LECTURES ON FOUR-MANIFOLDS AND TOPOLOGICAL GAUGE THEORIES

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ABSTRACT

I give an elementary introduction to the theory of four-manifold invariants and its relation with topological field theory. I review the recent developments in the theory of Donaldson and Seiberg-Witten invariants.

1. A Lecturer's Apology

Why another set of lectures on four-manifolds and topological field theory? Apart from the contractual obligations imposed upon the lecturer there might be some justification in the following. The field of four-dimensional topological field theory has seen some remarkable advances in the last year. This has brought the topic of four-manifolds and Donaldson theory to the attention of a much bigger audience in the physics community. More or less as a service to those who are interested but feel intimidated to read the mathematical literature or the recent research papers, I present an elementary introduction to the field in these lectures. I am very much aware of the limited scope of the material as treated here. I will have to refer to the literature for further reading. Luckily, there are some very good books [3, 4, 5] and recently some excellent review papers have appeared that I can warmly recommend [1, 11].

These lectures are organised as follows. I will first review in §2 some standard material about four-dimensional geometry. One could call these classical invariants. The introductory chapter in the book by Donaldson and Kronheimer is a good place to read more about this. I will then proceed in §3 by sketching the ideas of Donaldson theory and Seiberg-Witten theory. Since these invariants are closely related to quantum field theory, one can regard these as quantum invariants. Finally in §4 I provide a very brief introduction to topological field theories in general.

2. Classical Invariants

It is a remarkable fact that in many aspects dimension four is as distinctive from a mathematical perspective as it is from a physical perspective. Many theorems that can be proven in generality for manifolds of dimension $d \geq 5$, such as the generalized Poincaré conjecture (Smale 1961), meet fundamental difficulties in the case of four dimensions. With $d \geq 5$ generic and $d \leq 3$ often simple enough 'to do by hand', this leaves $d = 4$ as the most challenging arena in differential topology. It has been a beautiful development that the richness of four-dimensional physics, with asymptotically free non-abelian gauge theories, can be brought to bare upon these difficult mathematical problems.

The classification problem of 4-manifolds is very much different in nature than, say, the classification of surfaces. Compact, orientable surfaces are topologically classified by their genus $g \in \mathbb{N}$. That the situation is not that simple in four dimensions we can already see by considering the most important invariant of any manifold — the fundamental group π_1. It is a classical theorem in topology that any finitely representable group (*i.e.* a group that can be represented by a finite number of generators satisfying a finite number of relations — not a very severe restriction) can appear as the fundamental group of a four dimensional manifold. There exist a surgery algorithm that constructs the required manifold starting from the generators and the relations. Markov's solution of the word problem shows that the question whether two finitely representable groups are actually isomorphic is undecidable. That is, there is no computing algorithm that can decide this question within a guaranteed finite time. This theorem has therefore rather dramatic consequences for the classification problem of manifolds in dimensions $d \geq 4$. There is simply not a conceivable 'list' of four-manifolds. This fundamental problem can however easily be circumvented by assuming that the four-manifold X is simply connected

$$\pi_1(X) = 0. \tag{2.1}$$

This we will often assume in the following.

2.1. The intersection form

After the fundamental group, the next important invariant is the second cohomology group $H^2(X)$. We recall that De Rham cohomology groups $H^k(X)$ of a compact manifold X are defined as the space of closed differential forms of degree k modulo exact forms

$$d\alpha = 0, \qquad \alpha \sim \alpha + d\beta. \tag{2.2}$$

The Betti numbers b_k of X are the dimensions of these groups

$$b_k = \dim H^k. \tag{2.3}$$

To understand better the role played by the second cohomology of a four-manifold, it might be instructive to consider first the analogous situation for two-dimensional surfaces, if only because it is so much more easily visualized.

Fig. 1: *A canonical homology basis for a surface of genus g.*

For any two-dimensional surface X we can consider the first homology group $H_1(X)$ of homology cycles or equivalenty the dual cohomology group $H^1(X)$ of 1-cocycles. We can think of a 1-cocycle physically as a flat abelian gauge field A, $F = dA = 0$. The pairing between a cycle $C \in H_1$ and a cocycle $A \in H^1$ is the Wilson line

$$\int_C A. \tag{2.4}$$

For a genus g surface H^1 has rank $b_1 = 2g$. It is naturally a symplectic space by the intersection form

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta. \tag{2.5}$$

Q is an integer, unimodular *anti-symmetric* form. By a standard theorem in symplectic linear algebra there exists a canonical symplectic base $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g$, satisfying

$$Q(\alpha_i, \beta_j) = -Q(\beta_i, \alpha_j) = \delta_{ij}. \tag{2.6}$$

with all other intersections vanishing. The dual homology basis a_i, b_i we can picture as in *fig.* 1.

If we have a orientation-preserving diffeomorphism on the surface, it will act on the first cohomology by a symplectic transformation. That is, there is a map (actually a surjection)

$$\mathrm{Diff}^+(X) \to Sp(2g, \mathbb{Z}). \tag{2.7}$$

In four dimensions the analogue object to consider is the second cohomology group $H^2(X)$. A physical picture to keep in mind is that, instead of considering Wilson lines, we now look at magnetic fluxes F (satisfying the Bianchi identity $dF = 0$) through a two-cycle Σ (a linear combination of surfaces)

$$\int_\Sigma F. \tag{2.8}$$

In four dimensions we have an analogous intersection form

$$Q : H^2 \times H^2 \to \mathbb{R} \tag{2.9}$$

defined by

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta =: \alpha \cdot \beta. \tag{2.10}$$

In the four-dimensional case Q is *symmetric* and non-degenerate. Over $\mathbb{R}$ such a form can be diagonalized and thus is easily classified by its rank

$$rank\, Q = b_2 = \dim H^2 \tag{2.11}$$

and signature (b_2^+, b_2^-) (the number of positive respectively negative eigenvalues). The signature $\sigma(X)$ of the 4-manifold X is defined as the difference of positive and negative eigenvalues of Q

$$\sigma = b_2^+ - b_2^-. \tag{2.12}$$

It has a local expression due to Hirzebruch's signature theorem

$$\sigma(X) = \int_X \frac{1}{3} p_1, \tag{2.13}$$

with the first Pontryagin class p_1 given in terms of the Riemann curvature R as

$$p_1 = -\frac{1}{8\pi^2} \mathrm{Tr}\, R \wedge R. \tag{2.14}$$

The signature σ and the Euler character χ are the two classical invariants of a four-manifold. The Euler character is of course defined as

$$\chi = \sum_k (-1)^k b_k = 2 - 2b_1 + b_2 \tag{2.15}$$

and has a local expression

$$\chi = \int_X e, \qquad e \sim \mathrm{Pf}\, R. \tag{2.16}$$

However, there is more sophisticated information hidden in the intersection matrix Q since it is actually defined on the lattice

$$H^2_{\mathbb{Z}} := H^2(X, \mathbb{Z}). \tag{2.17}$$

We can think of the elements of this lattice as *quantized* fluxes, in the sence that

$$F \in H^2_{\mathbb{Z}} \Rightarrow \int_\Sigma F \in \mathbb{Z} \tag{2.18}$$

for all surfaces Σ.

