the fibre in the fibration

$$X_{g,d} \to H_{g,d} \to E_{2g-2}, \tag{2.6}$$

where $H_{g,d} = \mathcal{M}_g(E,d)$ is the Hurwitz space of simple branched covers and E_n is the configuration space of n points on E.

We now define the number $N_{g,d}$ of genus g, degree d curves on E as

$$N_{g,d} = \sum_{\xi \in X_{g,d}} \frac{1}{\#\mathrm{Aut}\,\xi}. \tag{2.7}$$

Here $\mathrm{Aut}\,\xi$, the group of automorphisms of a homomorphism ξ, is the product of the centralizer of the image $\xi(\pi_1^b) \subset S_d$ and the group S_b permuting the branch points. Alternatively, one can say we have defined $N_{g,d}$ as the *volume* of the Hurwitz space $\mathcal{M}_g(E,d)$ with respect to the normalized Kähler volume form induced from E. A definition as the Euler characteristic does not make sense here, since that vanishes identically.

The generating functions F_g for $g > 1$ are now given by

$$F_g(q) = \sum_{d=1}^{\infty} N_{g,d}\, q^d, \qquad q = e^{2\pi i t}. \tag{2.8}$$

The case $g = 1$ should be treated separately, since these covers are unbranched and there is consequently also a contribution of degree zero (constant) maps. These maps are not stable and, as explained in §1, the prescription is to first compute the first derivative dF_1/dt. Here we count the constant maps by $-\frac{1}{24} = \chi(\mathcal{M}_1)$. After integration F_1 is obtained as

$$F_1(q) = -\frac{1}{24}\log q + \sum_{d=1}^{\infty} N_{1,d}\, q^d. \tag{2.9}$$

In these counting functions F_g we consider only *connected* covers. However, as already remarked by Hurwitz [12], it is convenient to combine all functions F_g in a two-variable partition function Z that counts *all* covers

$$Z(q,\lambda) = \exp \sum_{g=1}^{\infty} \lambda^{2g-2} F_g(q). \tag{2.10}$$

If we write $Z(q,\lambda) = q^{-\frac{1}{24}}\widehat{Z}(q,\lambda)$, then by nature of the exponential function the partition function $\widehat{Z}$ has an expansion

$$\widehat{Z}(q,\lambda) = \sum_{g,d=1}^{\infty} \widehat{N}_{g,d}\, q^d \lambda^{2g-2}, \tag{2.11}$$

where $\widehat{N}_{g,d}$ is the weighted number of all, not necessarily connected covers of E with Euler number $2-2g$ and degree d,

$$\widehat{N}_{g,d} = \frac{\#\mathrm{Hom}(\pi_1^{2g-2}, S_d)}{d!(2g-2)!}, \tag{2.12}$$

where the holonomy around the b punctures is a cycle of length two in S_d.

3. THREE THEOREMS

We are now in a position to state three theorems about the function Z that originated in quantum field theory. We will briefly sketch the relation with physics.

Theorem 1: 'Yang-Mills' [8] — *Let $G = U(N)$ act on itself by conjugation, let $\mathcal{H} = L^2(G)^G$ be the Hilbert space of invariant, square-integrable functions on G, and let Δ be the Laplacian on G (as constructed from the Haar measure) considered as a self-adjoint operator on $\mathcal{H}$, then*

$$\operatorname{Tr} q^{\Delta/N} = Z(q, 1/N)^2. \tag{3.1}$$

The idea to treat the classical Lie groups in perturbation theory around infinite rank is a very productive idea in physics conceived of by 't Hooft [13]. The left-hand side of the equation is actually the partition function of quantum Yang-Mills theory with gauge group G. That is, if A is a connection on a (trivial) principal G bundle over E, with curvature F, then Z is given by a path-integral

$$Z = \int [dA] e^{-NS(A)}, \qquad S = \int_E \operatorname{Tr}(F \wedge *F). \tag{3.2}$$

We will not say anything here about the relation with gauge theory in the large N limit. The material is excellently covered in [11]. The second, closely related theorem gives an explicit representation for Z:

Theorem 2: 'Fermions' [14] —

$$Z(q,\lambda) = q^{-\frac{1}{24}} \oint \frac{dz}{2\pi z} \prod_{p \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (1 + z q^p e^{\lambda p^2})(1 + z^{-1} q^p e^{-\lambda p^2}). \tag{3.3}$$

This second result is due to an alternative formulation of the partition function as a path-integral in terms of free fermions $b, c \in \Gamma(E, K^{1/2})$,

$$Z^* = \int [dbdc] e^{-S(b,c)}, \qquad S = \int_E b\overline{\partial} c + \lambda b \partial^2 c. \tag{3.4}$$

