PUPT-1184
IASSNS-HEP-90/48
May 1990

Loop Equations and Virasoro Constraints in Non-perturbative 2-D Quantum Gravity

Robbert Dijkgraaf* , Herman Verlinde †

Joseph Henry Laboratories

Princeton University

Princeton, NJ 08544

and

Erik Verlinde ‡

School of Natural Sciences

Institute for Advanced Study

Princeton, NJ 08540

Abstract

We give a derivation of the loop equation for two-dimensional gravity from the KdV equations and the string equation of the one matrix model. We find that the loop equation is equivalent to an infinite set of linear constraints on the square root of the partition function satisfying the Virasoro algebra. We give an interpretation of these equations in topological gravity and discuss their extension to multi-matrix models. For the multi-critical models the loop equation naturally singles out the operators corresponding to the primary fields of the minimal models.

*(rhd@pupthy.princeton.edu)
†(hlv@pupthy.princeton.edu)
‡(verlinde@iassns)

1. Introduction

Two dimensional quantum gravity can be formulated as a sum over random surfaces. In the matrix model approach to 2-d gravity [1, 2] this partition function is defined by an appropriate double scaling limit of the matrix integral

$$Z = \int d\phi \, e^{-N \mathrm{tr} V(\phi)} \tag{1.1}$$

where ϕ is an $N \times N$ hermitean matrix and $V(\phi) = \sum_n g_n \phi^n$ is some potential. This double scaling limit, discovered in [3, 4, 5], amounts to taking the large-N limit combined with the continuum limit in which the couplings in V approach their critical values. The usual $1/N$-expansion of the free energy $F = -\log Z$ then goes over into the genus expansion $F = \sum_g \lambda^{2g-2} F_g$ in the string coupling constant λ.

The multi-critical behaviour of the matrix model was first studied by Kazakov [12], who made the important observation that these multi-critical points describe 2-d gravity coupled to conformal matter. In the one matrix model these points are labeled by an integer k and are believed to correspond to the $(p, q) = (2, 2k-1)$ minimal models. In order to describe the general (p, q) minimal model coupled to gravity, one has to consider multi-matrix models [8]-[11]. For this general case, Douglas has proposed a solution in terms of the generalized KdV equations [11]. In this proposal, the non-perturbative partition function of 2-d gravity is given by the square of the τ-function of the KdV hierarchy, satisfying an additional equation called the string equation.

Although there are many points of agreement, several of the matrix model results are still poorly understood from the continuum point of view. Important features, such as the presence of an infinite number of scaling operators and the relation with integrable systems, have not been satisfactory explained up to now. Also, a clear geometrical interpretation of the string equation is still lacking. The purpose of this paper is to gain some insight into these issues.

As our central result we will show that the string equation of [3]-[6] for the one-matrix model can be translated into the form of a loop equation [13, 12]. The advantage of this reformulation is that, in contrast with the string equation, the loop equation has a very clear geometrical interpretation in terms of joining and splitting of loops. We also show that our loop equation, which is a small modification of that found by David [14], can be written as an infinite set of *linear* constraints on the square root $\tau = \sqrt{Z}$ of the partition function Z. Rather surprisingly, these constraints, which uniquely characterize Z, generate a Virasoro algebra [15].

An interesting aspect of these 'Virasoro conditions' is that in the higher multi-critical points they naturally single out the finite number of scaling fields corresponding to the primary fields of the minimal model. The other scaling operators turn out to be redundant fields, in the sense that their correlation functions are determined by the loop equation of motion. Our results therefore give a useful refinement of Kazakov's original analysis of the multi-critical points [12], which in fact was also based on the loop equation.

An elegant interpretation of the results of [3, 4, 5] has been given by Witten [17]. He showed that the amplitudes at genus $g \leq 1$ of the one-matrix model are identical to those of two-dimensional topological gravity (in a non-trivial background), and conjectured that this was true to all orders. In [19] this result was generalized to the n-matrix models, which were found to have the structure of topological gravity coupled to topological matter. In this paper we will show that the loop equations of the one-matrix model imply certain recursion relations for the amplitudes, which strongly indicate that in topological gravity all operators interact purely via contact terms. Motivated by this result, it has been shown recently that such a formulation of pure topological gravity can indeed be given [20], and furthermore that one can derive the same recursion relations as in the one-matrix model, thereby establishing the equivalence of the two systems.

For the n-matrix model there are n different types of loops, and one therefore expects as many loop equations. We have not been able to find a derivation of these general loop equations, but we conjecture that the appropriate generalization of the Virasoro constraints in the one-matrix case is given by a similar set of constraints on $\tau = \sqrt{Z}$ satisfying the W_n algebra. In section 5 we present some evidence supporting this conjecture, and speculate about a possible topological interpretation.

The organization of this paper is as follows. In section 2, after a short review of the results of [4]-[6], we write down the loop equation of the one-matrix model. Its reformulation in terms of Virasoro constraints is given in section 3. In section 4 we discuss the multi-critical points and topological gravity; the multi-matrix models are discussed in section 5. We end with some concluding remarks. In the Appendix we describe the derivation of the loop equation from the string equation.

2. The loop equation in two-dimensional gravity

In this section we will present an exact loop equation for the double scaling limit of the one-matrix model, which is compatible with its non-perturbative solution. First we will review the most important results of [4]-[6].

The partition function $Z(x)$ at these critical points is determined as a function of the (renormalized) cosmological constant x through the so-called string equation. This is a differential equation written in terms of the 'specific heat'

$$u(x) = -\lambda^2 D^2 F(x) \; ; \qquad D \equiv \frac{\partial}{\partial x} \tag{2.1}$$

The specific heat u can be identified with the two-point function $\langle PP \rangle$ of the puncture operator. For the k^{th} critical point the string equation reads

$$R_k[u] = x \tag{2.2}$$

where $R_k[u] = R_k(u, u', u'', \ldots)$ are the so-called Gelfand-Dikii differential polynomials of the KdV hierarchy. These are defined through the recursion relations

$$DR_{k+1}[u] = (\tfrac{1}{2}\lambda^2 D^3 + 2uD + Du)R_k[u] \tag{2.3}$$

The first few polynomials are $R_0 = 1$, $R_1 = u$, $R_2 = \frac{1}{2}(3\lambda^2 u^2 + u'')$.

One can interpolate between the different multi-critical points by switching on sources $t_n, n \geq 1$ which couple to scaling operators σ_n. These operators σ_n can be thought of as creating microscopic loops. Insertion of σ_n correspond to differentiation with respect to the coupling t_n, and is identified with the n^{th} KdV flow of u. That is, we have*

$$\langle \sigma_n PP \rangle = \frac{\partial u}{\partial t_n} = DR_{n+1}[u], \tag{2.4}$$

and the equations (2.3) imply therefore certain recursion relations for the two-point functions $\langle \sigma_n P \rangle = R_{n+1}[u]$. The differential equations (2.4) determine the specific heat u and therefore the partition function Z for arbitrary values of the couplings t_n. The string equation for a general massive model is [6]

$$-x = \sum_{n=1}^{\infty} (2n+1) t_n R_n[u] \tag{2.5}$$

*Here the operators σ_n differ from those in [4]-[6] by a factor $(2n+1)!!/n!$

Here the couplings are chosen such that $t_k = -1/(2k+1),\ t_n = 0, n \neq k$ for the k^{th} multi-critical model.

