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String Propagation in a Black Hole Geometry

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Abstract

We consider string theory in the background of the two-dimensional black hole as described by the $SL(2,R)/U(1)$ coset theory recently introduced by Witten. We study the spectrum of this conformal field theory, and give explicit representations for the tachyon vertex operators in terms of $SL(2,R)$ matrix elements. This is used to compute the scattering of strings off the black hole and to show that the string propagator exhibits Hawking radiation. We further discuss the role of winding states and the appearance of bound states in the Euclidean solution. We find that target space duality in the Lorentzian theory interchanges the black hole horizon with the space-time singularity. We conclude with a comparison with the non-critical $c = 1$ string and its formulation as a gauged $SL(2,R)$ WZW model.

1. Introduction

Much of the recent progress in string theory has been based on the insight that consistent string theories can be constructed in dimensions much lower than the critical dimension. This observation has allowed us to study toy models of string theory, for example in one spatial dimension, for which in particular the matrix model approach has proved to be very successful [1]-[6]. These 'non-critical' string theories are conventionally formulated on the worldsheet as a $c = 1$ conformal field theory coupled to two-dimensional gravity. However, their space-time interpretation is more clear if one thinks of these models as critical string theories in a non-trivial $1 + 1$ dimensional background [3].

The general sigma model for a $1 + 1$ dimensional target space with coordinates $x = (x^1, x^2)$ has an action of the form*

$$S = \frac{1}{2\pi} \int d^2z \Big(G_{ij}(x)\overline{\partial} x^i \partial x^j + \Phi(x) R^{(2)} + T(x) \Big), \tag{1.1}$$

where $G_{ij}(x)$, and $\Phi(x)$ are the space-time metric and dilaton field and $T(x)$ describes the tachyon. The dynamics of the background fields is governed by the condition that the β-functions of the worldsheet field theory vanish. At one-loop these β-functions can be identified with the field equations of the target space action [7]

$$S_{eff} = \int d^2x \sqrt{G} e^{\Phi} \Big(R + (\nabla\Phi)^2 + (\nabla T)^2 - 2T^2 - 8 \Big). \tag{1.2}$$

What makes string theory special in $d = 1 + 1$ is that, because there are no transversal directions, the string can not vibrate and classically is completely described by its center of mass coordinates. As a consequence 1+1 dimensional string theory contains essentially only one field theoretic degree of freedom, which by exploiting the gauge invariances of the string action may be identified with the tachyon field $T(x)$. The metric and dilaton field (and all the other massive modes) only have global modes, and serve as a background for the propagating tachyon.

A particularly interesting solution of the graviton/dilaton field equations has recently been found in [8, 9]. It is given by ($x^i = (u, v)$)

$$ds^2 = \frac{du\, dv}{1 - uv}, \tag{1.3}$$

$$\Phi = \log\Big(1 - uv\Big). \tag{1.4}$$

The surprising observation of [8] is that this solution in fact exhibits all the characteristic

*We have not included an anti-symmetric tensor field $B_{ij}(x)$, because in the $1 + 1$ dimensional target spaces we will consider it can be gauged away.

Fig. 1: The two-dimensional black hole in (u,v)-coordinates. The horizon is located at $uv=0$, and the future and past singularities at $uv=1$. The regions I and II describe the space-time outside of the horizon, whereas regions III and IV correspond to the future and past interior of the horizon.

features of a black hole geometry. In particular, as seen from the space-time diagram in fig. 1, it possesses a horizon at $uv=0$ as well as a curvature singularity at $uv=1$ just as the Schwarzschild black hole in terms of Kruskal coordinates. What makes the solution (1.3) particularly interesting is that, as shown in [8], it in fact corresponds to an exact conformal field theory, in the form of the $SL(2,R)/U(1)$ gauged WZW-model. Thus the exciting possibility opens up of addressing important issues in string theory, such as the nature of space-time singularities, with the aid of conformal field theory techniques.

Our aim in this paper is to investigate the spectrum of the string theory described by the $SL(2,R)/U(1)$ coset model, and, in this way illuminate some of the physical aspects of string propagation in a black hole geometry. What makes this problem non-trivial is the fact that there is no global time coordinate t such that the metric (1.3) is time-independent. There is, however, a non-trivial Killing vector given by $u\partial_u + v\partial_v$, but this is only timelike in the regions I and II of fig. 1. and spacelike in the regions III and IV. In point particle theories this leads to physically interesting phenomena like Hawking radiation, and one would like to know whether in string theory there is a similar effect. To answer this question we will study the string propagator, which we define following Hartle and Hawking [10] via analytic continuation to the Euclidean black hole. This analytic continuation can be performed in region I where we can express u and v in terms of a radial coordinate r and a time coordinate t via

$$u = \sinh r\, e^{t}, \qquad v = -\sinh r e^{-t} \tag{1.5}$$

Fig. 2: The Euclidean black hole geometry in which the momentum modes of the tachyon propagate, as parametrized by the coordinates (r, θ). This geometry is regular at the point $r = 0$.

Then by rotating the time-coordinate to imaginary values $t = -i\theta$ one obtains the Euclidean black hole [9, 8]

$$ds^2 = dr^2 + \tanh^2 r \, d\theta^2, \tag{1.6}$$

$$\Phi = \log \cosh^2 r. \tag{1.7}$$

This metric describes a regular manifold having the shape of an semi-infinite cigar when the θ-coordinate is chosen to be periodic modulo 2π (see fig. 2). Also the Euclidean black hole allows an exact CFT description in terms of a $SL(2, R)/U(1)$ coset model [8], where now the $U(1)$ is compact.