The above expression is simply the Hamiltonian representation. The integrand in (3.3) is a natural generalisation of the usual theta-function. More generally one can define for $t(p) = \sum t_k p^k$ an arbitrary polynomial (with $z = e^{t_0}$ and $q = e^{t_1}$)

$$\vartheta[t] = \prod_{n \in \mathbb{Z}_{>0}} (1 - q^n) \prod_{p \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} (1 + e^{t(p)})(1 + e^{-t(-p)}). \tag{3.5}$$

These generalized theta-functions appear naturally as characters of modules of the $W_{1+\infty}$ algebra [15, 18]. The familar theta-function identity

$$\vartheta(z, q) = \sum_{n \in \mathbb{Z}} z^n q^{n^2}, \tag{3.6}$$

immediately shows that

$$F_1(q) = -\log \eta(q), \tag{3.7}$$

with $\eta(q)$ Dedekind's eta-function. In fact, the expansion of Z in terms of the functions F_g can be considered as a generalization of (3.6). The modular properties of F_g are given by the following corollary to Theorem 2 [16, 17, 18]

Corollary — *The functions $F_g(q)$ for $g \geq 2$ are quasi-modular forms of weight $6g - 6$, $F_g \in \mathbb{Q}[E_2, E_4, E_6]$.*

This corollary is the realization of the first claim of mirror symmetry, namely that the counting functions have modular properties in terms of the mirror manifold, which here happens to be again an elliptic curve. The first few cases are given by

$$\begin{aligned} F_2(q) &= \frac{1}{103680}(10E_2^3 - 6E_2E_4 - 4E_6), \\ F_3(q) &= \frac{1}{35831808}(-6E_2^6 + 15E_2^4E_4 - 12E_2^2E_4^3 + 7E_4^3 \\ &\qquad + 4E_2^3E_6 - 12E_2E_4E_6 + 4E_6^2). \end{aligned} \tag{3.8}$$

Here the familar Eisenstein series $E_k(q)$ are defined for even $k \geq 2$ as

$$E_k(q) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}. \tag{3.9}$$

If $k > 2$, they are modular forms of weight k. As is well-known E_4 and E_6 generate the ring of modular forms. E_2 is 'quasi-modular' of weight two

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau + d). \tag{3.10}$$

E_2 can be easily made into a proper modular form by allowing a mild anholomorphicity and defining

$$E_2^*(\tau,\overline{\tau}) = E_2(\tau) - \frac{3}{\pi\tau_2}. \tag{3.11}$$

By replacing E_2 by E_2^* we similarly get an anholomorphic partition function F_g^* with

$$F_g^* \in \Gamma(\mathcal{M}_1, L^{2g-2}). \tag{3.12}$$

Here L is the line bundle over $\mathcal{M}_1$ with fibre $L_E = H^0(E, T_E^3)$.

The third theorem expresses the fact that the functions F_g can be computed by Feynman diagrams.

Theorem 3: 'Bosons' [14] — *F_g can be expressed as a sum over cubic graphs*

$$F_g = \sum_{\Gamma \in G_g} \frac{I_\Gamma}{|\mathrm{Aut}\,\Gamma|} \tag{3.13}$$

with weights

$$I_\Gamma = \prod_{vertices\ v} \oint_{z_v \in \gamma_v} \prod_{edges\ e} P(z_{v_+(e)} - z_{v_-(e)}). \tag{3.14}$$

Here the $3g-3$ vertices v of Γ are labeled by points $z_v \in E$. To each of the $2g-2$ edges e correspond two vertices $v_\pm(e)$. The contours γ_v are to be taken non-intersecting and in the homotopy class of the cycle $[0,1]$. The propagator $P(z)$ is given in terms of the Weierstrass $\wp$-function as

$$P(z) = \begin{cases} \frac{1}{4\pi^2}\wp(z) + \frac{1}{12}E_2, & \text{if } z \neq 0, \\ \frac{1}{12}E_2, & \text{if } z = 0. \end{cases} \tag{3.15}$$

This theorem is due to the famous boson/fermion correspondence in two dimensions, which gives a third path-integral expression for the partition function as

$$Z^* = \int [d\varphi] e^{-S(\varphi)}, \qquad S(\varphi) = \int \frac{1}{2}\partial\varphi\overline{\partial}\varphi + \frac{\lambda}{6}(-i\partial\varphi)^3. \tag{3.16}$$

We will now sketch how these results are proven, with emphasis on theorem 2. For more details see [14, 18].