Let us briefly review some general facts about lattices. A lattice Γ is by definition a $\mathbb{Z}$-vector space $\Gamma \cong \mathbb{Z}^n$ together with a non-degenerate, symmetric bilinear form

$$Q : \Gamma \times \Gamma \to \mathbb{Z}. \tag{2.19}$$

The dual lattice Γ^* is defined as the set

$$\Gamma^* = \{x \in \mathbb{R}^n,\ Q(x, y) \in \mathbb{Z},\ \forall y \in \Gamma\}. \tag{2.20}$$

A lattice is self-dual, $\Gamma^* = \Gamma$ if and only if the intersection form is unimodular

$$\det Q = \pm 1. \tag{2.21}$$

By Poincaré duality this is indeed the case for $H^2_{\mathbb{Z}}$, which is therefore always a self-dual lattice. A lattice Γ is called even if

$$\alpha^2 \in 2\mathbb{Z}, \qquad \forall \alpha \in \Gamma \tag{2.22}$$

and odd otherwise. For even lattices the signature satisfies necessarily

$$\sigma \equiv 0 \pmod 8. \tag{2.23}$$

The classification of self-dual lattices now proceeds depending on whether the intersection form is even/odd and whether is it is (positive or negative) definite or not. The various possibilities are listed in the following table

	odd	even
indef	$p\mathbf{1} \oplus q(-\mathbf{1})$	$pH \oplus qE_8$
def	$n\mathbf{1}$, *exotic*	*exotic:* E_8, D_{16}, Leech, etc.

Here H denotes the two-dimensional hyperbolic lattice

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.24}$$

and E_8 the root lattice of the Lie algebra of the same name. The most remarkable fact is that there exists (for given rank and signature) a *unique* indefinitive self-dual lattice — a theorem by Hasse and Minkowski. In the odd case it is simply diagonal, in the even case a sum of hyperbolic and E_8 lattices.

Unfortunately, the situation in the definite case is much more complicated, because there are many more possibilities apart from the obvious cubic lattice $n\mathbf{1}$. These are usually referred to as exotic lattices. Although the number of possibilities remains finite for given rank, it grows rather dramatically: for rank $8, 16, 24, 32, \ldots$ we find $1, 2, 24, > 10^7, \ldots$ inequivalent even definite self-dual lattices.

It might be helpful to mention some concrete examples of well-known 4-manifolds and their intersection forms

X	Q
$\mathbb{CP}^1$	$\mathbf{1}$
$\overline{\mathbb{CP}^1}$	$-\mathbf{1}$
$S^2 \times S^2$	H
T^4	$\mathbf{31} \oplus 3(-\mathbf{1})$
$K3$	$3H \oplus 2(-E_8)$

Here a bar indicates a complex manifold with opposite orientation. (Every complex manifolds comes with a preferred orientation.)

Just as in the case of Riemann surfaces we can study the representation of the diffeomorphism group on $H^2_{\mathbb{Z}}$. Since this action should preserve the intersection form, there is a homomorphism of the group of orientation preserving diffeomorphisms $\mathrm{Diff}^+(X)$ into the arithmetic group $O(Q,\mathbb{Z}) = \mathrm{Aut}\, H^2_{\mathbb{Z}}$, the group of rotations that leaves the lattice fixed,

$$\mathrm{Diff}^+(X) \to O(Q,\mathbb{Z}). \tag{2.25}$$

The relevant question is whether this map is actually onto (as was the case for the modular group for a Riemann surface). We will see that this is an important issue for Donaldson theory, where we construct differential invariants out of classes in $H^2_{\mathbb{Z}}$. Since a manifold invariant should by definition be invariant under diffeomorphisms, these Donaldson polynomials will necessarily be constructed out of tensors on H^2 that are invariant under the action of the image of Diff^+. In case that this image is the full group $O(Q,\mathbb{Z})$ standard invariant theory teaches us that the only invariants are tensors constructed out of the intersection form Q itself.

There is one more set of characteristic classes that we have not mentioned yet, the Stiefel-Whitney classes

$$w_i \in H^i(X, \mathbb{Z}_2). \tag{2.26}$$

For the case of a compact, simply-connected orientable manifold we have automically $w_1 = w_3 = 0$. This leaves w_2 as the only new characteristic class. It is the obstruction to a spin-structure. The class w_2 plays an important role if we consider the lattice reduced modulo 2, because of Wu's formula that states that for all $x \in H^2_{\mathbb{Z}}$

$$w_2 \cdot x = x^2 \pmod 2. \tag{2.27}$$

This implies that Q is necessarily even if $w_2 = 0$ or equivalently if X is spin.

A central question in four-manifold theory is to which extend a simply connected manifold is determined by its intersection form. Remarkably there are two results, both announced in 1982, that give a very different picture. The mathematicians involved in this, Michael Freedman and Simon Donaldson, have gone on to win 1986 Fields Medals for their exceptional work.

It all depends on whether we consider the classification question in the category of topological manifolds or smooth manifolds. That is, do we allow transition functions that are only continuous or do we insist that they are actually $\mathcal{C}^\infty$. In the topological category Freedman's theorem gives a very simple classification. Roughly speaking there is a four-manifold for every self-dual lattice. More precisely, we have to distinguish the cases of $w_2 = 0$ or $w_2 \neq 0$. The 4-manifolds with $w_2 = 0$ are in one-to-one correspondence with even, self-dual lattices, but the for the case $w_2 \neq 0$ there are actually two different manifolds with a given Q (distinguished by the Kirby-Siebenmann invariant that determines if $X \times S^1$ is smooth or not). So, if we work in the topological category, that is, do not worry if the transition functions are smooth, there is a beautiful and simple classification of the simply-connected 4-manifolds: they are just classified by their intersection form (up to a two-fold ambiguity in the non-spin case.)

In the smooth category the situation is much more complicated. Not all intersection forms are realized and there can be many non-equivalent manifolds with the same intersection form. This became only fully clear by the work of Donaldson, but was anticipated by Rohlin's theorem (1952) that states that a smooth spin manifold satisfies

$$\sigma \equiv 0 \pmod{16}. \tag{2.28}$$

(Recall that topologically we only have $\sigma \equiv 0 \pmod 8$.) This can be understood by considering the index theorem of the Dirac operator.

2.2. The Lorentz group $SO(4)$

It might be convenient to also review some standard facts concerning the Lorentz group $SO(4)$ and its representations. Consider a 4-dimensional vector space $V \cong \mathbb{R}^4$ and the corresponding space of two-forms $\Lambda^2 \cong \mathbb{R}^6$. If we pick a volume form $e \in \Lambda^4$, then we can define the intersection form

$$q : \Lambda^2 \times \Lambda^2 \to \mathbb{R} \tag{2.29}$$

by

$$\alpha \wedge \beta = q(\alpha, \beta) e, \qquad \alpha, \beta \in \Lambda^2. \tag{2.30}$$

It is easily verified by explicit computation that the symmetric bilinear form q has signature $(3,3)$. The linear transformations $SL(4,\mathbb{R})$ preserve the volume form and induce linear maps on Λ^2 that preserve q. Thus we have a natural map

$$SL(4,\mathbb{R}) \to SO(3,3). \tag{2.31}$$

This is actually a double cover of the identity component of $SO(3,3)$.