This paper is organized as follows. In section 2 we will review the formulation of the $SL(2, R)/U(1)$ coset model as a gauged WZW-model. Following the treatment of Gawedzki and Kupiainen [11] we adopt a gauge-fixing procedure which allows an exact quantization. In section 3 we determine the spectrum of primary fields of the coset model and construct explicitly the corresponding string vertex operators in terms of $SL(2, R)$ matrix elements. The target space interpretation of these vertex operators for the Euclidean black hole is discussed in section 4. In particular we study the string winding modes, which are related to the momentum modes via a duality transformation. An interesting physical phenomenon is the occurence of bound states of the string and black hole.

In section 5 we move to the Minkowskian black hole. From the explicit form of the vertex operators we derive the reflection coefficient for the scattering of strings off the black hole. We also discuss a curious self-duality of the black hole CFT which interchanges the singularity and the horizon. In section 6 we use our results for the string vertex operators to study the string propagator, and show that the black hole indeed emits Hawking radiation. Finally, in section 7 we discuss the relation between the two-dimensional black hole string theory and the more conventional $c = 1$ non-critical string theory. In an appendix we have collected some useful properties of $SL(2,R)$-representations and their matrix elements.

2. The $SL(2,R)/U(1)$ coset model

In this section we review the Lagrangian formulation of the conformal field theory describing the $2d$ black hole and discuss its quantization. As explained in [8] the black hole conformal field theory can be formulated as a gauged WZW-model based on the non-compact group $G = SL(2,R)$. In this formulation the theory is represented in terms of a field $g(z,\overline{z}) \in SL(2,R)$ and a gauge field A which gauges the symmetry

$$g \to hgh, \tag{2.1}$$

with h in an appropriately chosen abelian subgroup H of $G = SL(2,R)$. If one chooses H to be compact, the coset manifold G/H describes a Euclidean version of the two-dimensional black hole, while the construction with H non-compact leads to a black hole target space of Lorentzian signature. The two constructions are related to each other via analytic continuation. In this section we will for the sake of clarity restrict our attention to the conformal field theory of the Euclidean black hole, and comment on the necessary modifications for the Lorentzian case.

To write the action for the gauged $SL(2,R)$ WZW model we will parametrize the group manifold by Euler angles. We introduce real coordinates $r, \theta_{_L}, \theta_{_R}$ and write $g \in SL(2,R)$ as

$$g = e^{\frac{i}{2}\theta_L\sigma_2} e^{\frac{1}{2}r\,\sigma_1} e^{\frac{i}{2}\theta_R\sigma_2} \tag{2.2}$$

with σ_i the Pauli matrices and the coordinates range over $0 \le r < \infty$, $0 \le \theta_{_L} < 2\pi$, $-2\pi \le \theta_{_R} < 2\pi$. This parametrization is convenient for describing the conformal field theory of the Euclidean black hole; for the Lorentzian case one should simply replace $\sigma_2 \to \sigma_3$ and $\theta_{_{L,R}} \to t_{_{L,R}}$.

The generator of the abelian subgroup H is taken to be σ_2, so that the local gauge transformations correspond to shifting $\theta_{_{L,R}} \to \theta_{_{L,R}} + \alpha$. In the parametrization (2.2) the gauged WZW action takes the form

$$\begin{aligned} S \;=\; S_{wzw}[r,\theta_{_L},\theta_{_R}] + \frac{k}{2\pi}\int d^2z\Big[A(\overline{\partial}\theta_{_R} + \cosh r\,\overline{\partial}\theta_{_L}) \\ +\overline{A}(\partial\theta_{_L} + \cosh r\,\partial\theta_{_R}) - \overline{A}A\,(\cosh r + 1)\Big] \end{aligned} \tag{2.3}$$

with

$$S_{wzw}[r,\theta_L,\theta_R] = \frac{k}{4\pi}\int d^2z\,(\overline{\partial}r\partial r - \overline{\partial}\theta_L\partial\theta_L - \overline{\partial}\theta_R\partial\theta_R - 2\cosh r\,\overline{\partial}\theta_L\partial\theta_R) \tag{2.4}$$

Since the gauge field appears quadratically, one can, at least formally, integrate out A, which yields a sigma model action for the metric (1.6) [8]. This procedure, however, is only correct in first order in $1/k$, and, since we are interested in the exact properties of the theory, we will proceed in a different fashion.

As a preparation for the discussion of the quantization and physical spectrum of the coset theory, we now describe the gauge fixing of the action (2.3). Following [11] we make use of the observation the gauge field $A = (A, \overline{A})$ can be parametrized as*

$$\begin{aligned} A &= \partial\phi_L, \\ \overline{A} &= \overline{\partial}\phi_R. \end{aligned} \tag{2.5}$$

with ϕ_L and ϕ_R complex scalar fields that satisfy the reality condition $\phi_L = (\phi_R)^*$. Then, when we also shift the fields

$$\begin{aligned} \theta_L &\rightarrow \theta_L + \phi_L \\ \theta_R &\rightarrow \theta_R + \phi_R \end{aligned} \tag{2.6}$$

we find that due to the gauge invariance, the action S will only depend on the difference