4. Counting Covers

Let G be any finite group, R the set of irreducible representations of G, and C the set of conjugacy classes of G. We denote the character of an element in a class $c \in C$ in a representation $r \in R$ as $\chi_r(c)$. It is furthermore convenient to follow Schur and introduce the notation

$$f_r(c) = \frac{\#c \cdot \chi_r(c)}{\dim r} \tag{4.1}$$

Now let Σ be a closed, oriented, topological surface of genus h. Pick N marked points $P_1, \ldots, P_N \in \Sigma$ and conjugacy classes $c_1, \ldots, c_N \in C$. Let X be the set of equivalence classes of principal G-bundles over Σ with holonomies around the point P_i in the class c_i. That is, if we recall that the fundamental group π_1^N of the N times punctured surface is freely generated by elements $\alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h, \gamma_1, \ldots, \gamma_N$ with the single relation

$$\prod_{i=1}^{h} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = \prod_{j=1}^{N} \gamma_j, \tag{4.2}$$

then X is defined as

$$X = Y/G, \qquad Y = \{\xi \in \mathrm{Hom}(\pi_1^N, G) \mid \xi(\gamma_j) \in c_j\}, \tag{4.3}$$

where the G action is by conjugation. We now want to count all such bundles by computing the 'partition function'

$$Z_h(c_1, \ldots, c_N) = \sum_{\xi \in X} \frac{1}{\#\mathrm{Aut}\,\xi} = \#Y/\#G. \tag{4.4}$$

The automorphism group $\mathrm{Aut}\,\xi$ of a bundle ξ is by definition the centralizer of the image of π_1^N under ξ. There is an elegant formula according to the following lemma:

Lemma [19, 20] —

$$Z_h(c_1, \ldots, c_N) = \sum_{r \in R} \left[\left(\frac{\#G}{\dim r} \right)^{2h-2} \prod_{j=1}^{N} f_r(c_j) \right] \tag{4.5}$$

The proof follows essentially the argument of the Verlinde formula [21]. Consider the center $\mathcal{H}$ of the group algebra $\mathbb{C}[G]$, generated as a vector space by the elements

$$z_c = \sum_{g \in c} g, \qquad c \in C. \tag{4.6}$$

This so-called class algebra is a commutative, associative algebra with identity e and invariant inner product η given by a linear form $\langle \cdot, \cdot \rangle$,

$$\eta(z, z') = \langle z \cdot z' \rangle, \qquad \langle z_c \rangle = \frac{1}{\# G} \delta_{c,e}. \tag{4.7}$$

The class algebra $\mathcal{H}$ is semi-simple and diagonalized by going to an orthogonal basis $\{z_r\}$ labelled by the irreducible representations $r \in R$,

$$z_r = \sum_{g \in G} \chi_r(g) g \tag{4.8}$$

with multiplication

$$z_r \cdot z_{r'} = \delta_{r,r'} \frac{\# G}{\dim r} z_r. \tag{4.9}$$

(One should be careful to distinguish $\mathcal{H}$ with this multiplication from the usual representation ring $z_r \cdot z_{r'} = \sum_{r'' \in r \otimes r'} z_{r''}$.) The calculation of the function Z for a general punctured surface now follows from Verlinde's argument by decomposing the surfaces in $2g - 2 + N$ pairs of pants. One should carefully check that it takes into account the right automorphism groups [20].

The above data define a two-dimensional topological field theory, with 'Hilbert space' $\mathcal{H}$ and 'correlation functions' $Z_h(c_1, \ldots, c_N)$. Note that we have for genus zero

$$Z_0(c_1, \ldots, c_N) = \langle z_{c_1} \cdots z_{c_N} \rangle, \tag{4.10}$$

and — more relevant to our interests — for genus one

$$Z_1(c_1, \ldots, c_N) = \mathrm{Tr}_{\mathcal{H}}(z_{c_1} \cdots z_{c_N}). \tag{4.11}$$

One can think of this model as a two-dimensional gauge theory for the finite group G.

5. FERMIONS AND BOSONS

We now apply the lemma of §4 to the case of a simple d-fold covering of an elliptic curve. That is, we choose G to be the symmetric group S_d on d elements and write C_d, R_d and $\mathcal{H}_d$ for the set of conjugacy classes, irreducible representations and the class algebra. We further put the genus $h = 1$, all conjugacy classes $c_j = c$, where c is the conjugacy class of a simple transposition, and $N = b = 2g - 2$, the number of branch points and minus the Euler number of the cover. We now want to compute (with $Z = q^{-1/24}\widehat{Z}$)

$$\widehat{Z}(q, \lambda) = \sum_{d,b=0}^{\infty} \frac{q^d \lambda^b}{d!b!} \#\mathrm{Hom}(\pi_1^b, S_d). \tag{5.1}$$