The intersection form q is not enough to decompose Λ^2 into maximal positive and negative subspaces

$$\Lambda^2 \cong \Lambda^2_+ \oplus \Lambda^2_-. \tag{2.32}$$

In fact, there is a moduli space of such decompositions parametrized by the grassmannian

$$SO(3,3)/SO(3) \times SO(3). \tag{2.33}$$

To find such an explicit decomposition we can pick a metric (positive inner product) g on V. This induces a metric on Λ^2 (also denoted by g) which decomposes Λ^2 into orthogonal eigenspaces of the Hodge $*$-operator, defined by

$$g(\alpha,\beta) = q(\alpha, *\beta). \tag{2.34}$$

The $*$-operator squares to one

$$*^2 = 1, \tag{2.35}$$

and the spaces $\Lambda^2_\pm$ can now be defined as the eigenspaces with $* = \pm$: the self-dual (SD) respectively anti-self-dual (ASD) forms. The Hodge star only depends on the conformal class of the metric g, which can be identified with the grassmannian (2.33). The choice of a metric g gives rise to a natural homomorphism of the groups

$$\begin{aligned} SO(4,\mathbb{R}) \quad\to\quad & SO(3) \times SO(3) \\ & = SO(3,3) \cap SO(6), \end{aligned} \tag{2.36}$$

which is again a two-fold cover.

This local picture can be repeated globally by considering the tangent bundle TX. There is a splitting of the 2-forms into SD and ASD pieces

$$\Omega^2 = \Omega^2_+ \oplus \Omega^2_-. \tag{2.37}$$

Descending to cohomology, we find a similar decomposition

$$H^2 = H^2_+ \oplus H^2_- \tag{2.38}$$

with

$$b_{\pm}^2 = \dim H_{\pm}^2. \tag{2.39}$$

We also notice that (anti)self-dual cocycles are necessarily harmonic

$$d\alpha = 0,\ *\alpha = \pm\alpha \Rightarrow d^*\alpha = 0. \tag{2.40}$$

Since we are also interested in fermions, we will in general consider representations of the universal cover group of $SO(4)$

$$Spin(4) \cong SU(2)_+ \times SU(2)_-. \tag{2.41}$$

Its irreducible representations are labeled by the dimensions (n_+, n_-) with $n_\pm \in \mathbb{N}$. In particular we have

$$\begin{array}{rcl}
\textit{scalar}: & \psi & (\mathbf{1},\mathbf{1}), \\
\textit{chiral spinor}: & \psi_\alpha & (\mathbf{2},\mathbf{1}), \\
\textit{anti-chiral spinor}: & \psi_{\dot\alpha} & (\mathbf{1},\mathbf{2}), \\
\textit{vector}: & \psi_{\alpha\dot\alpha} = \psi_\mu & (\mathbf{2},\mathbf{2}), \\
\textit{SD 2-form}: & \psi_{\alpha\beta} = \psi^+_{\mu\nu} & (\mathbf{3},\mathbf{1}), \\
\textit{ASD 2-form}: & \psi_{\dot\alpha\dot\beta} = \psi^-_{\mu\nu} & (\mathbf{1},\mathbf{3}).
\end{array}$$

We already saw in detail how the (A)SD 2-forms transform as vectors under the two factors of $SO(3) \times SO(3) \cong SO(4)/\mathbb{Z}_2$.

The various fields that we will meet later will be sections of the associated vector bundles. As we already mentioned, the spinor bundles only exists if the 4-manifold is spin $w_2 = 0$.

2.3. Complex vector bundles

We will also need some basic knowledge about characteristic classes of complex vector bundles. Let

$$E \to X \tag{2.42}$$

be a complex vector bundle over X of $rank\, E = k$, $E_x \cong \mathbb{C}^k$. Such a bundle has Chern classes

$$c_i \in H^{2i}(X, \mathbb{Z}), \qquad i = 1, \ldots, k. \tag{2.43}$$

There are local expression for the Chern classes. If we pick a connection A with curvature $F = dA + A^2$ we have the following expressions. In the case of a line bundle, $k = 1$,

$$c_1(L) = \frac{iF}{2\pi} \in H^2_{\mathbb{Z}}. \tag{2.44}$$

(Note that a mathematical abelian connection A is pure imaginary.) More general the total Chern class is defined as

$$c = \sum_i c_i = \det\left(\mathbf{1} + \frac{i}{2\pi}F\right). \tag{2.45}$$

We will also make use of the Chern character

$$ch = \sum ch_i = \mathrm{Tr}\,\exp\left(\frac{i}{2\pi}F\right) \tag{2.46}$$

with in particular

$$\begin{aligned} ch_0 &= rank, \\ ch_1 &= c_1, \\ ch_2 &= \tfrac{1}{2}c_1^2 - c_2 = -\frac{1}{8\pi^2}\mathrm{Tr}\,F \wedge F. \end{aligned} \tag{2.47}$$

The instanton charge of a bundle is defined to be

$$n = -\int_X ch_2. \tag{2.48}$$

We also introduce the concept of a monopole charge. This is an integer m associated to every surface Σ given by the total magnetic flux

$$m = \int_\Sigma c_1. \tag{2.49}$$

2.4. Complex surfaces

We will also be interested in four-manifolds that carry a complex structure. These are manifold of *complex* dimension two and are therefore usually referred to as complex surfaces. Locally, we can introduce two complex coordinates

$$z^1 = x^0 + ix^1, \qquad z^2 = x^2 + ix^3, \tag{2.50}$$

and the complexified tangent bundle splits

$$TX \otimes \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}. \tag{2.51}$$

From now on T_X will always denote the holomorphic tangent bundle. We have a similar decomposition

$$d = \partial + \overline{\partial}, \qquad \partial = dz^i \frac{\partial}{\partial z^i}. \tag{2.52}$$

This allows us to define more sophisticated cohomology groups — the Dolbeault cohomology groups

$$H^{p,q} = H^q_{\overline{\partial}}(\Lambda^p T^*), \tag{2.53}$$

where we consider the cohomology of the $\overline{\partial}$-operator on (p, q)-forms . The Hodge numbers $h^{p,q}$ are defined as their dimensions

$$h^{p,q} = \dim H^{p,q}. \tag{2.54}$$

On a complex surface a *divisor* is a linear combination of holomorphic curves

$$D = \sum n_i C_i, \qquad n_i \in \mathbb{Z}. \tag{2.55}$$

Divisors are necessarily dual to classes in the Picard lattice

$$H^{1,1}_{\mathbb{Z}} := H^{1,1} \cap H^2_{\mathbb{Z}}. \tag{2.56}$$

A *holomorphic* line bundle $L \to X$ is defined by holomorphic transition functions and its curvature is necessarily of type $(1, 1)$. Line bundles therefore correspond to cohomology classes

$$c_1(L) \in H^{1,1}_{\mathbb{Z}}. \tag{2.57}$$

The corresponding divisor can be represented by picking a meromorphic section s of L with zeroes at D_0 and poles at D_∞ and writing

$$D = D_0 - D_\infty. \tag{2.58}$$

A complex surface X is Kähler if the metric is of type $(1, 1)$ and the associated Kähler form

$$\omega = -ig_{j\bar{\jmath}} dz^j \wedge d\overline{z}^{\bar{\jmath}} \tag{2.59}$$

is closed

$$d\omega = 0. \tag{2.60}$$

Kähler forms therefore also necessarily reside in $H^{1,1}$. For Kähler manifolds we have the Hodge structure

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X) \tag{2.61}$$

and the hermiticity condition

$$H^{p,q} = \overline{H^{q,p}}, \tag{2.62}$$

so that $h^{p,q} = h^{q,p}$. In the case of surfaces this implies the decomposition

$$H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}. \tag{2.63}$$

So, apart from the Betti number b_2, we have a second invariant $h^{2,0} = h^{0,2}$.