$$\phi = \phi_L - \phi_R. \tag{2.7}$$

The above change of integration variables, replacing the $U(1)$ gauge field A with the variable ϕ, can also be thought of as the two step procedure of first imposing the gauge condition $\partial_\alpha A^\alpha = 0$, and then parametrizing the gauge slice via $A^\alpha = \epsilon^{\alpha\beta}\partial_\beta\phi$. In this way we find that the complete gauge fixed action is given by

$$S_{gf} = S_{wzw}[r,\theta_L,\theta_R] + S[\phi] + S[b,c] \tag{2.8}$$

where

$$S[\phi] = -\frac{k}{4\pi}\int d^2z\,\partial\phi\overline{\partial}\phi \tag{2.9}$$

describes a (wrong sign) free scalar field and

$$S[b,c] = \int d^2z\,(b\overline{\partial}c + \overline{b}\partial\overline{c}), \tag{2.10}$$

*We are assuming that world sheet has trivial topology. On a general surface this parametrization can only be done locally.

describes a spin $(1,0)$ ghost system representing the Jacobian of the redefinition (2.5).

The quantization of the gauge fixed theory (2.8) is straightforward, since everything is expressed either in free fields, or in terms of an (ungauged) $SL(2,R)$ WZW-model. The derivatives of the fields r, θ_L and θ_R can be combined into the $SL(2,R)$-currents

$$\begin{aligned} J^3(z) &= k\,(\partial\theta_L + \cosh r\,\partial\theta_R), \\ J^\pm(z) &= k\,e^{\pm i\theta_L}\Big(\partial r \pm i\sinh r\,\partial\theta_R\Big), \end{aligned} \tag{2.11}$$

whose modes satisfy the $SL(2,R)$ Kac-Moody algebra. These currents can be used to construct other operators in the theory, for example, the stress tensor of the complete model is given by

$$T(z) = \frac{1}{k-2}\eta_{ab}J^aJ^b + \frac{k}{4}(\partial\phi)^2 + b\partial c \tag{2.12}$$

where η_{ab} denotes the metric on the lie algebra $sl(2,R)$. The modes of the stress tensor $T(z)$ generate the Virasoro algebra with central charge

$$c = \frac{3k}{k-2} - 1$$

To find the operators that are part of the coset theory one needs to use the BRST-symmetry of the combined system (2.8), which is generated by the nilpotent BRST charge

$$Q_{BRST} = \oint dz\, c\,(J^3 + \tfrac{i}{2}k\partial\phi) + \text{c.c.} \tag{2.13}$$

The physical operators of the coset model are then characterized by the requirement that they commute with the BRST-charge and are defined up to BRST-commutators. in particular, the chiral algebra of the coset model will be a subalgebra of the $SL(2,R)$ Kac-Moody algebra, consisting of those combinations of J^+ and J^- that commute with J^3. It is at this point that one makes contact with the standard GKO coset construction [12]. From this point of view this model has been previously considered in [13].

The discussion so far can exactly be taken over to the Lorentzian black hole, by making the above mentioned analytic continuation from $\theta_{L,R}$ to $it_{L,R}$. One difference, however, is that the coset model is believed to describe a unitary CFT when the subgroup H is compact, but is clearly non-unitary when H is non-compact. This is reflected in the fact that the action in the second case has an indefinite signature. A second important difference is that the fields $\theta_{L,R}$ are periodic, while the 'time' variables $t_{L,R}$ can take any real value. As a consequence the field ϕ representing the gauge degree of freedom is compactified (modulo 2π) in the Euclidean case, but uncompactified in the Lorentzian case. As we will see this fact will have important implications for the spectrum of the two coset theories.

3. Vertex Operators in the Black Hole CFT

In this section we will determine the spectrum of primary fields of the $SL(2,R)/U(1)$-coset theory. As will become clear later on, the primary fields of the coset chiral algebra represent the vertex operators for the tachyon field. The string theory described by this model possibly has other physical vertex operators representing 'global modes' for the other fields. These are given by the Virasoro primary fields, but they correspond to descendents of the coset chiral algebra and will not be considered in this paper (but see *e.g.* [13] and [14]). In this section we will consider both the Lorentzian and the Euclidean signature.

3.1. The spectrum of primary fields

The $SL(2,R)/U(1)$ primary operators are given by local expressions in the fields r, θ_L, θ_R and ϕ, but do not contain derivatives of these fields. The most general Ansatz for the coset primary fields is therefore

$$V(z,\overline{z}) = T\Big(r(z,\overline{z}), \theta_L(z,\overline{z}), \theta_R(z,\overline{z})\Big) e^{iq_L\varphi(z)+iq_R\overline{\varphi}(\overline{z})} \tag{3.1}$$

where we used the fact that the field $\phi(z,\overline{z}) = \varphi(z) + \overline{\varphi}(\overline{z})$ describes a (possibly compactified) free scalar, so that its vertex operators have the usual exponential form. The allowed values for the charges q_L and q_R are fixed by the compactification radius of ϕ.