The equations (2.5) and (2.4) encode the information about all the correlation functions of the scaling fields σ_n. We now want to show that from these results one can extract the exact loop equation of the one-matrix model.

The expectation value of a loop $w(\ell)$ is defined in continuum gravity as the sum over all surfaces with a boundary of fixed length ℓ. In the matrix model macroscopic loops are represented by operators $w(\ell) = \mathrm{tr}\,\phi^M$ where the length $\ell = Ma$ is kept fixed in the continuum limit. Here $a^{-2-\frac{1}{k}} = \lambda N$ for the k^{th} critical point. The macroscopic loop $w(\ell)$ can be expanded in terms of the scaling operators σ_n as [7, 6]

$$w(\ell) = \sum_{n=0}^{\infty} \frac{\ell^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2})} \sigma_n \tag{2.6}$$

Loop equations are relations between the correlation functions of one or more loops. Before taking the continuum limit, they can be derived very directly as Schwinger-Dyson equations from the matrix integral [13]. Here we will be interested in the equations for the continuum model.

We will discuss the loop equations first for the multi-critical models. In order to write them in a compact way it is convenient to introduce a source $J(\ell)$ for the loop $w(\ell)$, so that

$$\Big\langle w(\ell_1) \ldots w(\ell_s) \Big\rangle = \lambda^2 \frac{\delta^s \log Z[J(\ell)]}{\delta J(\ell_1) \ldots \delta J(\ell_s)} \tag{2.7}$$

Here the expectation value $\langle \ldots \rangle$ is defined in the presence of the source $J(\ell)$, where $J(\ell) = 0$ correspond to, say, the k^{th} multi-critical point. Now let us re-express the KdV recursion relations and the string equation in terms of the macroscopic loops. Combining (2.3), (2.4) and (2.6) we find after a short calculation that the expectation value of the loop $w(\ell)$ satisfies the differential equation.

$$D^2 \frac{\partial}{\partial \ell} \Big\langle w(\ell) \Big\rangle = \Big(\tfrac{1}{2} \lambda^2 D^4 + 2uD^2 + (Du)D \Big) \Big\langle w(\ell) \Big\rangle + \frac{Du}{\sqrt{\pi \ell}} \tag{2.8}$$

This relation can be used to express $\langle w(\ell) \rangle$ in terms of the function u and its derivatives, and is equivalent to the KdV equations (2.3) and (2.4). From the string equation (2.5) we can derive

$$\tfrac{1}{2}\Big(\frac{\partial}{\partial\ell}\Big)^{k-\frac{1}{2}}\,\Big\langle w(\ell)\Big\rangle\Big|_{\ell=0} = \int_0^\infty d\ell'\,\ell'\,J(\ell')\Big\langle w(\ell')\Big\rangle + \tfrac{1}{4}x^2 \tag{2.9}$$

where we substituted (2.4) into (2.5) and integrated once with respect to cosmological constant x. The source term on the right-hand side arises by expanding the general string equation (2.5) around the k^{th} critical point. The half-integer power of $\partial/\partial\ell$ is defined by

$$\Big(\frac{\partial}{\partial\ell}\Big)^{k-\frac{1}{2}}\frac{\ell^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})} = \frac{\ell^{n-k}}{(n-k)!}\ , \qquad\qquad n\geq k \tag{2.10}$$

while for $n<k$ the result is zero. The two equations (2.8) and (2.9) contain all known information about the continuum limit of the one-matrix model, and in particular they imply the string equation (2.5).

Let us now turn to the loop equation. In fact, equation (2.9) is already a special case of the loop equation, namely for $\ell = 0$. The idea is to use the KdV relation (2.8) to perturb away from $\ell = 0$ and derive the complete equation for arbitrary values of ℓ. We give the details of this derivation in the Appendix. The exact non-perturbative loop equation for the k^{th} critical point is

$$\begin{aligned}\tfrac{1}{2}\Big(\frac{\partial}{\partial\ell}\Big)^{k-\frac{1}{2}}\Big\langle w(\ell)\Big\rangle \;=\; &\int_0^\infty d\ell'\,\ell'\,J(\ell')\Big\langle w(\ell+\ell')\Big\rangle + \tfrac{1}{8}\lambda^2\ell + \tfrac{1}{4}x^2 \\ &+\int_0^\ell d\ell'\Big(\tfrac{1}{2}\lambda^2\Big\langle w(\ell')w(\ell-\ell')\Big\rangle + \tfrac{1}{4}\Big\langle w(\ell')\Big\rangle\Big\langle w(\ell-\ell')\Big\rangle\Big)\end{aligned} \tag{2.11}$$

¿From (2.11) one can derive equations for the correlation functions of any number of loops by differentiating with respect to the source $J(\ell)$. The physical interpretation of the loop equation (2.11) is as follows. The left-hand side represents the infinitesimal variation of the loop $w(\ell)$. Because one integrates over all geometries, this will have no effect except when the loop touches other loops or itself. The first term on the right-hand side describes the 'contact term' with the other loops, while the last two terms describe the splitting of the loop $w(\ell)$ itself. The terms $\frac{1}{8}\lambda^2\ell + \frac{1}{4}x^2$ may at first appear a bit out of place, but find their origin in the fact that the sphere and the torus admit global conformal transformations. Their presence has the important consequence that the loop equation is inhomogeneous.

In (2.11) there is an unusual relative factor of 2 between the connected and the disconnected splitting term. This factor is one of the many indications that in the one-matrix model there is in fact a doubling of all degrees of freedom, that is, it appears to consist of two identical subsystems. In [21] this doubling effect is attributed to the fact that the string equation is derived for a matrix model with only even potentials. For nonsymmetric critical potentials the partition function will be $\sqrt{Z}$ and the factor 2 disappears. We will continue to work with the doubled partition function Z.

For a general massive model with couplings $t_n = \bar{t}_n$ one finds an identical loop equation, except that the differential operator on the left-hand side is replaced by

$$\tfrac{1}{2}\Big(\frac{\partial}{\partial \ell}\Big)^{k-\frac{1}{2}} \longrightarrow -V'\Big(\frac{\partial}{\partial \ell}\Big) \tag{2.12}$$

where the potential has the expansion $V(z) = \sum \bar{t}_n z^{n+\frac{1}{2}}$. This form of the loop equation is, up to a slight modification, identical to that derived by David from the loop equation in the discrete matrix model.

Notice that by shifting the source $J(\ell)$ we can go from one model to another. In fact, one can formally absorb the potential V into the definition of the J. This is most easily done by reformulating the loop equation in terms of the Laplace transform of the loop

$$w(z) = \int_0^\infty d\ell \, e^{-\ell z} \, w(\ell) = \sum_{n=0}^{\infty} z^{-n-\frac{3}{2}} \sigma_n \tag{2.13}$$

By shifting the Laplace transform of the source $J(z) \to J(z) + V(z)$ we find that the potential term indeed drops out of the loop equation, which takes the form

$$\Big[J'(z)\Big\langle w(z)\Big\rangle\Big]_< + \tfrac{1}{2}\lambda^2\Big\langle w^2(z)\Big\rangle + \tfrac{1}{4}\Big\langle w(z)\Big\rangle^2 + \frac{\lambda^2}{8z^2} + \frac{x^2}{4z} = 0 \tag{2.14}$$

where the subscript $<$ denotes the truncation to the Laurent powers z^n with $n \leq -1$. Correlation functions of a finite number of loops for a particular potential $V(z)$ are obtained by taking derivatives with respect to $J(z)$ and then putting $J(z) = V(z)$. The Laplace transform of the source has the expansion

$$J(z) = \sum_{n=0}^{\infty} t_n z^{n+\frac{1}{2}} \tag{2.15}$$

in terms of the couplings t_n, where from now on we denote the cosmological constant x by $t_0 \equiv x$. In comparing eqn. (2.14) with the results of [6, 7, 14] one has to redefine the loop operator as $\hat{w}(z) = w(z) + V'(z)$.