We can now ask how the Hodge star for a given Kähler metric acts on the Dolbeault groups. It turns out that $*$ has eigenvalue $+1$ on $H^{2,0}$ and $H^{0,2}$ and has signature $(1, b_2^-)$ on $H^{1,1}$. In fact, the positive direction in $H^{1,1}$ corresponds to the Kähler form, the negative subspace is the so-called primitive cohomology. If we take the projections we find

$$H^{1,1}_{\mathbb{R}} = \mathbb{R}\omega \oplus H^2_-. \tag{2.64}$$

and

$$H^2_+ = H^{2,0}_{\mathbb{R}} \oplus \mathbb{R}\omega \oplus H^{0,2}_{\mathbb{R}}. \tag{2.65}$$

So the Hodge and Betti numbers are related by

$$b_2^+ = 2h^{2,0} + 1. \tag{2.66}$$

Since we have a complex structure, we have an extra characteristic class, the first Chern class of the tangent bundle T_X

$$c_1(X) = c_1(T_X). \tag{2.67}$$

It is closely related to the canonical bundle which is defined as

$$K_X = \det T_X^* = \Lambda^2 T_X^*. \tag{2.68}$$

Its sections are $(2,0)$-forms (holomorphic volume forms) that locally look like

$$f(z^1, z^2) \cdot dz^1 \wedge dz^2. \tag{2.69}$$

This line bundle gives a distinguished divisor, the *canonical* divisor often also denoted by the symbol K. It is represented by minus the first Chern class of the manifold

$$K = c_1(K) = -c_1(X). \tag{2.70}$$

We also have

$$K = w_2 \pmod 2. \tag{2.71}$$

We can think of the divisor K as the curve on which a meromorphic $(2,0)$ form has its zeroes and/or poles. If $h^{2,0} > 0$ then we can pick a holomorphic form. Note that the second Chern class equals the Euler class, and therefore does not give a new invariant

$$c_2(X) = e. \tag{2.72}$$

Note also that the condition $c_1 = 0$ tells us that the canonical bundle is trivial and that the manifold X therefore is a Calabi-Yau space. In four-dimensions this is equivalent to being hyper-Kähler and there are only two compact examples: the four-torus T^4 and $K3$.

There is also a holomorphic index theorem, Noether's formula (Todd genus, $\overline{\partial}$-index) which reads

$$h^{0,0} - h^{0,1} + h^{0,2} = \frac{1}{4}(\chi + \sigma) = \int_X \frac{1}{12}(c_1^2 + c_2). \tag{2.73}$$

This can be used to express the length of the canonical class K as

$$K^2 = c_1^2 = 2\chi + 3\sigma \equiv \sigma \pmod 8. \tag{2.74}$$

Finally, we want to mention the concept of *Kodaira dimension.* In the case of Riemann surfaces or algebraic curves there are many reasons to distinguish the cases of the sphere S^2, the torus T^2 and genus $g > 1$. For example from the perspective of uniformisation, automorphism groups etc. One way to make this distinction is to look at the number of holomorphic n-differentials, *i.e.* sections of the tensor powers of the canonical bundle K^n, as a function of the weight $n \in \mathbb{N}$. If this grows as

$$h^0(K^n) \sim n^\kappa, \tag{2.75}$$

we call κ the Kodaira dimension. If $h^0 = 0$ we declare $\kappa = -\infty$. We easily check with the Riemann-Roch theorem in hand that

$$\kappa = \begin{cases} -\infty, & g = 0, \\ 0, & g = 1, \\ 1, & g > 1. \end{cases} \tag{2.76}$$

We can keep this definition of κ in higher dimensions. In the case of surfaces this gives us the possibilities $\kappa = -\infty, 0, 1, 2$. This is a classification analogous to the one of complex curves. The manifolds with $\kappa = 2$ are called of general type and should be compared with the Riemann surfaces of genus $g > 1$.

As an example (or exercise) of all the various classical invariants we have introduced in this section let us consider a 4-manifold (complex surface) of the form $X = C_g \times C_h$ with C_g a Riemann surface of genus g. We find the following invariants:

$$\begin{aligned}
\chi &= 4(1-g)(1-h), \\
\sigma &= 0, \\
Q &= \bigoplus_{2gh+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
K &= (2g-2)[C_h] + (2h-2)[C_g], \\
\kappa &= \begin{cases} -\infty & g=0 \text{ or } h=0, \\ 0 & g=h=1, \\ 1 & g=1,\ h>1 \text{ or vice versa}, \\ 2 & g,h>1. \end{cases}
\end{aligned} \tag{2.77}$$

3. Quantum Invariants

We now turn to the manifold invariants first studied by Donaldson. For more information see the books [3, 4, 5]. As we will see in the next section, these are actually correlation functions in a topological field theory, as was first realised by Witten [12]. We also discuss very briefly Seiberg-Witten theory.

3.1. Donaldson theory

The starting point in Donaldson theory is non-abelian gauge theory with gauge group $SU(2)$. (Apart from the closely related case of $SO(3)$ here is very little known about the case of a general Lie group.) So let A be a $SU(2)$ gauge field on a principal bundle P over the four-manifold X. It will have curvature

$$F = dA + A^2. \tag{3.78}$$

We denote by $\mathcal{A}$ the space of all connections on P. Since the difference between two connections is a 1-form with values in the Lie algebra $su(2)$, $\mathcal{A}$ is an affine space associated to the vector space $\Omega^1(su(2))$. It naturally decomposes in connected components labeled by the instanton number n of P

$$n = \int_X ch_2 = -\int_X \frac{1}{8\pi^2} \mathrm{Tr}\, F^2 \in Z. \tag{3.79}$$

The gauge group

$$\mathcal{G} = \mathrm{Aut}\,(P) \tag{3.80}$$

acts on the connection in the usual way. In a local trivilisation a gauge transformation is a map $h : U \subset X \to SU(2)$ and A transform as

$$h: \; A \to h^{-1}Ah + h^{-1}dh. \tag{3.81}$$

The relevant space we are interested in is actually the quotient space

$$\mathcal{B} := \mathcal{A}/\mathcal{G}^0, \tag{3.82}$$

where $\mathcal{G}^0$ are the gauge transformations that leave one fiber fixed. This group acts freely on $\mathcal{A}$, so the quotient $\mathcal{B}$ is a smooth infinite-dimensional manifold.

After we pick a metric g on X, we can consider anti-self-dual (ASD) connections or instantons, satisfying the following non-linear equation

$$F_+ = F + *F = 0. \tag{3.83}$$

The moduli space of ASD connections is defined as the space of solutions to this equation modulo gauge equivalence

$$\mathcal{M}_n = \{A \in \mathcal{A} \mid F_+ = 0\}/\mathcal{G}^0. \tag{3.84}$$

Although we do not emphasize this in the notation, it is important to realise that the definition of $\mathcal{M}_n$ depends on the choice of metric g on X.