Our goal is to determine the functions $T(r,\theta_L,\theta_R)$, which for the moment we take to be arbitrary functions on the universal cover of $SL(2,R)$. It is a well-known fact that a complete basis of functions on a group consists of the matrix elements of the different representations. In our situation we will be interested in the unitary representations of the universal cover $\widetilde{SL}(2,R)$. The vectors $|l,\omega\rangle$ in these infinite dimensional representations are labeled by the eigenvalue ω of the generator of the abelian subgroup H. According to the values of the $SL(2,R)$ isospin l and ω the unitary representation can be divided into three different series:

- principal continuous series: $l = -\frac{1}{2} + i\lambda$; λ, ω real;
- principal discrete series: $l < -\frac{1}{2}$; $|\omega| + l$ non-negative integer;
- complementary series: $l \in [-1, -\frac{1}{2}]$, ω real.

We can now represent the function T in (3.1) by the matrix elements

$$T(r,\theta_L,\theta_R) = \langle l, \omega_L | g(r,\theta_L,\theta_R) | l, \omega_R \rangle. \tag{3.2}$$

The quantum numbers ω_L and ω_R are the eigenvalues of the left- and right-moving charges J_0^3 and $\overline{J}_0^3$ respectively. To see this we recall that the current $J^3(z)$ and $\overline{J}^3(\overline{z})$ are the generators of the symmetry $g(z,\overline{z}) \to h_L(z)g(z,\overline{z})h_R(\overline{z})$.

We are now ready to impose the condition that the vertex operators (3.1) commutes with the BRST-charge (2.13). It is sufficient to require that the total charge of the current $J^3 + ik\partial\phi$ and its complex conjugate are zero. This leads to the relations

$$\begin{aligned} \omega_L + q_L &= 0, \\ \omega_R - q_R &= 0. \end{aligned} \qquad (3.3)$$

To work out the implications of these relations for the spectrum of the theory we will now separately discuss the Minkowskian and Euclidean models.

The spectrum of the Minkowskian Black Hole CFT

For the Lorentzian theory the scalar field ϕ is uncompactified. This means that q_L and q_R must have the same value, and so (3.3) then implies that $\omega_L = -\omega_R = \omega$. It turns out that with this condition the matrix elements (3.2) are only non-zero for the continuous series with $l = -\frac{1}{2} + i\lambda$. Finally, by undoing the shift (2.6) we find that the vertex operators are independent of the field ϕ and take the form

$$T^l_\omega(r,t) = \langle l, \omega | g(r, t_L, t_R) | l, -\omega \rangle, \qquad (3.4)$$

where $l = -\frac{1}{2} + i\lambda$ and $t = \frac{1}{2}(t_L - t_R)$ is the time-variable. This result is intuitively clear because these are just the primary fields of the original $SL(2,R)$-WZW model that are invariant under the gauge symmetry $g \to hgh$.

The stress-tensor of this Lorentzian model is expressed in terms of the Casimir on $SL(2,R)$, similar as in (2.12). Using this fact the conformal dimension of the vertex operators (3.4) are easily determined. We find

$$L_0|T^l_\omega\rangle = \Big(-\frac{l(l+1)}{k-2} - \frac{\omega^2}{k}\Big)|T^l_\omega\rangle, \qquad (3.5)$$

where we used the relation (3.3). This is in accordance with what one expects from the GKO coset construction [12]. Notice that for all the unitary representations the Casimir $l(l+1)$ is real, but that only for the continuous series $-l(l+1) = \lambda^2 + \frac{1}{4} > 0$. As we see from (3.5) the continuous variable λ plays the role of the radial momentum. The physical interpretation of the vertex operators will be further discussed in section 5.

The spectrum of the Euclidean Black Hole CFT

In the case of the Euclidean model the scalar field ϕ is compactified. To determine its compactification radius we use the fact that even on the universal cover $\widetilde{SL}(2,R)$ the difference $\theta = \frac{1}{2}(\theta_L - \theta_R)$ is a periodic modulo 2π. This implies that $\omega_L - \omega_R$ is integer and from (3.3) we see that therefore $q_L + q_R$ is also integer. From this we can conclude that ϕ is periodic modulo 2π; this is also consistent with (2.6). Because ϕ is compactified q_L and q_R do not have to be equal. Instead, as is well known for the gaussian model, their difference is quantized, in this case in multiples of k: $q_L - q_R = n \cdot k$. Combining all these conditions we find that $\omega_{L,R}$ take their values on the lattice

$$\omega_L = \tfrac{1}{2}(m + nk), \qquad \omega_R = -\tfrac{1}{2}(m - nk), \tag{3.6}$$

with n and m integers. The quantum number m is to be interpreted as the discrete momentum of the string in the θ direction, while n is the winding number. Again we find that by shifting the fields $\theta_L \to \theta_L - \varphi$, $\theta_R \to \theta_R + \overline{\varphi}$ the ϕ-field disappears from the vertex operator. The final form of the vertex operators in the Euclidean coset theory can be written in the form

$$T^l_{mn}(r, \theta_L, \theta_R) = \mathcal{P}^l_{\omega_L \omega_R}(\cosh r) e^{i\omega_L \theta_L + i\omega_R \theta_R} \tag{3.7}$$

where the functions $P^l_{\omega,\omega'}(x)$ are the so-called Jacobi functions; some useful properties of these are given in the Appendix. The vertex operators (3.7) seem to depend on three independent fields. However, because the WZW-theory is gauged, the fields effectively satisfy the constraints $J^3 = \overline{J}^3 = 0$. Therefore it is more appropriate to think of θ_L and θ_R as the left- and right-moving parts of a single field θ.