With the use of the above lemma that computation reduces simply to to

$$\begin{aligned} \widehat{Z}(q, \lambda) &= \sum_{d,b=0}^{\infty} \frac{q^d \lambda^b}{b!} \sum_{r \in R(S_d)} (f_r(c))^b \\ &= \sum_{d=0}^{\infty} q^d \sum_{r \in R(S_d)} \exp(\lambda f_r(c)). \end{aligned} \tag{5.2}$$

Equivalently, we can express the partition function as

$$\widehat{Z}(q, \lambda) = \sum_{d=0}^{\infty} q^d \mathrm{Tr}_{\mathcal{H}_d} \left(e^{\lambda z_c} \right). \tag{5.3}$$

If we define universal objects $C = \bigcup_{d=0}^{\infty} C_d$, $R = \bigcup_{d=0}^{\infty} R_d$, and the infinite-dimensional, graded algebra

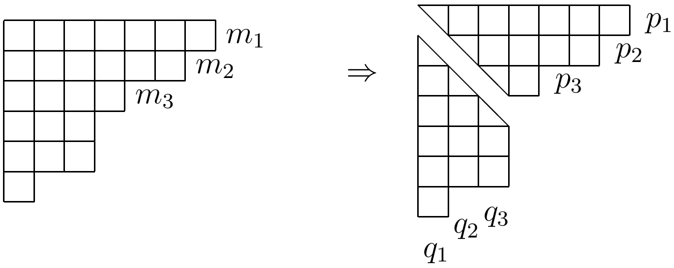
$$\mathcal{H} = \bigoplus_{d=0}^{\infty} \mathcal{H}_d, \tag{5.4}$$

then the partition function can be written as

$$\widehat{Z}(q, \lambda) = \mathrm{Tr}_{\mathcal{H}} \left(q^L e^{\lambda W} \right) \tag{5.5}$$

with L the degree operator ($L = d$ on $\mathcal{H}_d$) and $W = \oplus_d z_c$. The vector space $\mathcal{H}$ has two natural bases: the representation basis $\{z_r\}_{r \in R}$ and the conjugacy class basis $\{z_c\}_{c \in C}$. Both are naturally labeled by partitions. Physically they correspond respectively to fermions and bosons.

More precely, let in the following I be the set of positive half-integers, $I = \mathbb{Z}_{\geq 0} + \frac{1}{2} = \{\frac{1}{2}, \frac{3}{2}, \ldots\}$. Every irreducible representation r of the permutation group S_d is given by a partition of d, or equivalently a Young diagram with d boxes with rows of length $m_1 \geq m_2 \geq \ldots > 0$. Such a Young diagram gives us two subsets $P, Q \subset I$, $P = \{p_1 > p_2 > \cdots\}$, $Q = \{q_1 > q_2 > \cdots\}$, by slicing the diagram through the middle and counting the fraction of boxes respectively in the rows and columns of the two halves, as indicated below



For a representation r labelled by such a pair of subsets P, Q we can define the numbers

$$w_r^k = \sum_{p \in P} p^k - \sum_{p \in Q} (-p)^k. \tag{5.6}$$

These numbers have surprisingly interesting properties

$$\begin{aligned} w_r^0 &= \#P - \#Q = 0, \\ w_r^1 &= d, \\ w_r^2 &= f_r(c), \end{aligned} \tag{5.7}$$

where c is the class of transpositions, cycles of length two. This last formula was first derived by Schur (?). (For general k the quantity w_r^k is again expressed in terms of characters.) We can now replace the sum of over all representation of all symmetric groups S_d by a sum over all subsets $P, Q \subset I$ with $\#P = \#Q$, *i.e.* with $w^0 = 0$. Since every element $p \in I$ either appears once or not, we have a simple formula for the generating functions. If we use the notation

$$t(p) = \sum_k t_k p^k, \qquad w_r = \sum_k t_k w_r^k = \sum_{p \in P} t(p) - \sum_{p \in Q} t(-p), \tag{5.8}$$

then we find a simple generating function identity

$$\sum_{d=0}^{\infty} \sum_{r \in R(S_d)} e^{w_r} = \prod_{p \in I} (1 + e^{t(p)})(1 + e^{-t(-p)}). \tag{5.9}$$

Using the result (5.2) and specializing to $z = e^{t_0}, q = e^{t_1}$ and $\lambda = t_2$ gives immediately the proof of Theorem 2.

To obtain Theorem 3, we have to consider bosons. That is, we have to evaluate the trace (5.5) in the conjugacy class basis. The problem is now that the operator W is no longer diagonal. The corresponding expression (Theorem 3) is therefore much more complicated. Unfortunately, we do not have the space to explain this relation precisely, but have to refer to [14, 18].

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