The aim of Donaldson theory is to study the four-manifold X in an indirect way by using the properties of the moduli space $\mathcal{M}_n$ considered as a subspace of the space of all connections modulo gauge transformations

$$\mathcal{M}_n \subset \mathcal{B}. \tag{3.85}$$

Even though the choice of a metric g enters the definition of the equations, "miraculously" certain properties of $\mathcal{M}_n$ translate into interesting results about the four-manifold. We will now review some important properties of the moduli space without proofs:

1. The moduli space $\mathcal{M}_n$ is only non-empty for positive instanton number $n \geq 0$. This follows directly by considering the Yang-Mills action functional

$$S = \frac{1}{8\pi^2} \int \mathrm{Tr}\,(F \wedge *F), \tag{3.86}$$

which has the important property that it is positive definite. Instantons minimalise S for given n. We can use the identity $*F = -F$ to show that

$$\begin{aligned} n = \; &= -\int \frac{1}{8\pi^2} \mathrm{Tr}\, F^2 \\ &= \int \frac{1}{8\pi^2} \mathrm{Tr}\,(F \wedge *F) = S \geq 0. \end{aligned} \tag{3.87}$$

2. Generically, $\mathcal{M}_n$ is an oriented manifold. Its virtual dimension can be computed by an index computation

$$\begin{aligned} d := \dim \mathcal{M}_n &= 8n - 3(1 - b_1 + b_2^+) \\ &= 8n - \frac{3}{2}(\chi + \sigma). \end{aligned} \tag{3.88}$$

It is actually a very non-trivial problem to show that it is not empty for positive d. Fortunately there is a theorem by Taubes that show that this is indeed the case.

3. The moduli space $\mathcal{M}_n$ is non-compact. This fundamental fact is familiar to physicists who know very well that the instanton equations allow solutions on flat space $\mathbb{R}^4$. Since the ASD equations depend on the choice of Hodge $*$, they are invariant under conformal transformations of the metric. So we can find a one-parameter family of rescaled solutions $x \longrightarrow t \cdot x$. Since the instanton in the limit $t \longrightarrow 0$ becomes a δ-function in the origin and therefore is no longer a smooth solution to the ASD equation, the moduli space on $\mathbb{R}^4$ is clearly non-compact.

This argument can be repeated for general four-manifolds. In the limit $t \longrightarrow 0$ we have almost point-like instantons on $\mathbb{R}^4$. We can now cut out a little disk containing most of the solution and "graft" this onto a point of X. Taubes has proven that the almost-solution we obtain in this way can actually be relaxed into a real smooth solution. We see that therefore also on a general manifold point-like instantons occur in the limit $t \longrightarrow 0$.

4. The moduli space can have singularities. These singularities occur whenever $\mathcal{G}$ does not act freely. This is the case if the holonomy group of the connection is $U(1)$ instead of $SU(2)$. In that case we can restrict the gauge group consistently to $U(1)$ and we are dealing with an abelian connection. So the only singularities are the abelian instantons. In that case the curvature is just a $\mathbb{R}$-valued 2-form $F \in H^2(X, \mathbb{R})$. It must however satisfy a quantisation rule. The first Chern class of the $U(1)$ bundle should be an integer cohomology class,

$$c_1 = \frac{F}{2\pi} \in H^2_{\mathbb{Z}}. \tag{3.89}$$

Of course, the instanton number $n = ch_2$ can now be computed in terms of c_1 as

$$n = ch_2 = -\tfrac{1}{2} c_1^2. \tag{3.90}$$

Fig. 2: *If we try to interpolate between two metrics g_0, g_1 with smooth moduli spaces, we can meet a singular moduli space for a value g_* in the case $b_2^+ = 1$.*

If we now also impose the ASD condition, we see that abelian instantons $*F = -F$ correspond one-to-one with elements of the lattice

$$x \in H^2_- \cap H^2_{\mathbb{Z}} \tag{3.91}$$

with length

$$x^2 = -n. \tag{3.92}$$

When form these singularities a serious problem? First of all, we should remember that we are free to pick a metric. If we choose a generic metric and thus a generic positioning of the subspace $H^2_- \subset H^2$, the intersection with the integer lattice $H^2_{\mathbb{Z}}$ will only be non-zero if $H^2_+ = 0$, *i.e.* if

$$b_2^+ = 0. \tag{3.93}$$

However, the case $b_2^+ = 1$ is also interesting.In that case the subspace H^2_- has codimension one. Now generically it will not intersect the lattice $H^2_{\mathbb{Z}}$. But if we want to rotate H^2_- from one such generic position to another, we might have to intersect somewhere the integer lattice. So in this case we will have problems finding a family of metrics g_t $(0 \leq t \leq 1)$ interpolating between two metrics g_0 and g_1 with smooth moduli spaces. Somewhere along the path we can have a moduli space with singularities as is illustrated in *fig.* 2.

An example. All of these aspects are illustrated by the following famous example, see *fig.* 3. Consider the case of instanton number $n = 1$ on a simply-connected

Fig. 3: *The moduli space for* $b_2^+ = b_1 = 0$ *and instanton charge 1 is a five-dimensional space, whose boundary equals the original four-manifold* X. *It has singularities corresponding to abelian solutions.*

four-manifold with $b_2^+ = 0$. In this case the intersection form Q is (negative) definite. So this is precisely the case where in the topological category we can have exotic manifolds. Donaldson considered this case in the smooth category and was able to prove that these exotic variants cannot occur. We easily compute from equation (3.88) that the moduli space $\mathcal{M}_1$ is in this case 5-dimensional. These 5 parameters have a similar interpretation as in the case of $\mathbb{R}^4$. Four of them correspond to the position of the instanton, and the fifth correspond to the scale or size. In the limit of point-like instantons the remaining four coordinates parametrize the four-manifold X, so we have

$$\partial \mathcal{M}_1 = X. \tag{3.94}$$

Furthermore, $\mathcal{M}$ has singularities corresponding to elements

$$x \in H^2_{\mathbb{Z}}, \qquad x^2 = -1. \tag{3.95}$$

There will be $2b_2$ of such classes in the case of a standard intersection form. It can be shown that in this case the singularities are cones on $\mathbb{C}P^2$. That is, locally they look like $\mathbb{C}^3/S^1$.

3.2. Donaldson invariants

We now want to compute some characteristic numbers of the moduli spaces $\mathcal{M}_n$, which is the crux of Donaldson theory. The first ingredient is a map that associates to any cohomology class α on the 4-manifold X a cohomology class $\hat{\alpha}$ on the moduli space

$$\alpha \in H^k(X) \to \hat{\alpha} \in H^k(\mathcal{M}). \tag{3.96}$$

The definition of this operation is roughly as follows. There is a natural bundle on the product space $X \times \mathcal{M}$. This bundle has a second Chern class $\hat{c}_2$. We can now consider the differential form $\alpha \wedge \hat{c}_2$, which is of degree $4 + k$, and integrate it over the fiber X

$$\hat{\alpha} = \int_X a \wedge \hat{c}_2. \tag{3.97}$$

We see that in the particular example of the identity $\mathbf{1}$ we have

$$\hat{\mathbf{1}} = \int_X c_2 = n \in \mathbb{Z}. \tag{3.98}$$

We would now like to integrate these cohomology classes $\hat{\alpha}$ over $\mathcal{M}$. However, this is not a trivial matter, since we have seen that the moduli space is non-compact. In fact, a tremendous amount of work goes in proving that the heuristic definitions below actually make sense in some precise way. With this remark out of the way, we will continue our discussion as if the moduli space is nice and compact.