The zero-modes J^a_0 of the $SL(2,R)$ currents can, when acting on the vertex operators $T^l_{mn}(r, \theta_L, \theta_R)$, be represented as the differential operators

$$\begin{aligned} J^3 &= i\frac{\partial}{\partial \theta_L}, \\ J^\pm &= e^{\pm i\theta_L}\left[\frac{\partial}{\partial r} \pm \frac{i}{\sinh r}\left(\frac{\partial}{\partial \theta_R} - \coth r \frac{\partial}{\partial \theta_L}\right)\right] \end{aligned} \tag{3.8}$$

Similar expressions exist for the zero modes $\overline{J}^a$ of the right moving currents. The vertex operators T^l_{mn} are eigenfunctions of the Virasoro operators L_0 and $\overline{L}_0$, which, given their expression in terms of the currents, can also be expressed as differential operators. We find

$$\begin{aligned} L_0 &= -\frac{\Delta_0}{k-2} - \frac{1}{k}\frac{\partial}{\partial \theta_L^2}, \\ \overline{L}_0 &= -\frac{\Delta_0}{k-2} - \frac{1}{k}\frac{\partial}{\partial \theta_R^2}, \end{aligned} \tag{3.9}$$

where Δ_0 is the Casimir on the group given by

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\left(\frac{\partial^2}{\partial \theta_{_L}^2} - 2\cosh r \frac{\partial^2}{\partial \theta_{_L} \partial \theta_{_R}} + \frac{\partial^2}{\partial \theta_{_R}^2}\right) \tag{3.10}$$

In fact, the matrix elements T^l_{mn} are simultaneous eigenfunctions of $\partial/\partial\theta_{_{L,R}}$ and the Casimir operator Δ_0, which is the natural Laplacian on $\widetilde{SL}(2,R)$. The eigenvalues are the conformal dimensions, which in this case are not equal. We have

$$L_0\,|T^l_{mn}\rangle = h^l_{mn}|T^l_{mn}\rangle, \qquad \overline{L}_0\,|T^l_{mn}\rangle = \overline{h}^l_{mn}|T^l_{mn}\rangle, \tag{3.11}$$

where

$$\begin{aligned} h^l_{mn} &= -\frac{l(l+1)}{k-2} + \frac{(m+nk)^2}{4k}, \\ \overline{h}^l_{mn} &= -\frac{l(l+1)}{k-2} + \frac{(m-nk)^2}{4k}. \end{aligned} \tag{3.12}$$

Here we notice that the spectrum of the Euclidean black hole CFT coincides with that of a Liouville field with momenta $\alpha = \sqrt{\frac{2}{k-2}}l$ and background charge $Q = \sqrt{\frac{2}{k-2}}$ coupled to a scalar field compactified at a radius

$$R = \sqrt{k}R_0, \tag{3.13}$$

where R_0 is the self-dual radius. However, we would like to stress that this does not imply that the two theories are equivalent.

3.2. Completeness and inner product.

In this subsection we will examine the allowed spectrum of isospins l carried by our vertex operators T^l_{mn} in more detail, and discuss the issue of completeness. For this we will borrow some basic results from the mathematical theory of harmonic analysis on $SL(2,R)$ (see e.g. [15]). In the next section we will provide a more physical and intuitive derivation of the spectrum.

To determine which matrix elements actually occur in the spectrum of the CFT, we impose the condition that the vertex operators $T(g) = T(r, \theta_{_L}, \theta_{_R})$, which are functions on the group $\widetilde{SL}(2,R)$, lead to normalizable operators. It is reasonable to assume that the CFT inherits this inner product, that is usually defined in terms of the two-point function on the sphere, from the natural inner product of the $\widetilde{SL}(2,R)$ representations. This is in turn defined via integration over the group

$$\langle T_1|T_2\rangle = \int dg\, T_1(g)^* T_2(g), \tag{3.14}$$

where the group invariant measure is

$$dg = \frac{1}{4\pi^2} \sinh r\, dr\, d\theta_{\scriptscriptstyle L} d\theta_{\scriptscriptstyle R}. \tag{3.15}$$

The normalizable operators correspond to the square integrable functions on $\widetilde{SL}(2,R)$, which, due to the factor $\sinh r$ in the measure, have to decrease like $e^{-\alpha r}$ with $\mathrm{Re}\,\alpha > \frac{1}{2}$ for $r \to \infty$. If we also allow square-integrable wave packets, we can include $\mathrm{Re}\,\alpha = \frac{1}{2}$. Since matrix elements with spin l behave as e^{lr}, we thus find a restriction to representations with $\mathrm{Re}\,l \leq -\frac{1}{2}$. Note that, in particular, this implies that, unlike any known unitary CFT, the identity operator in our model does *not* correspond to a normalizable state in the Hilbert space. This is one of several correspondences between the $SL(2,R)/U(1)$ model and Liouville theory [16, 17].

To determine the complete list of normalizable vertex operators we can simply use the known results for the square integrable functions on $\widetilde{SL}(2,R)$, which have been well studied in the mathematical literature. We will describe the spectrum of allowed values for the isospin l for fixed values of $\omega_{\scriptscriptstyle L}$ and $\omega_{\scriptscriptstyle R}$ as given by the lattice (3.6). A complete basis for the square integrable functions is provided by: (i) The matrix elements of the principal continuous representations with $l = -\frac{1}{2} + i\lambda$ with $\lambda \in \mathbf{R}$. Here λ can be taken to be positive. We denote these matrix elements as T^{λ}_{mn}. (ii) The matrix elements of the discrete series with l real, which conventionally are written as D^{l}_{mn}.