In [14] David has shown that non-perturbative solutions to the string equation with double poles on the real axis are inconsistent with the loop equations. Since we deduced these loop equations from the string equation, our results seem to be in contradiction with [14]. However, it is important to realize that our proof (as given in the appendix), although non-perturbative in the string coupling constant λ, only applies to the asymptotic expansion of the loop operator in ℓ or z [22]. This implies that the string equation is consistent with the *microscopic* loop equations, formulated in terms of the local scaling operators σ_n, which we will consider in the next section.

3. Virasoro constraints and the τ-function

In this section we will study the loop equation and its consequences for the partition function Z of the continuum one matrix model in more detail. We will find that the loop equation is equivalent to an infinite set of linear constraints on the square root of Z.

¿From the partition function Z one can derive the correlation functions of the scaling operators σ_n by expanding $\log Z$ in the couplings $t_n, n \geq 0$. More precisely, for a general massive model with couplings $t_n = \bar{t}_n$ we have

$$\left\langle \sigma_{n_1} \dots \sigma_{n_s} \right\rangle_{\bar{t}_n} = \lambda^2 \frac{\partial^s}{\partial t_{n_1} .. \partial t_{n_s}} \log Z(t_0, t_1, \dots)\big|_{t_n = \bar{t}_n} \tag{3.1}$$

Here we should note that the point $\bar{t}_n = 0$ is not a suitable point for an expansion in the couplings t_n, because the partition function Z is highly singular for $t_n \to 0$. This point would correspond to a matrix model without a potential $V(\phi) \equiv 0$, which is of course ill defined.

It has been suggested in [11] that the partition function of the one-matrix model is related to the τ-function of the KdV-hierarchy. The precise relation appears to be that the τ-function is equal to the square root of Z

$$Z(t_0, t_1, \ldots) = \tau^2(t_0, t_1, \ldots) \tag{3.2}$$

The statement that $\tau(t_0, t_1, \ldots)$ is a τ-function of the KdV-hierarchy means that the function $u = 2D^2 \log \tau$ is a solution of the generalized KdV equation (2.4). Using (2.3) we can re-write the KdV equation in the form of a recursion relation for τ

$$D^2 \frac{\partial}{\partial t_{n+1}} \log \tau = \Big(\tfrac{1}{2}\lambda^2 D^4 + 2uD^2 + (Du)D \Big) \frac{\partial}{\partial t_n} \log \tau \tag{3.3}$$

In fact we are dealing with a very special τ-function namely one for which u also satisfies the string equation (2.5). This fact uniquely fixes $\tau(t_0, t_1, \ldots)$. An interesting geometric interpretation of the τ-function and the string equation has been given in [16].

Now let us see what we can learn about the τ-function from the loop equation. By expanding the Laplace transformed loop equation (2.14) as a Laurent series in z we obtain an infinite set of relations for the one- and two-point functions of the operators σ_n. Using (3.1) and (3.2) we find that these relations can be expressed as linear, homogeneous differential equations for the τ-function. They take the suggestive form

$$L_n \tau = 0 \qquad (n \geq -1). \tag{3.4}$$

where L_n denotes the differential operator

$$\begin{aligned} L_{-1} &= \sum_{m=1}^{\infty} (m + \tfrac{1}{2}) t_m \frac{\partial}{\partial t_{m-1}} + \tfrac{1}{8}\lambda^{-2} t_0^2 \\ L_0 &= \sum_{m=0}^{\infty} (m + \tfrac{1}{2}) t_m \frac{\partial}{\partial t_m} + \tfrac{1}{16}, \\ L_n &= \sum_{m=0}^{\infty} (m + \tfrac{1}{2}) t_m \frac{\partial}{\partial t_{m+n}} + \tfrac{1}{2}\lambda^2 \sum_{m=1}^{n} \frac{\partial^2}{\partial t_{m-1} \partial t_{n-m}}, \end{aligned} \tag{3.5}$$

The fact that these relations are not linear on the partition function Z but on its square root $\tau = \sqrt{Z}$ is a consequence of the relative factor of 2 in (2.14) between the connected and disconnected two-point function of w.

The equations (3.4) are a consistent set of constraints by virtue of the fact that operators L_n satisfy a closed algebra, namely the Virasoro algebra, truncated to

$n \geq -1$

$$[L_n, L_m] = (n-m)L_{n+m} \tag{3.6}$$

Hence we can think of the τ-function as an $SL(2, \mathbf{C})$ invariant highest weight state of the Virasoro algebra!

The first Virasoro constraint $L_{-1}\tau = 0$ is, given that τ is a τ function of the KdV-hierarchy, equivalent to the string equation. As mentioned before, this already uniquely characterizes τ and indeed is sufficient to derive the other Virasoro constraints. Namely, the KdV relation (3.3) implies the following recursion relation

$$D^2\Big(\frac{L_{n+1}\tau}{\tau}\Big) = \Big(\tfrac{1}{2}\lambda^2 D^4 + 2uD^2 + (Du)D\Big)\Big(\frac{L_n\tau}{\tau}\Big) \tag{3.7}$$

The proof of this relation is described in the Appendix. Equation (3.7) shows that starting from the L_{-1} constraint one can use KdV to obtain the other Virasoro constraints in (3.4) by induction.

An interesting question is whether conversely the Virasoro constraints contain enough information to derive the KdV equations. In the next section we will show that the Virasoro constraints (3.4) determine τ uniquely. Therefore, the KdV equations must somehow be contained in them, but we have not been able to find a direct derivation of this result.

The expressions (3.5) have the same form as the Virasoro operators of a free bosonic scalar field $\varphi(z)$ in two dimensions with anti-periodic boundary conditions $\varphi(e^{2\pi i}z) = -\varphi(z)$. In that case the field φ has a mode expansion in half-integer powers of z

$$\partial\varphi(z) = \sum_{n\in\mathbf{Z}} \alpha_{n+\frac{1}{2}} z^{-n-\frac{3}{2}}, \tag{3.8}$$

If we make the correspondence (here $n \geq 0$)

$$\alpha_{-n-\frac{1}{2}} = \lambda^{-1}(n+\tfrac{1}{2})t_n\ , \qquad \alpha_{n+\frac{1}{2}} = \lambda\frac{\partial}{\partial t_n} \tag{3.9}$$

the Virasoro generators L_n in (3.5) become identified with the negative Laurent coefficients of the stress-tensor

$$T(z) =: \tfrac{1}{2}\partial\varphi(z)^2 : + \frac{1}{16z^2} \tag{3.10}$$

The presence of the $\frac{1}{16}$ in the expression for L_0 is directly related to the fact that the ground state in the twisted sector has conformal dimension $\frac{1}{16}$. The τ-function defines a state $|\Omega_{\bar{t}}\rangle$ in this sector for each value of the coupling $t_n = \bar{t}_n$ via

$$\tau(t_0, t_1, \ldots) = \langle\, t - \bar{t} \,|\, \Omega_{\bar{t}} \,\rangle \tag{3.11}$$

where $\langle t|$ denotes the coherent state

$$\langle\, t \,| = \langle\, 0 \,|\exp\Big(\sum_{n=0}^{\infty} t_n \alpha_{n+\frac{1}{2}}\Big) \tag{3.12}$$

Here $|\,0\,\rangle$ is the ground state in the twisted sector satisfying $\alpha_{n+\frac{1}{2}}|\,0\,\rangle = 0$ $(n \geq 0)$.