If we assume that X is simply connected, then the only non-trivial classes are two-forms $v \in H^2(X)$ and a volume form $\lambda \in H^4(X)$. We now define the famous Donaldson polynomials as

$$D_n(v, \lambda) = \int_{\mathcal{M}_n} \exp(\hat{v} + \hat{\lambda}). \tag{3.99}$$

It is a polynomial since only contributions of degree $d = \dim \mathcal{M}_n$ contribute (with $\deg(\hat{v}) = 2$, $\deg(\hat{\lambda}) = 4$.) It is also convenient to introduce a generating function by summing over all instanton numbers

$$D(v, \lambda) = \sum_{n \geq 0} D_n(v, \lambda) =: \langle \exp(\hat{v} + \hat{\lambda}) \rangle, \tag{3.100}$$

where we already anticipate the interpretation in terms of correlation functions. The main theorem of Donaldson states that the polynomials D_n are diffeomorphism invariants if $b_2^+ > 1$.

The idea behind the proof of this result is the following: We are computing an integral of the form

$$\int_{\mathcal{M}} \hat{\alpha}. \tag{3.101}$$

Perturbing the metric changes the moduli space $\mathcal{M}$ considered as a subspace of the space $\mathcal{B}$ of connections modulo gauge equivalence. But we can consider $\hat{\alpha}$ as an element of $H^2(\mathcal{B})$. So $\int_{\mathcal{M}} \alpha$ is invariant under deformations of $\mathcal{M}$ by applying Stokes' theorem. The only problems occur if in this deformation process we hit a metric for which the moduli space is singular. As we explained before, this can always be avoided if $b_2^+ > 1$.

The Donaldson invariants have some striking properties. For example, they vanish whenever the manifold X is obtained as the connected sum of two manifolds X_1 and

Fig. 4: *If a manifold can be considered as the connected sum of two manifolds with $b_2^+ > 1$ the Donaldson polynomials vanish.*

X_2 both with $b_2^+ > 0$, see *fig.* 4. They can therefore be used to check whether a manifold is 'decomposable.' One might get worried that the invariants are actually identical zero. Fortunately, it can be proven that $D \neq 0$ if X is a Kähler surface.

We now turn to a remarkable result obtained by Kronheimer and Mrowka [6]. First a manifold X is declared to be of simple type if the expectation value of $\hat{\lambda}^2 = 4$. In terms of the generating function D this means that D satisfies the linear differential equation

$$\frac{\partial^2 D}{\partial \lambda^2} = 4D. \tag{3.102}$$

Under this assumption the only interesting invariants can be summarized in the new generating function

$$\mathbb{D}(v) = D(v,0) + \frac{1}{2}\frac{\partial}{\partial \lambda} D(v,0). \tag{3.103}$$

According to the theorem of Kronheimer and Mrowka, this expression has a remarkable simple form

$$\mathbb{D}(v) = e^{v^2/2} \sum_{x \in H^2_{\mathbb{Z}}} e^{x \cdot v} n_x. \tag{3.104}$$

The classes x with $n_x \neq 0$ are called basic classes. Only a finite number of such basic classes occur and they satisfy

$$x \equiv w_2 \pmod 2 \tag{3.105}$$

and, conjecturally,

$$x^2 = 2\chi + 3\sigma. \tag{3.106}$$

Concrete examples are $K3$ where

$$\mathbb{D} = e^{v^2/2} \tag{3.107}$$

or an elliptic surface (wich allows a map $\pi : X \rightarrow \mathbb{P}^1$ with generic fiber $\pi^{-1}(p) \cong T^2$) with

$$\mathbb{D} = e^{v^2/2}(\sinh x)^n, \tag{3.108}$$

with

$$x = fiber,\ \ n = h^{2,0} - 1. \tag{3.109}$$

This simple form of the Donaldson invariants (for manifolds of simple type) is a priori quite mysterious. It is however explained beautifully by the new work of Seiberg and Witten to which we now turn.

3.3. Seiberg-Witten theory

As we will see in the next section, any $N = 2$ supersymmetric field theory can be twisted to give a topological field theory. Up to now we have considered Yang-Mills theory that computes the Donaldson invariants. We will now turn to theories which also include matter fields. Actually the model will be topological version of quantum electrodynamics (QED). This model arose in the fundamental work of Seiberg and Witten in their study of the low energy effective field theory of $SU(2)$ Yang-Mills [9]. This explains why these new invariants are related to the Donaldson invariants. However, here we take another point of view, and just study these equations for their own sake. I can very much recommend the original paper [15] and the lectures [1, 11] for more details.

In Seiberg-Witten theory the ingredients are an abelian gauge field $A \in \mathcal{A}$ and a charged spinor field M. So our principal bundle is a $U(1)$ fiber bundle with associated line bundle L. Topologically, $U(1)$ bundles are determined by their first Chern class

$$x = c_1 = [F/2\pi] \in H^2_{\mathbb{Z}}. \tag{3.110}$$

What do we mean by a charged spinor field? A (chiral) spinor would be a section of the spin bundle S^+. However, if X is not a spin manifold, we are unable to define the spin bundle S^+ consistently over X. Now a *charged* spinor would be a section of the bundle $W^+ = L \otimes S^+$. Even if X is not spin, one can make sense of the bundle W^+, which is referred to as a *spin*^{c} bundle [8]. It does however places a constraint on the first Chern class of L. It has to satisfy

$$x \equiv w_2 \pmod 2, \tag{3.111}$$

which we recognise as the condition the basic classes of Kronheimer and Mrowka satisfied. We can now write down the equations that follow from twisting $N = 2$ QED

$$\begin{aligned} F_+ &= (\overline{M}M)_+, \\ DM &= 0, \end{aligned} \tag{3.112}$$

with the twisted Dirac operator $D : W^+ \to W^-$. The moduli space $\mathcal{M}_x$ with fixed monopole number x is in this case defined as

$$\mathcal{M}_x = \{A \in \mathcal{A}, M \in \Gamma(W^+) \mid eqns\ (3.112)\}/\mathcal{G}^0. \tag{3.113}$$

We will now, again without proof, very briefly review its properties and compare them to the case of Donaldson theory:

1 $\mathcal{M}_x$ is non-empty for only a finite number of $x \in H^2_{\mathbb{Z}}$. This is a startling difference with Donaldson theory, where one could only argue that the instanton number n was bounded from below. In the Seiberg-Witten theory the curvature satisfies a bound of the form

$$\int |F|^2 \leq r. \tag{3.114}$$

This implies directly that only for a finite number of $x \in H^2_{\mathbb{Z}}$ there will be actual solutions.

2.The moduli space $\mathcal{M}_x$ is an oriented manifold of (virtual) dimension

$$d = \frac{1}{4}(x^2 - 2\chi - 3\sigma). \tag{3.115}$$

This follows again from an index computation.

3. The moduli space is compact. One can show that $|M|$ is bounded by the scalar curvature of the manifold. There are no L^2-solutions on $\mathbb{R}^4$.