Of course, these matrix elements may not always exist. This is not a problem for the continuous representations, whose matrix elements for arbitrary $\omega_{\scriptscriptstyle L}$ and $\omega_{\scriptscriptstyle R}$ are generically non-vanishing. In contrast, the discrete representations have various restrictions on their matrix elements. First, $\omega_{\scriptscriptstyle L}$ and $\omega_{\scriptscriptstyle R}$ should have the same sign. Furthermore, $|\omega_{\scriptscriptstyle L,R}| + l$ should be a non-negative integer. So, translated to our lattice (3.6) of allowed matrix elements, parametrized by the integers m and n, we find contributions of the discrete representations for

$$|nk| > |m|, \qquad\qquad l = r - \tfrac{1}{2}|nk| + \tfrac{1}{2}|m| < -\tfrac{1}{2} \tag{3.16}$$

where r runs over the non-negative integers. So for given m and n, there are at most a finite number of l values that contribute.

The above results for the spectrum can be inferred from the so-called Plancherel Formula, which is one of the central results of harmonic analysis on $\widetilde{SL}(2,R)$. It guarantees us that we have not missed any states. It is likely, however, that in the conformal field theory the spectrum of discrete representations is truncated like in WZW-models based on compact groups [18]. Indeed, in a discrete representation of spin l the state with $|\omega| = -l$ has conformal dimension $h = -l(2l+k)/k(k-2)$, which becomes negative for $l < -k/2$. So, if we would assume unitarity, we find that the isospin l is restricted to [13]

$$-\tfrac{1}{2}k < l < -\tfrac{1}{2}. \tag{3.17}$$

However, since the issue of unitarity of this CFT is still open, it is not yet clear whether this truncation is actually obeyed.

Since the vertex operators correspond to matrix elements, they acquire a canonical normalization, and it is meaningful to consider their inner product. The results are the following for the continuous series

$$\langle T^{\lambda}_{mn} | T^{\lambda'}_{mn} \rangle = \frac{1}{\rho_\epsilon(\lambda)} \delta(\lambda - \lambda') \tag{3.18}$$

where $\epsilon = \omega_{L,R}$ (mod 1), and $\rho_\epsilon(\lambda)$ is the so-called Plancherel measure

$$\rho_\epsilon(\lambda) = \pi\lambda \, Re \, \tanh \pi(\lambda + i\epsilon). \tag{3.19}$$

For the discrete series the inner product is

$$\langle D^{l}_{mn} | D^{l}_{mn} \rangle = \frac{1}{l - \frac{1}{2}} \frac{\Gamma(l + \omega_R + 1)\Gamma(l - \omega_R + 1)}{\Gamma(l + \omega_L + 1)\Gamma(l - \omega_L + 1)} \tag{3.20}$$

with $\omega_{L,R}$ given in (3.6). All other inner products vanish.

4. Target Space Geometry of the Euclidean Black Hole

In the previous section we treated the $SL(2,R)/U(1)$ model as a coset conformal field theory. Now we want to shift our point of view and approach the model as a critical string theory describing the propagation of a string in the Euclidean black hole geometry.

To this end we have to put $k = 9/4$ in order to obtain the critical central charge $c = 26$. However, we will mostly write our expressions for general k.

Quite generally, the physical state equations for the vertex operators $T(r, \theta_L, \theta_R)$ read

$$(L_0 + \overline{L}_0 - 2)|T(r, \theta_L, \theta_R)\rangle = 0, \tag{4.1}$$

$$(L_0 - \overline{L}_0)|T(r, \theta_L, \theta_R)\rangle = 0. \tag{4.2}$$

Here L_0 and $\overline{L}_0$ represent the differential operators given in (3.9) acting on the field $T(r, \theta_L, \theta_R)$. Since we have a Euclidean signature there will be no propagating on-shell modes. However, we will be interested in the off-shell modes, which are the non-zero modes of the kinetic operator $L_0 + \overline{L}_0 - 2$. So, we will relax condition (4.1) and only impose the $h = \overline{h}$ condition (4.2). In this way we expect to find modes that only depend on two space-time variables instead of three, in accordance with our intuitive sigma-model point of view.

Indeed, since

$$L_0 - \overline{L}_0 = \frac{1}{k}\Big(\frac{\partial^2}{\partial\theta_R^2} - \frac{\partial^2}{\partial\theta_L^2}\Big), \tag{4.3}$$

equation (4.2) tells us that physical states are of the form

$$T(r, \theta_L, \theta_R) = T(r, \theta) + \widetilde{T}(r, \widetilde{\theta}), \tag{4.4}$$

where

$$\theta = \tfrac{1}{2}(\theta_L + \theta_R), \qquad \widetilde{\theta} = \tfrac{1}{2}(\theta_L - \theta_R). \tag{4.5}$$

We can thus introduce two separate tachyon fields $T(r, \theta)$ and $\widetilde{T}(r, \widetilde{\theta})$ that depend only on r and one θ-coordinate, and are two-dimensional scalar fields. According to the discussion of the previous section, the physical modes are either pure momentum modes ($n = 0$) or pure winding modes ($m = 0$). The tachyon field $T(r, \theta)$ can be expanded in matrix elements of the continuous series:

$$T(r, \theta) = \sum_m \int d\lambda\, \rho(\lambda)\, a_m^l\, T_m^\lambda(r, \theta), \tag{4.6}$$

with

$$T_m^\lambda(r, \theta) = \langle l, \tfrac{1}{2}m|g(r, \theta, -\theta)|l, -\tfrac{1}{2}m\rangle, \tag{4.7}$$

and $l = -\frac{1}{2} + i\lambda$. We see from this spectrum that the θ coordinate is periodic with period 2π.