The Virasoro constraints (3.4) imply for the state $|\,\Omega_{\bar{t}}\,\rangle$ the following relations

$$L_n|\,\Omega_{\bar{t}}\,\rangle = \lambda^{-1}\sum_m (m+\tfrac{1}{2})\bar{t}_m \alpha_{m+n+\frac{1}{2}}|\,\Omega_{\bar{t}}\,\rangle \tag{3.13}$$

Note when all couplings $\bar{t}_n$ vanish these constraints have no solution: there is no state in the twisted sector which is annihilated by all Virasoro operators L_n with $n \geq -1$. Indeed, as mentioned before, the theory with all $\bar{t}_n = 0$ makes no sense. As all couplings $\bar{t}_n$ approach zero the state $|\,\Omega_{\bar{t}}\,\rangle$ will disappear from the Fock space.

It is interesting to elaborate on the relation between the scalar field $\varphi(z)$ and the loop operator $w(z)$. Combining (2.13), and (3.9) gives

$$\frac{\langle\, 0\,|\partial\varphi(z)|\,\Omega_{\bar{t}}\,\rangle}{\langle\, 0\,|\,\Omega_{\bar{t}}\,\rangle} = \lambda\Big\langle w(z)\Big\rangle_{\bar{t}} \tag{3.14}$$

We may interprete this equation as the statement that the positive frequency modes of the field $\partial\varphi(z)$ create a loop $w(z)$. Correspondingly, the (canonically conjugated) negative frequency modes of $\varphi(z)$ annihilate a loop. It appears, therefore, that the scalar field $\varphi(z)$ plays the role of a second quantized string field. It is tempting to speculate that this may be an explicit realization of the wormhole idea [28] that the effect of topology change in quantum gravity is that the coupling constants become dynamical variables, or, in other words, that classical string backgrounds become dynamical because of string loop effects [29]. There also seems to be a close relation between our considerations and the work of Polchinsky [24] and Das and Jevicki [25] on $d = 1$ string theory, who consider a (two-dimensional) string field very similar to our $\varphi(z)$.

4. Multi-critical points and topological gravity

The Virasoro constraints (3.4)-(3.5) are recursion relations for the amplitudes of the scaling operators σ_n. We will now analyse these relations in the multi-critical points and discuss their geometrical interpretation.

Let us recall that in the k^{th} multi-critical point all t_n's vanish except for $t_k = -1/(2k+1)$ and the cosmological constant $t_0 = x$. Hence, to obtain the recursion relations for the σ_n in this theory, we expand (3.4)-(3.5) in the couplings around this point and in the string coupling constant λ. We find

$$\begin{aligned}\langle\sigma_{n+k}\prod_{m\in S}\sigma_m\rangle_g \;&=\; x\langle\sigma_n\prod_{m\in S}\sigma_m\rangle_g + \sum_{j\in S}(2j+1)\langle\sigma_{j+n}\prod_{m\neq j}\sigma_m\rangle_g \\ &+\sum_{j=1}^{n}\Big\{\langle\sigma_{j-1}\sigma_{n-j}\prod_{m\in S}\sigma_m\rangle_{g-1}+\tfrac{1}{2}\sum_{\substack{S=X\cup Y\\ g=g_1+g_2}}\langle\sigma_{j-1}\prod_{m\in X}\sigma_m\rangle_{g_1}\langle\sigma_{n-j}\prod_{m\in Y}\sigma_m\rangle_{g_2}\Big\}\end{aligned} \tag{4.1}$$

which holds for all $n \geq -1$ (if we define $\sigma_{-1} \equiv 0$), and on any surface. Exceptions to (4.1) are some 1-, 2- and 3-point functions on the sphere and 1-point functions on the torus. Namely, one has $\langle\sigma_{k-1}\rangle_{g=0} = \frac{1}{2}x^2$ and $\langle\sigma_k\rangle_{g=1} = \frac{1}{4}$.

An important observation is that the relations (4.1) can be used to eliminate all operators σ_n with $n \geq k-1$ from the correlator. Thus we can express all correlation functions in the theory in terms of those of a finite set of operators, namely the σ_n with $n \leq k$–2. Notice that these are precisely the operators which are identified with the primary fields in the $(2, 2k-1)$ minimal model! So we find the (4.1) naturally singles out the dressed primary fields of the CFT among the infinite set of operators present in the matrix models. In fact, it makes sense to call the other, non-primary fields *redundant*, since their correlators are determined by the above 'equations of motion'. This identification of the σ_n's with $n \geq k-1$ as redundant operators is also supported by the form of the recursion relations. Namely, (4.1) strongly suggests that their correlation functions vanish except for 'contact interactions' with the other operators (represented by the second term on the r.h.s), or with the possible nodes of the surface (corresponding to the last two terms). The primary fields, on the other hand, are the true physical fields, in the sense that their correlation functions are supported everywhere in moduli space.

Additional evidence for this interpretation is found in recent results obtained in the study of two-dimensional topological gravity, which has been shown to correspond to the first critical point, $k = 1$, [17]-[20]*. In this case, equation (4.1) for $n = -1$ is known as the puncture equation, and has been derived from the topological viewpoint in [19]. It indeed expresses the fact that the puncture operator P only interacts through contact terms. However, since for $k = 1$ all operators are redundant (in the sense of equation (4.1)) it is a natural suggestion that in fact *all* interactions in topological gravity are essentially contact interactions.

In [20] it is shown that such a formulation can indeed be given, and this result is furthermore used to give a field theoretical derivation of the above recursion relations.† The key step in this derivation is that the contact interactions between the σ_n's define a *non-commutative* algebra, isomorphic to the Virasoro algebra. This non-commutativity is consequence of the fact that the operators σ_n 'create' curvature, as well as 'measure' curvature. Related to this is that the contact term of σ_n at σ_m, given by $(2m+1)\sigma_{m+n-1}$, is not symmetric in n and m. As shown in [20], the recursion relations (4.1) follow from the presence of this 'contact term algebra' via a simple consistency requirement. This result, together with the derivation of (4.1) given here, establishes the equivalence, conjectured by Witten, of the one-matrix model with topological gravity.

The geometric picture suggested by these results is that the measure of 2-d topological gravity may be thought of as being fully concentrated on degenerate surfaces. From this viewpoint, the fact that the Virasoro constraints are quadratic in the t_n and $\partial/\partial t_n$ follows from the specific structure of the compactification of the moduli space $\mathcal{M}_{g,s}$ of s-punctured surfaces. Namely, if a puncture (say at x_s) approaches one of the other punctures or a node of the surface, this is described via the formation of a new node, splitting off a sphere with three punctures (one of which is the point x_s) from the rest of the surface. The other two points on this sphere can be either attached to other components of the surface or represent the other operator insertion. These two different possibilities correspond to respectively annihilation ($\partial/\partial t_n$) and creation operators (t_n) in the expressions of the L_n's. The fact that these relations are always *bilinear* in the creation and annihilation operators is because, besides the point x_s, there are always exactly *two* other points on the sphere. This observation will become useful in the next section, where we will discuss a possible geometrical interpretation of the multi-matrix models.