4. The moduli space can have singularities. These occur for the same reasons as in Donaldson theory. A necessary condition is $M = 0$, so we are left again with abelian instantons

$$F_+ = 0. \tag{3.116}$$

If $b_2^+ > 1$ we do not have to worry about the singularities; they can be avoided by picking a generic metric.

We can now proceed and try to write down interesting cohomology classes on $\mathcal{M}$ in complete analogy with Donaldson theory. In this case there exist a universal first Chern class $\hat{c}_1$ on the space $X \times \mathcal{M}$. This gives a map

$$\alpha \in H^i(X) \to \hat{\alpha} = \int_X \alpha \wedge \hat{c}_1 \in \mathcal{M}. \tag{3.117}$$

In this case we find for $v \in H^2$

$$\hat{v} = \int_X v \wedge x = v \cdot x \tag{3.118}$$

and for $\lambda \in H^4(X)$ a two-cocycle

$$\hat{\lambda} \in H^2(\mathcal{M}) \tag{3.119}$$

the universal c_1 restricted to $\mathcal{M}$. We can now define the Seiberg-Witten invariant by

$$SW(x) = \int_{\mathcal{M}_x} \exp(\hat{\lambda}). \tag{3.120}$$

This expression makes sense since the moduli space is nice and compact. The main theorem of Seiberg-Witten theory is that these expressions are again differential invariants of the four-manifold X.

We will be mainly interested in the case where $\mathcal{M}_x$ is actually zero-dimensional, *i.e.* consists of a collection of points. The equivalent notion of a manifold of simple type for Seiberg-Witten theory is defined to be the case where all the expectation values of $\hat{\lambda}$ vanish, *i.e.* only $d = 0$ contributes. Here we have the simplified definition

$$SW(x) = \sum_{x \in \mathcal{M}_x} \pm 1, \tag{3.121}$$

where the $\pm$ sign is computed as a ratio of determinants. We note that the condition $d = 0$ gives

$$x^2 = 2\chi + 3\sigma, \tag{3.122}$$

which was a (conjectured) property of the basic classes. In fact the mysterious invariant n_x that appeared in the work of Kronheimer and Mrowka is conjectured to be given by the Seiberg-Witten invariant up to a constant factor. The precise conjecture reads (for X simple and $b_2^+ > 1$)

$$\mathbb{D}(v) = c \cdot e^{v^2/2} \sum_x e^{v \cdot x} SW(x), \tag{3.123}$$

with constant

$$c = 2^{(7\chi + 11\sigma)/4 + 2}. \tag{3.124}$$

What are the results obtained with these new invariants. The list is growing rapidly, but some striking examples are the following:

- Just as in Donaldson theory the invariants $SW(x)$ vanish whenever the 4-manifold is composed as the connected sum $X = X_1 \# X_2$ of two manifolds with $b_2^+ > 0$.

- If X allows a complex structure, so that the canonical class K is well-defined, then $SW(\pm K) = \pm 1$.

- If X is a minimal surface of general type ($\kappa = 2$ in the Kodaira classification) the only basic classes are $\pm K$.
- Taubes has managed to proof that if X is symplectic, then SW is non-trivial [10]. This is a much stronger result than the result by Donaldson that D is non-zero for Kähler manifolds. In fact, symplectic manifolds can be considered as a generalization of Kähler manifolds where one only demands an almost-complex structure.
- The Thom conjecture has been proved [7]. If we want to find a smoothly embedded surface that realizes a homology class C, then the following lower bound on the genus of the surface holds

$$C \cdot C + |C \cdot x| \leq 2g - 2, \tag{3.125}$$

  for all basic classes x.
- Perhaps the most striking result (also by Taubes) concerns symplectic manifolds. In this case we have another powerful invariant, the so-called Gromov-Witten invariant, that computes how many (pseudo)holomorphic curves realize a certain homology class $x \in H^2$. What Taubes was able to prove is that this Gromov-Witten invariant actually equals the Seiberg-Witten invariant.

4. Topological Field Theories

There are roughly two applications of topological field theories. They either can be used to describe the global degrees of freedom of physical quantum field theories, or they can be used to give a physical setting of mathematical moduli problems. Here we will be mainly concerned with the latter application. To understand how topological field theories can be applied to 4-manifolds we have to understand two concepts: localisation and twisting. For general introduction to TFT see for example [2, 13].

4.1. Localisation

A crucial ingredients in topological field theories is the idea of localisation. This allows us to express partition and correlation functions as integrals over finite dimensional (moduli) spaces instead of the infinite field configuration space that features in the path-integral. Of course, it is the unique structure of topological models that allows such a drastic reduction in degrees of freedom.

The mathematical idea of localisation has a rich application. The most familiar one is the calculation of the Euler characteristic of a manifold. On the one hand this be computed by picking a Riemannian metric and integrating the Euler density $\mathrm{Pf}\,(R)$ constructed out of the Riemann curvature R. On the other hand the Euler character can be computed by counting the number of zeroes of a generic vector field on the manifold.

In quantum mechanics the localisation principle is well-known as the phenomenon of the Nicolai map. This can be nicely illustrated by a zero-dimensional example. Consider a polynomial $s(x)$ and the "path-integral"

$$Z = \int dx\, e^{-s^2/2}. \tag{4.126}$$

As it stands this integral has no interesting invariances and cannot be computed in any simplified way. If we instead consider the modified integral

$$Z = \int dx\, e^{-s^2/2} \frac{\partial s}{\partial x} \tag{4.127}$$

we can compute it without much effort, since it can be rewritten as

$$Z = \int ds\, e^{-s^2/2} = \begin{cases} 0, & \text{if } s \text{ even,} \\ 1, & \text{if } s \text{ odd.} \end{cases} \tag{4.128}$$

So we see Z computes a simple invariant of s: the degree of the map $s : \mathbb{R} \to \mathbb{R}$, which is either zero or one depending on whether s is an even or odd polynomial. The partition function Z is thus invariant under all deformations of s that leave the boundary conditions invariant.

Now that we have established the "topological invariance" of Z we can make use of this by deforming the action s in such a way that the localisation becomes evident. To do this, we rescale $s \to t \cdot s$ and take the limit $t \to \infty$. In this limit the gaussian factor will damp the integral for all values for x except for the zeroes of the function s. Around these points we can perform a semi-classical approximation. In this way the computation localizes to a finite number of ppoints, and we compute Z as sum over the zeroes of s of a factor ± 1, very much in analogy with the Euler character,

$$Z = \sum_{s(x)=0} \operatorname{sgn} \det(\partial s), \tag{4.129}$$

where we already adapted our notation to the situation of more than one variable x. One way to express this localisation is that the semi-classical approximation gives an exact result.