Similarly, $\widetilde{T}(r, \widetilde{\theta})$ is built out of the matrix elements

$$\widetilde{T}_n^l(r, \widetilde{\theta}) = \langle l, \tfrac{1}{2}nk|g(r, \widetilde{\theta}, \widetilde{\theta})|l, \tfrac{1}{2}nk.\rangle \tag{4.8}$$

Here we see from the eigenvalues that occur, that the coordinate $\widetilde{\theta}$ should be considered to have period $2\pi/k$. In the expansion of this tachyon field the discrete representations do contribute; their interpretation will be discussed in detail in section 4.3. First, let us take a closer look at the momentum states.

4.1. The momentum modes and their target space action.

In the conventional sigma-model approach a tachyon field is described by a target-space effective action of the form

$$S[T] = \int d^2x \, e^{\Phi} \sqrt{G} (G^{ij} \partial_i T \partial_j T - 2T^2), \tag{4.9}$$

where G_{ij} and Φ are the background metric and dilaton field. We now like to combine this point of view with the more abstract group theoretical discussion of the last section. The link between these two approaches is provided by the L_0 operator, which we would like to identify with the target space Laplacian of the sigma model. That is, we would like to have that

$$L_0 = -\frac{1}{2e^{\Phi}\sqrt{G}} \partial_i e^{\Phi} \sqrt{G} G^{ij} \partial_j \tag{4.10}$$

The precise form of L_0 for the momentum modes follows from our discussion of the previous section. When acting on the tachyon field $T(r,\theta)$ the L_0 operator given in (3.9) takes the simpler form

$$L_0 = -\frac{1}{k-2} \left[\frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + (\coth^2 \frac{r}{2} - \frac{2}{k}) \frac{\partial^2}{\partial \theta^2} \right] \tag{4.11}$$

We are now in a position to determine the effective metric and dilaton fields by comparing this result with (4.10). It is a simple exercise to determine the background that reproduces this Laplacian. The solution for G_{ij} and Φ is of the form

$$\begin{aligned} ds^2 &= \tfrac{1}{2}(k-2) \Big[dr^2 + \beta^2(r) \, d\theta^2 \Big], \\ \Phi &= \log\Big(\sinh r / \beta(r) \Big). \end{aligned} \tag{4.12}$$

where $\beta(r)$ is given by

$$\beta(r) = 2 \left(\coth^2 \frac{r}{2} - \frac{2}{k} \right)^{-\frac{1}{2}} \tag{4.13}$$

In leading order in $1/k$ this coincides with the results (1.6) obtained in [9, 8]. It is of course tempting to conjecture that the above metric and dilaton field represent the exact solution to the σ model β-function equations, but clearly a more direct analysis is needed to confirm this. Notice that in the critical case, $k = 9/4$, the $1/k$ corrections are substantial.

In this momentum sector there is one mode $T_0(r)$ which satisfies the on-shell condition $(L_0 - 1)T_0(r) = 0$. It can be given the following integral representation

$$T_0(r) = \int_0^{2\pi} d\varphi \sqrt{\cosh r + \cos\varphi \sinh r}. \tag{4.14}$$

It represents a marginal operator in the conformal field theory [9]. Its coupling constant plays an analogous role as the cosmological constant in Liouville theory.

4.2. The winding modes and duality

We have seen that the momentum modes build up a conventional scalar field propagating in the Euclidean black hole geometry. It represents the zero-mode of the string. The discussion of the winding modes might be of more interest, since it represents a genuine 'stringy' effect. (As we have stressed in $1+1$ dimensions the characteristic towers of propagating massive string modes are absent, and string theory looses many of its obvious distinctions from ordinary field theory.) These winding modes are not described by the effective action (4.9) for the tachyon field and represent in fact another field-theoretic degree of freedom, which clearly reflect the nature of strings as extended objects.

From the conformal field theory point of view the momentum and winding vertex operators are treated on equal footing. They only differ in the relevant sign between the left and right quantum numbers ω_L and ω_R. The conformal dimensions and all other properties of the vertex operators are invariant if we change the sign of, say, ω_R. This is the general way in which target space duality, like the familiar $R \to 1/R$ symmetry of a compactified boson, is implemented in conformal field theory: the right-moving representations are replaced by their complex conjugates, whereas the left-movers are left invariant.

In this particular case, the $SL(2,R)$ currents of the coset CFT transform under duality as

$$J^a \to J^a, \qquad \overline{J}^a \to -\overline{J}^a, \tag{4.15}$$

which is clearly a symmetry of the theory. Thus, in terms of gauged WZW models, duality relates the two possible anomaly free gauge symmetries: the axial symmetry

$$g \to hgh, \tag{4.16}$$

and the vector symmetry

$$g \to hgh^{-1}. \tag{4.17}$$

Fig. 3: This geometry, that is related to fig. 2 of the Introduction by duality, represents the background in which the string winding modes propagate. In coordinates $(r, \widetilde{\theta})$ it has a curvature singularity, which for $k \to \infty$ occurs at $r = 0$.