*In fact, the correct correspondence is that the total partition sum $Z = e^{-F}$ of the one-matrix model is the *square* of the partition function of topological gravity [27]

†In (4.1) the normalization of the σ_n's differs from that in [20] by a factor 3^n.

Notice that in the $k = 1$ critical point with $x = 0$, the equations (4.1) are in fact *recursion* relations, in the sense that the right-hand side contains correlation functions with either one operator or one handle less than the left-hand side. Thus we can repeatedly use the above recursion relation to reduce any amplitude to a (unique) expression in terms of the basic building blocks $\langle PPP \rangle_{g=0} = 1$ and $\langle \sigma_1 \rangle_{g=1} = \frac{1}{8}$. Hence, (4.1) determines all correlation functions in the $k = 1$ critical point, and therefore the complete expansion of the τ-function in terms of the couplings. It is reasonable to assume that τ is analytic in this point, and thus we conclude that the loop equations uniquely determine the partition function of pure 2-d gravity to all finite orders in the string coupling.

Finally, we notice that we can re-introduce the loops $w(\ell)$ in topological gravity by the formal expansion (2.6) in terms of the operators σ_n. ¿From the above recursion relations (4.1) one then recovers the loop equations discussed in section 2. It is an intruiging fact that, although ℓ is introduced as a formal expansion parameter, via the loop equations it acquires the interpretation of the length of a boundary. For example, the L_{-1} and L_0 components of the loop equations read

$$\begin{aligned} \Big\langle P w(\ell_1) \ldots w(\ell_s) \Big\rangle &= \Big(\sum_{i=1}^{s} \ell_i \Big) \Big\langle w(\ell_1) \ldots w(\ell_s) \Big\rangle \\ \Big\langle \sigma_1 w(\ell_1) \ldots w(\ell_s) \Big\rangle &= \Big(-\tfrac{1}{2} x \frac{\partial}{\partial x} + \sum_{i=1}^{s} \ell_i \frac{\partial}{\partial \ell_i} \Big) \Big\langle w(\ell_1) \ldots w(\ell_s) \Big\rangle \end{aligned} \tag{4.2}$$

The first equation expresses the fact that the puncture operator measures the length of the boundary loops, whereas the second equation shows that σ_1 generates overall dilations of the surface as well as the loops. Thus, somewhat surprisingly, we see that in a topological theory, in which a priori there is no notion of length, one can nevertheless introduce loop operators. Note, however, that in pure topological gravity with zero cosmological constant x there is no length scale, but as soon as $x \neq 0$ one can measure length in units of $x^{-\frac{1}{2}}$.

5. Multi-Matrix Models and W-algebras

We will now turn to the extension of our results to multi-matrix models. According to Douglas [11] the $p-1$ matrix model is related to the p^{th} generalized KdV hierarchy.

The spectrum of scaling operators is given by an infinite set of operators Φ_n, with $n \geq 1$, $n \neq 0 \pmod p$. The q^{th} multi-critical point of the $p-1$ matrix model corresponds to the (p,q) minimal CFT coupled to gravity [11, 23]. The (dressed) primary fields $\phi_{r,s}$ of the CFT are identified with the operators

$$\phi_{r,s} \equiv \Phi_{-rp+sq} \; ; \qquad \begin{array}{l} 1 \leq r \leq [\frac{sq}{p}] \\ 1 \leq s \leq p-1 \end{array} \tag{5.1}$$

where $[y]$ is the integer part of y. The scaling dimensions of the $\phi_{r,s}$ are given by the KPZ-formula [30] $\Delta_{r,s} = (-rp+sq-1)/(p+q-1)$. The remaining, non-primary operators are conveniently labelled as

$$\Phi_{rp+sq} \; ; \qquad \begin{array}{l} r \geq 0 \\ 1 \leq s \leq p-1 \end{array} \tag{5.2}$$

The origin of these non-primary operators is not yet understood from the view point of continuum 2d gravity. On the other hand, from the view point of the matrix models there seems to be no clear reason to distinguish the primary fields from the other operators. In this section we propose a generalization of the loop equation, which, as we will argue, could somewhat clarify these issues.

Similarly as for the one matrix model the partition function is related to a τ-function of p^{th} KdV hierarchy by

$$Z(t_1, t_2, \ldots) = \tau^2(t_1, t_2, \ldots) \tag{5.3}$$

Here t_n is the coupling constant of the operator Φ_n. (Note that in the case of the one-matrix model this implies we have relabeled the coupling constants $t_k \rightarrow t_{2k+1}$.)

The string equation proposed by Douglas can be represented as a linear constraint on τ. By an analogous calculation as described in the Appendix for the case of the one-matrix model, we find that the KdV equations allow us to derive an infinite set of Virasoro constraints.

$$L_{r-1}\tau = 0 \qquad\qquad r \geq 0 \tag{5.4}$$

where

$$L_{-1} = \sum_{n=p+1}^{\infty} \frac{n}{p} t_n \frac{\partial}{\partial t_{n-p}} + \lambda^{-2} \sum_{n=1}^{p-1} \frac{n(p-n)}{2p^2} t_n t_{p-n},$$

$$
\begin{aligned}
L_0 &= \sum_{n=1}^{\infty} \frac{n}{p} t_n \frac{\partial}{\partial t_n} + \frac{p^2-1}{24p}, \\
L_r &= \sum_{n=1}^{\infty} \frac{n}{p} t_n \frac{\partial}{\partial t_{rp+n}} + \tfrac{1}{2}\lambda^2 \sum_{n=1}^{rp-1} \frac{\partial^2}{\partial t_n \partial t_{rp-n}},
\end{aligned}
\tag{5.5}
$$

Here all summations run over $n \neq 0 \pmod p$. This $c = p-1$ Virasoro algebra is the coherent state realization of the Virasoro algebra of $p-1$ free bosons φ_s, twisted by the different non-trivial elements of $\mathbf{Z}_p$. Such scalar fields have a mode expansion

$$
\partial\varphi_s(z) = \sum_{r\in\mathbf{Z}} \alpha_{r+\frac{s}{p}}\, z^{-r-\frac{s}{p}-1}
\tag{5.6}
$$

with $\alpha_{r+\frac{s}{p}} = \partial/\partial t_{rp+s}$, $\alpha_{-r-\frac{s}{p}} = (r+\frac{s}{p})t_{rp+s}$ $(k>0)$. The Virasoro generators (5.5) are the components of the stress-tensor

$$
T(z) = \sum_{s=1}^{p-1} :\tfrac{1}{2}\partial\varphi_s\partial\varphi_{p-s}(z): + \frac{p^2-1}{24pz^2}
\tag{5.7}
$$

Note that the intercept $(p^2-1)/24p$ equals the sum of the conformal dimensions of the $\mathbf{Z}_p$ twist fields.