A third way to compute the integral Z will be a metaphor for the actual computations we are going to do in the case of four-dimensional gauge theories. We start by rewriting the factor $\partial s/\partial x$ as a fermionic gaussian integral. We introduce two fermionic variables ψ, ρ and rewrite Z as

$$Z = \int dx d\psi d\rho \, e^{-S}, \tag{4.130}$$

with

$$S = s^2/2 + \rho \partial s \psi. \tag{4.131}$$

The action S has a BRST symmetry Q given by

$$\begin{aligned} Qx &= \psi, \\ Q\psi &= 0, \\ Q\rho &= s(x), \end{aligned} \tag{4.132}$$

As it stands this symmetry is not nilpotent. This property we obtain if we introduce a further bosonic auxiliary field H and write

$$Z = \int dx dH d\psi d\rho \, e^{-S}, \tag{4.133}$$

with

$$S = isH + \tfrac{1}{2}H^2 + \rho \partial s \psi. \tag{4.134}$$

The symmetry is now extended as

$$\begin{aligned} Qx &= \psi, \\ QH &= 0, \\ Q\psi &= 0, \\ Q\rho &= H. \end{aligned} \tag{4.135}$$

With this extra auxiliary field H the BRST symmetry squares to zero 'off- shell"

$$Q^2 = 0. \tag{4.136}$$

Now there is a simple localisation argument for this BRST charge due to Witten. Our integral Z is expressed as an integral over a $2|2$ dimensional superspace. On this space we have the action of a fermionic symmetry Q generated by a vector field ξ, that is, $Q = \mathcal{L}_\xi$. The odd vector field ξ squares to zero, $\xi^2 = \frac{1}{2}[\xi, \xi] = 0$, a non-trivial property for an odd vector field. The orbits of this group action are $0|1$ dimensional curves parametrized by an odd coordinate θ. Because of the fundamental identities of grassmannian calculus

$$
\begin{aligned}
\int d\theta\, 1 &= 0, \\
\int d\theta\, \theta &= 1,
\end{aligned}
\tag{4.137}
$$

the integral of a constant function, such as the action density e^{-S}, along the orbit will automatically give zero. Therefore the only non-vanishing contributions to the integral can come from the zero-dimensional orbits, that is, the fixed points of Q or equivalently the zeroes of the vector field ξ.

4.2. Twisting

The second ingredient in the topological properties of supersymmetric gauge theories is the twisting procedure. It has been known for a long time that certain quantities in supersymmetric theories have topological properties. They do not depend on the metric on space-time or the positions of the operator insertions. Unfortunately, it is a nontrivial matter to have a global realized supersymmetry transformation. We need covariant constant spinors. This usually imposed highly restrictive conditions on the topology of the space-time manifold.

Witten's twisting procedure mends this deficiency. By changing the spins of the various fields one produces a scalar supersymmetry transformations. Since it is elementary to find a covariant constant function (just a constant function) on an arbitrary manifold, this produces automatically field theories with topological correlation functions. The price we pay is that we are (in general) now dealing with a different model and so have to be careful to draw any conclusions about the physical model that we started with.

In four dimensions the twisting procedure starts with a $N = 2$ supersymmetric model. This has a symmetry group

$$
Spin(4) \times U(2)_R,
\tag{4.138}
$$

where $Spin(4)$ is the double cover of the Lorentz group $SO(4)$ and the internal R-symmetry $U(2)_R$ acts on the two supersymmetry currents, which transform as a doublet. This group is (locally) isomorphic to

$$
SU(2)_+ \times SU(2)_- \times SU(2)_R \times U(1)_{gh},
\tag{4.139}
$$

where we recognize the two factors . Irreducible representations (n_+, n_-, n_R, q). The twisting procedures replaces the component $SU(2)_+$ by the diagonal subgroup of $SU(2)_+ \times SU(2)_R$. That is, the new spins are

$$
(n_+ \otimes n_R, n_-).
\tag{4.140}
$$

4.3. Donaldson theory

The topological field theory interpretation is due to Witten [12, 14]. It fits perfectly in the general framework we sketched. Starting point is $N=2$ supersymmetric Yang-Mills theory. This field theory consists of a multiplet (A, λ^I, ϕ) consisting of a connection A_μ, two spinors $(\lambda^I_\alpha, \lambda^I_{\dot\alpha})$ $(I = 1, 2)$ and a complex scalar field ϕ all taking their values in $su(2)$. After twisting we recover the following fields

$$\begin{aligned} A_\mu &\rightarrow A_\mu \\ \lambda^I_\alpha &\rightarrow \psi_\mu \\ \lambda^I_{\dot\alpha} &\rightarrow \rho^+_{\mu\nu}, \eta \\ \phi &\rightarrow \phi \\ \overline{\phi} &\rightarrow \overline{\phi} \end{aligned} \tag{4.141}$$

We recognise the fundamental topological multiplet (A, ψ, ρ) together with an equivariant multiplet $(\eta, \phi, \overline{\phi})$. The BRST transformations can be derived from the $N=2$ supersymmetry algebra and read

$$\begin{aligned} \delta A_\mu &= \psi_\mu \\ \delta \psi_\mu &= D_\mu \phi \\ \delta \rho^+_{\mu\nu} &= F^+_{\mu\nu} \\ \delta \overline{\phi} &= \eta \end{aligned} \tag{4.142}$$

The section is identified as

$$s(A) = F^+ \tag{4.143}$$

and its zero locus is the moduli space $\mathcal{M}$.

The observables are the cohomology classes of the BRST operator. Let

$$\mathcal{O}^{(0)} = \mathrm{Tr}\, \phi^2 \tag{4.144}$$

and construct the descendents

$$d\mathcal{O}^{(i)} = \delta \mathcal{O}^{(i+1)}. \tag{4.145}$$

We can describe the Donaldson map as

$$\begin{aligned} \alpha \in H^i(X) \Rightarrow \quad \hat\alpha &= \int \alpha \wedge \mathcal{O}^{(4-i)} \\ &= \int_{C_\alpha} \mathcal{O}^{(4-i)}, \end{aligned} \tag{4.146}$$

where $C_\alpha \in H_{4-i}$ denotes the homology class Poincaré dual to α. In more detail we find

$$v \in H^2(X) \Rightarrow \hat{v} = \int_{C_v} \mathrm{Tr}\,(\phi F + \psi^2) \tag{4.147}$$

and

$$\lambda \in H^4(X) \cong \mathbb{R} \Rightarrow \hat{\lambda} = \lambda \cdot \mathrm{Tr}\,\phi^2 \tag{4.148}$$

4.4. Seiberg-Witten theory

In this case the physical theory is $N = 2$ QED. This means we have added a $N = 2$ matter multiplet (M^I, ψ). The twisting modifies the spin as

$$\begin{aligned} M^I &\rightarrow M_\alpha \\ \psi_\alpha &\rightarrow \psi_\alpha \\ \psi_{\dot\alpha} &\rightarrow \rho_{\dot\alpha} \end{aligned} \tag{4.149}$$

We obtain a commuting bosonic variable M that transforms as a spinor! The BRST transformations are

$$\begin{aligned} \delta M_\alpha &= \psi_\alpha \\ \delta \rho_{\dot\alpha} &= (DM)_{\dot\alpha} \end{aligned} \tag{4.150}$$

Finally, because of the coupling of the matter multiplet to the gauge multiplet, the transformation for the field $\rho^+_{\mu\nu}$ gets modified to

$$\delta \rho^+_{\mu\nu} = F^+_{\mu\nu} - (\overline{M}M)^+_{\mu\nu} \tag{4.151}$$

Looking at the transformation rules of the fields $(\rho^+_{\mu\nu}, \rho_{\dot\alpha})$ we see that in this case the section is given by

$$s = (F^+ - (\overline{M}M)^+, DM) \tag{4.152}$$

Its zero locus is the Seiberg-Witten moduli space.

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