Again we would like to interprete the fields $\partial\varphi_s(z)$ as creating and annihilating loops. Because we now have $p-1$ different scalar fields we should also have $p-1$ loops $w_s(\ell)$. (Note that this number equals the number of matrices in the matrix model.) The Virasoro constraints can in the q^{th} critical point be cast into the form of a loop equation for one of the loops $w_s(\ell)$, namely $s = q\,(\mathrm{mod} p)$. For example for the topological point $q=1$ equation (5.4) gives the following loop equation for the first loop $w_1(\ell)$

$$
\begin{aligned}
\Big(\frac{\partial}{\partial\ell}\Big)^{\frac{1}{p}} \Big\langle w_1(\ell) \Big\rangle &= \sum_{r=1}^{p-1} \int_0^{\ell} d\ell' \Big(\tfrac{1}{2}\lambda^2 \Big\langle w_{p-r}(\ell') w_r(\ell-\ell') \Big\rangle \\
&\qquad + \tfrac{1}{4} \Big\langle w_{p-r}(\ell') \Big\rangle \Big\langle w_r(\ell-\ell') \Big\rangle \Big) + \frac{p^2-1}{12p}\ell\lambda^2
\end{aligned}
\tag{5.8}
$$

where we have put all sources $J_s(\ell)$ equal to zero. However, one expects that there are also loop equations describing the effect of infinitesimal deformations of the other loops $w_s(\ell)$ with $s \neq 1$. We will now give a concrete proposal for these loop equations.

It is reasonable to expect that the additional loop equations are again given by certain linear constraints on the τ-function. Furthermore, these constraints should be compatible with the Virasoro conditions (5.4). Hence, we are looking for a natural extension of the Virasoro algebra, which can be expressed in terms of the twisted scalar fields $\varphi_r(z)$. We conjecture that the appropriate extension is the W_p-algebra. We will present some arguments supporting this conjecture below.

Let us first recall some facts about the W_p algebra. Its generators are most easily obtained via the free field realization of the $k=1$ $SL(p,\mathbf{R})$ current algebra in terms of $p-1$ (twisted) bosons, via the Casimir construction of [31]. The algebra W_p has $p-1$ generators $W^{(s+1)}(z)$, $1 \leq s \leq p-1$, of conformal spin $s+1$, where $W^{(2)}(z)$ is the stress-tensor $T(z)$. For instance, in the first nontrivial case, $p=3$, the W_3 algebra is realized on two ($\mathbf{Z}_3$-twisted) scalar fields $\varphi_1(z), \varphi_2(z)$ and is generated by $T(z)$ and the W_3-generator $W^{(3)} =: (\partial\varphi_1)^3 + (\partial\varphi_2)^3 :$. The general W-generator $W^{(s+1)}$ has the Laurent expansion

$$W^{(s+1)}(z) = \sum_n W_n^{(s+1)} z^{-n-s-1}.$$

We now propose that the loop equations in the general multi-matrix model are equivalent to the following W-constraints on the corresponding τ-function ($\tau = \sqrt{Z}$)

$$W_r^{(s+1)}\,\tau = 0\ ; \qquad \begin{array}{l} r \geq -s \\ 1 \leq s \leq p{-}1 \end{array} \tag{5.9}$$

This set of constraints is a closed subset, that is, their commutators do not contain any new constraints. ¿From (5.9) one can derive the corresponding loop equations via the identification $\partial\varphi_s(z) = w_s(z) + J'_{p-s}(z)$.

It is very interesting to consider these W identities in the q^{th} critical point of a $p-1$ matrix model, with p,q relatively prime. In this way we will obtain some more insight into the operator identification of the (p,q) minimal CFT coupled to gravity. At this critical point the coupling $t_{p+q} \neq 0$, while other t_n vanish. In this case the linear terms in the W relations describe insertions the operators Φ_n, according to

$$W_{r-s}^{(s+1)}\,\tau = \langle \Phi_{rp+sq} \rangle\, \tau + \ldots = 0\ ; \qquad \begin{array}{l} r \geq 0 \\ 1 \leq s \leq p{-}1 \end{array} \tag{5.10}$$

where the ellipsis denote higher n-point functions. Hence we see that in the linear term only the non-primary operators (5.2) appear. This therefore implies that, similarly as in the one matrix case, we may use these constraints to eliminate the non-primary operators from the correlator, and express all correlation functions in terms of those containing only the primary fields (5.1). We consider this result, that the W_p-constraints (5.9) precisely select the non-primary fields as the redundant operators, an important indication that our conjecture is correct.

As a further check of our conjecture we have calculated the matrix model partition function for some small values of p with only the couplings t_s, $s = 1, \ldots, 2p-1$ different from zero, and verified that it coincides with the result following from the string equation of [11]. (Note that it is sufficient to verify the condition $W^{(s+1)}_{-s}\tau = 0$, since the other equation are generated by commuting with the L_n's.)

We end this section with a discussion of the above equations in the context of topological gravity and indicate a possible geometrical interpretation. The multi-matrix models have been given an alternative interpretation as a topological matter system coupled to topological gravity in [19]. ¿From this point of view the primary fields are given by $\mathcal{O}_\alpha = \Phi_\alpha$ ($\alpha = 1, \ldots, p-1$), with $\mathcal{O}_1 = P$. The other scaling operators are identified with the (topological) descendants of the primary fields: $\Phi_n = \sigma_r(\mathcal{O}_\alpha)$, with $n = rp + \alpha$. The metric on the primary fields is $\eta_{\alpha\beta} = \delta_{\alpha,p-\beta}$.

Given the identification of the scaling operators with the generalized KdV flows it is easy to calculate that the ghost charge of the operator $\sigma_r(\mathcal{O}_\alpha)$ in the topological model equals $(r-1)p+\alpha-1$, with a background charge of $(2g-2)(p+1)$ on a genus g surface. If one multiplies all charges with $(p-1)$ one obtains as background charge the (real) dimension of the moduli space of flat $SL(p, \mathbf{R})$ connections. Indeed it has been conjectured in [19, 26] that for the multi-matrix model the appropriate moduli space underlying the problem is that of flat $SL(p, \mathbf{R})$ bundles. In this picture the different primary fields would correspond to different 'punctures' on the surface (e.g. related to different holonomies of the gauge field).

Let us first consider the Virasoro relations in the topological point. They can be translated into properties of the correlation functions, by expanding around the $q = 1$ critical point. In this way we obtain the following recursion relations

$$\begin{aligned}\langle \sigma_{n+1}(P) \prod_{(r,\alpha)\in S} \sigma_r(\mathcal{O}_\alpha)\rangle_g &= \sum_{(r,\alpha)\in S} (rp+\alpha)\langle \sigma_{r+n}(\mathcal{O}_\alpha) \prod_{(s,\beta)\neq(r,\alpha)} \sigma_s(\mathcal{O}_\beta)\rangle_g \\ &+ \sum_{s=1}^{n}\sum_{\beta,\gamma} \eta^{\beta\gamma}\Big\{\langle \sigma_{s-1}(\mathcal{O}_\beta)\sigma_{n-s}(\mathcal{O}_\gamma) \prod_{(r,\alpha)\in S} \sigma_r(\mathcal{O}_\alpha)\rangle_{g-1} \qquad (5.11)\end{aligned}$$

$$+ \sum_{\substack{S=X\cup Y \\ g_1+g_2=g}} \tfrac{1}{2}\langle\sigma_{s-1}(\mathcal{O}_\beta)\prod_{(r,\alpha)\in X}\sigma_k(\mathcal{O}_\alpha)\rangle_{g_1}\langle\sigma_{n-s}(\mathcal{O}_\gamma)\prod_{(r,\alpha)\in Y}\sigma_r(\mathcal{O}_\alpha)\rangle_{g_2}\Big\}$$

Here $S = \{(r_1,\alpha_1),\ldots,(r_n,\alpha_n)\}$. Since the form of these equations is very similar to the recursion relation (4.1), the natural interpretation again seems to be that the operators $\sigma_n(P)$ only interact via contact terms.

Finally, let us describe a possible geometrical interpretation along these lines of the W-constraints (5.9). For definiteness, let us consider the special case of the two-matrix model. Here the main novelty compared to the one-matrix case is that the W-constraint is *trilinear* in the t_n and $\partial/\partial t_n$, instead of bilinear. How do we interprete these trilinear terms? As mentioned in the previous section, the bilinear form the Virasoro recursion relations in pure topological gravity is related to the fact that contact terms are associated with splitting off spheres with three punctures. These thrice punctured spheres are rigid, *i.e.* have no moduli. In the two-matrix case, we have two types of punctures P and Q, [19]. We can now deduce from the topological correlation functions which punctured spheres are rigid in the (alleged) moduli space underlying the topological theory. Namely, as shown in [19], the nonvanishing correlators at genus zero are $\langle P^2Q\rangle$ and $\langle Q^4\rangle$, which implies that these are the (candidate) rigid punctured spheres.

Now we would again like to interprete the W-constraint as expressing the fact that the fields $\sigma_n(Q)$ only receive contributions from contact interactions. It is clear that the trilinearity of W then implies that there are now also contact terms if more than two fields and/or nodes come together. The point we would like to make is that this is in fact natural if we assume that, in the compactification of the moduli space, the special punctured spheres $\langle P^2Q\rangle$ and $\langle Q^4\rangle$ play a similar role as $\langle PPP\rangle$ in the ordinary case. Namely, in our geometrical picture, the correlation functions of the theory are concentrated on fully degenerate surfaces, which are now built up from spheres with three as well as *four* punctures. From this it indeed follows that the recursion relations associated with the operators $\sigma_n(Q)$ will contain bilinear as well as *trilinear* terms. The precise form of the recursion relations is furthermore restricted by ghost charge conservation and by consistency requirements of the type discussed in [20]. Thus it indeed seems possible to understand the W-constraints (5.9) geometrically, as being the implementation of (a suitable generalization of) a 'contact term algebra' on the amplitudes of a topological field theory.

6. Concluding remarks

In this paper we have shown that the matrix model results of [3]-[11] can be translated into the form of loop equations. These equations have a clear geometrical interpretation in terms of splitting and joining of loops, and in addition reveal an attractive algebraic structure. They also provide a natural way to divide the scaling operators of the matrix model into redundant and physical operators, by implying that the redundant operators interact solely through contact terms. Because of their naturalness, we believe that the loop equations contain important hints for how to make contact between the matrix models and the continuum formulation of two-dimensional quantum gravity.

An important open question is whether the loop equations can be given a space-time interpretation, perhaps as Schwinger-Dyson equations or Ward identities of some string field theory. The reformulation in terms of Virasoro constraints (or W-constraints) and the correspondence of the partition function with a state in a bosonic Fock space may be a first step in this direction. It is tempting to identify the twisted boson(s) $\varphi(z)$ as representing string field(s) on a one-dimensional target space, where the coordinate z is related to the Liouville field. A *dynamical* picture of string theory, however, does not seem to arise, which may be related to the fact that for $d < 1$ there is no room for a (continuous) time evolution. For this reason it would be interesting to extend our results to the case $d = 1$, where, as shown recently in [24, 25], the string field φ naturally depends on *two* coordinates and has a non-trivial time evolution.

Acknowledgements

We are grateful to T. Banks for valuable discussions and for first suggesting that our original results (eqns (3.4)-(3.5)) should be interpreted as loop equations. We would furthermore like to acknowledge discussions with F. David, J. Distler, P. DiFrancesco, W. Fischler, D. Gross, D. Kutasov, A. Migdal, A. Polyakov, S. Shenker, L. Susskind, and E. Witten. The research of R.D. is supported by DOE grant DE-AC02-76WRO3072, that of E.V. by NSF Waterman Grant 88-17521 and that of H.V. by NSF Grant PHY80-19754.

Note added

While this manuscript was being typed we were informed by J. Distler that results similar to ours have been obtained independently by M. Fukuma, H. Kawai, and R. Nakayama, [32].

Appendix: derivation of the loop equation

In this appendix we describe the proof of the loop equation, starting from the KdV and string equation. We will work with the Laplace transformed loop $w(z)$. The loop equation (2.11) can be written as

$$\mathcal{L}(z) \equiv \Big[J'(z)\Big\langle w(z)\Big\rangle\Big]_{<} + \tfrac{1}{2}\lambda^2\Big\langle w^2(z)\Big\rangle + \tfrac{1}{4}\Big\langle w(z)\Big\rangle^2 + \frac{\lambda^2}{8z^2} + \frac{t_0^2}{4z} = 0 \qquad \text{(A.1)}$$

Our proof of this loop equation is based on the fact that the KdV equations imply the following differential equation for $\mathcal{L}(z)$

$$\Big(\tfrac{1}{2}\lambda^2 D^4 + (2u - z)D^2 + (Du)D\Big)\mathcal{L}(z) = 0 \qquad \text{(A.2)}$$

Before discussing its derivation let us explain why this equation together with the string equation is sufficient to prove (A.1). This is most easily seen by representing $\mathcal{L}(z)$ as a Laurent-expansion in z. The Laurent coefficients can be expressed in terms of the τ-function and the Virasoro operators given in (3.4).

$$\mathcal{L}(z) = 2\lambda^2 \sum_{n\geq -1}\Big(\frac{L_n\tau}{\tau}\Big) z^{-n-2} \qquad \text{(A.3)}$$

Inserting this Laurent expansion (A.3) into (A.2) gives the recursion relation

$$D^2\Big(\frac{L_{n+1}\tau}{\tau}\Big) = \Big(\tfrac{1}{2}\lambda^2 D^4 + 2uD^2 + (Du)D\Big)\Big(\frac{L_n\tau}{\tau}\Big) \qquad \text{(A.4)}$$

Since we know from the string equation that the first Laurent coefficient is zero, $L_{-1}\tau = 0$, we may conclude that by induction all Laurent coefficients of $\mathcal{L}(z)$ vanish, and thus $\mathcal{L}(z) = 0$. We have checked explicitly for the first few constraints that there are no integration constants. For the higher L_n's the absence of integration constants follows automatically from the Virasoro algebra.

Remains to prove (A.2). For this we need to use the KdV relation (2.8) which after Laplace tranformation reads

$$\Big(\tfrac{1}{2}\lambda^2 D^4 + (2u - z)D^2 + (Du)D\Big)\Big\langle w(z)\Big\rangle + \frac{Du}{\sqrt{z}} = 0 \qquad \text{(A.5)}$$

The calculations are straightforward, but somewhat lengthy. In order to present some of the details we will use a short-hand notation. We introduce the differential operator Δ given by

$$\Delta \equiv \Big(\tfrac{1}{2}\lambda^2 D^4 + 2\tilde{u} D^2 + (D\tilde{u})D\Big)\ ; \quad \tilde{u} \equiv u - \tfrac{1}{2}z \tag{A.6}$$

and we drop the explicit z-dependence in de notation of the loop $w = w(z)$ and its source $J = J(z)$. In this short-hand notation the statement we want to prove is that

$$\Delta\mathcal{L} = \Delta\Big[J'\langle w\rangle\Big]_< + \Delta\,\tfrac{1}{2}\lambda^2\langle w^2\rangle + \Delta\,\tfrac{1}{4}\langle w\rangle^2 + \Delta\frac{t_0^2}{4z} \tag{A.7}$$

vanishes. For the separate terms we find

$$\begin{aligned}
\Delta\Big[J'\langle w\rangle\Big]_< &= \frac{1}{\sqrt{z}}\Big(\lambda^2 D^3 + 2\tilde{u}D + \tfrac{1}{2}D\tilde{u}\Big)\langle w\rangle - \tfrac{1}{2}t_0 D\tilde{u} \\
\Delta\,\langle w^2\rangle &= -\frac{1}{\sqrt{z}}D^3\langle w\rangle - 2(D^2\langle w\rangle)^2 - D\langle w\rangle D^3\langle w\rangle \\
\Delta\,\langle w\rangle^2 &= -2D\tilde{u}\langle w\rangle + 4\lambda^2 D\langle w\rangle D^3\langle w\rangle + 3\lambda^2\Big(D^2\langle w\rangle\Big)^2 + 4\tilde{u}\Big(D\langle w\rangle\Big)^2 \\
\Delta\, t_0^2 &= 2t_0 D\tilde{u} + 4\tilde{u}
\end{aligned} \tag{A.8}$$

In the first equation we used (A.5) and $L_{-1}\tau = 0$; the terms on the right-hand side arise from commuting Δ through the source $J'(z) \sim \frac{1}{2}t_0/\sqrt{z} + \ldots$ The second equation follows by taking the functional derivative of (A.5) with respect to the source J, where one uses $\delta u/\delta J = D^2\langle w\rangle$. Finally, to obtain the third equation one has to work out the left-hand side according to the chain rule and again use (A.5). Inserting the different contributions into (A.7) yields

$$\begin{aligned}
\Delta\mathcal{L} &= \frac{1}{z}\tilde{u} + \frac{1}{\sqrt{z}}(\tfrac{1}{2}\lambda^2 D^3 + 2\tilde{u}D)\langle w\rangle \\
&\quad + \tfrac{1}{2}\lambda^2 D\langle w\rangle D^3\langle w\rangle - \tfrac{1}{4}\lambda^2\Big(D^2\langle w\rangle\Big)^2 + \tilde{u}\Big(D\langle w\rangle\Big)^2
\end{aligned} \tag{A.9}$$

To see that this expression vanishes we take its derivative. The result is

$$D(\Delta\mathcal{L}) = \Big(D\langle w\rangle + \frac{1}{\sqrt{z}}\Big)\Big(\Delta\langle w\rangle + \frac{Du}{\sqrt{z}}\Big) = 0 \tag{A.10}$$

where we again used (A.5). Finally one can verify that there is no integration constant, which completes the proof of (A.2) and thus the loop equation (A.1).

References

[1] V. Kazakov, Phys. Lett. **159B** (1985) 303; F. David, Nucl. Phys. **B257** (1985) 45; V. Kazakov, I. Kostov, and A. Migdal, Phys. Lett. **157B** (1985) 295; J. Fröhlich, "The Statistical Mechanics Of Surfaces," in *Applications Of Field Theory To Statistical Mechanics*, ed L. Garrido (Springer, 1985).

[2] F. David, Phys. Lett. **159B** (1985) 303; D. Boulatov, V. Kazakov, and A. Migdal, Nucl. Phys. **B275 [FS117]** (1986) 543; A. Billoire and F. David, Phys. Lett. **186B** (1986) 279; J. Jurkievic, A. Krzywicki, and B. Peterson, Phys. Lett. **186B** (1986) 273; J. Ambjorn, B. Durhuus, J. Fröhlich, and P. Orland, Nucl. Phys. **B270 [FS16]** (1986) 457.

[3] E. Brézin and V. Kazakov, Phys. Lett. **B236** (1990) 144.

[4] M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635.

[5] D.J. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 127.

[6] T. Banks, M. Douglas, N. Seiberg, and S. Shenker, Phys. Lett. **238B** (1990) 279.

[7] D.J. Gross and A. Migdal, "A Nonperturbative Treatment Of Two Dimensional Quantum Gravity," Princeton preprint PUPT-1159 (1989).

[8] E. Brézin, M. Douglas, V. Kazakov, and S. Shenker, Phys. Lett. **237B** (1990) 43.

[9] D.J. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 717.

[10] C. Crnković, P. Ginsparg, and G. Moore, "The Ising Model, The Yang-Lee Edge Singularity, And 2D Quantum Gravity," Yale preprint YCTP-P20-89 (1989).

[11] M. Douglas, Phys. Lett. **238B** (1990) 176.

[12] V. Kazakov, Mod. Phys. Lett. **A4** (1989) 2125.

[13] A.A. Migdal, Phys. Rep. **102** (1983) 199

[14] F. David, "Loop equations and Non-perturbative effects in two dimensional quantum gravity"-Saclay-preprint SPhT/90-043, March 90.

[15] We would like to thank E. Witten for first suggesting the possibility of a Virasoro structure in the one-matrix model.

[16] G. Moore, "Geometry of the String Equations", Yale-preprint YCTP-PA-90.

[17] E. Witten, "On The Topological Phase Of Two Dimensional Gravity," IAS preprint IASSNS-HEP-89/66 (1989).

[18] J. Distler, "2D Quantum Gravity, Topological Field Theory And The Multicritical Matrix Models," Princeton preprint PUPT-1161 (1989).

[19] R. Dijkgraaf and E. Witten, "Mean Field Theory, Topological Field Theory, and Multi-Matrix Models," IAS/Princeton preprint IASSNS-HEP-90/18, PUPT-1166 (1990).

[20] E. Verlinde and H. Verlinde, "A Solution Of Two Dimensional Topological Quantum Gravity," IAS/Princeton preprint IASSNS-HEP-90/40, PUPT-1176 (1990).

[21] C. Bachas and P. Petropoulos, "Doubling of Equations and Universality in Matrix Models of Random Surfaces," CERN preprint CERN-TH.5714/90 (1990).

[22] We acknowledge discussions with T. Banks and S. Shenker on this point.

[23] P. DiFrancesco and D. Kutasov, "Unitary Minimal Models Coupled to Two Dimensional Quantum Gravity," Princeton preprint PUPT-1173 (1990).

[24] J. Polchinsky, "Critical Behavior of Random Surfaces in One Dimension," University of Texas preprint UTTG-15-90 (1990).

[25] S.R. Das and A. Jevicki, "String Field Theory and Physical Interpretation of D=1 Strings", Brown University preprint BROWN-HET-750 (1990)

[26] E. Witten, "Surprises With Topological Field Theories," IAS preprint IASSNS-HEP-90/37 (1990).

[27] E. Witten, private communication

[28] S. Coleman, Nucl. Phys. **B 307** (1988) 864.

[29] W. Fischler and L. Susskind, Phys. Lett. **171B** (1986) 383, **173B** (1986) 262.

[30] V. Kniznik, A. Polyakov, A. Zamolodchikov, Mod. Phys. Lett. **A 3** (1988) 819; F. David, Mod. Phys. Lett. **A 3** (198) 1651; J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509

[31] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. **B** 304 (1988) 348,371

[32] M. Fukuma, H. Kawai, and R. Nakayama, "Continuum Schwinger-Dyson Equations and Universal Structures in Two-Dimensional Quantum Gravity", Tokyo preprint UT-562 (1990).