However, when considered as sigma-models these two group actions lead to qualitatively different target spaces. The first action has no fixed points and therefore describes a regular target manifold. This is no longer true for the second group action, which has the identity element as a fixed point. Here we expect the manifold to develop a singularity. We would like to stress that as conformal field theories these two models are completely equivalent, and thus that the two target space interpretations are equally valid. We will now turn to the alternative dual target space, corresponding to (4.17), in which the role of the momentum and winding modes are interchanged.

One method to derive this alternative space-time picture, is to study the the dual field $\widetilde{T}(r, \widetilde{\theta})$ that represent the winding modes. For this dual tachyon the L_0 equation takes the form

$$L_0 = -\frac{1}{k-2}\left[\frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + (\tanh^2 \frac{r}{2} - \frac{2}{k})\frac{\partial^2}{\partial \widetilde{\theta}^2}\right] \tag{4.18}$$

Analogous to our discussion of the momentum modes, we can give an interpretation of this L_0 equation as a wave equation for a massless particle coupled to a specific G, Φ background. In this case the resulting geometry is again of the form (4.12), where we now find

$$\beta(r) = 2\left(\tanh^2 \frac{r}{2} - \frac{2}{k}\right)^{-\frac{1}{2}}. \tag{4.19}$$

This metric and dilaton field now are singular at $r = \text{arctanh}\sqrt{2/k}$. For large k we have

$$ds^2 = dr^2 + 4\coth^2 \frac{r}{2}\, d\widetilde{\theta}^2. \tag{4.20}$$

So, in this picture the vortices are moving on a dual manifold that looks more like a 'trumpet' (see fig. 3). In fact, in the $k \to \infty$ the two manifolds are related by the familiar $R \to 1/R$ duality transformation applied to the coordinate θ. Our analysis seem to suggest that $1/k$ corrections modify this transformation. Another way to relate the two solutions, that is valid to all orders in $1/k$, is by the transformation* $r \to r + i\pi/2$, $\theta \to \widetilde{\theta}$.

Notice that for this particular geometry there are no *a priori* reasons why $\widetilde{\theta}$ should be periodic with period $2\pi/k$. This brings us to the issue of the compactification radii in the different models. There is an important geometrical distinction between the role of the two variables θ and $\widetilde{\theta}$ on the group manifold $SL(2,R)$. The first coordinate θ is a truly angular variable, comparable to polar coordinates, and should have period 2π in order to avoid a conical singularity at $r = 0$. The same holds for the target space of section 4.1. On the other hand, $\widetilde{\theta}$ parametrizes a non-contractable loop in $SL(2,R)$, and need not have a definite period when we allow ourselves to work on the universal covering group $\widetilde{SL}(2,R)$. The period $2\pi/k$ of $\widetilde{\theta}$ is only indirectly determined by duality considerations.

It is possible to change the period of the Euclidean time by applying an orbifold construction. Since the CFT has a global $U(1)$ symmetry we can mod out by any $\mathbf{Z}_N$ subgroup. This introduces an orbifold singularity at $r = 0$. One can convince oneself that by repeating this orbifold construction one can build a theory for any rational multiple of the original compactification radius.

4.3. Discrete representations and bound states

The most qualitatively different aspect of the dual tachyon field $\widetilde{T}(r, \widetilde{\theta})$ is the existence of bound states of the string and the black hole. This issue is closely related to the spectrum of $SL(2,R)$ representations that occur in this sector. Indeed, we already saw in section 3.2 that the normalizable modes of $\widetilde{T}(r, \widetilde{\theta})$ can be decomposed on matrix elements of both the continuous and the discrete representations.

This spectrum of tachyon modes can be motivated in a more physically understandable way as follows†. If we rescale our field by

$$\widetilde{T} \to (\sinh r)^{-\frac{1}{2}} \widetilde{T}, \tag{4.21}$$

the L_0 operator takes the form of a Schrödinger operator

$$L_0 \sim -\frac{\partial^2}{\partial r^2} + V(r), \tag{4.22}$$

*We thank M. Roček for pointing this out to us.

†We like to thank S. Shenker for suggesting this interpretation.

Fig. 4: The potential which determines, for fixed ω, the spectrum of the L_0 operator. The eigenvalues are proportional to $-(l+\frac{1}{2})^2$, where l is the $SL(2,R)$ isospin. There are both discrete bound states with l real and quantized, and a continuum of wave-like eigenfunctions with $l=-\frac{1}{2}+i\lambda$, $\lambda \in \mathbf{R}$. The former correspond to discrete representations of $SL(2,R)$, while the latter are related to the representations in the principal continuous series.

where, for fixed 'angular momentum' $\omega = \frac{1}{2}nk$, the potential V is (up to a constant) given by

$$V(r) = (\omega^2 - \frac{1}{16}) \tanh^2 \frac{r}{2} - \frac{1}{16} \coth^2 \frac{r}{2}. \tag{4.23}$$

With the redefinition (4.21) of the tachyon field, the measure is simply given by dr. The spectrum is now identical to that of a one-dimensional quantum mechanical particle moving in the effective potential $V(r)$. This potential behaves as an *attractive* $-1/r^2$ potential around $r=0$ (see fig. 4). By general arguments, it will have both oscillating modes with continuous positive L_0 eigenvalues, and bound states with discrete negative eigenvalues. Since the L_0 eigenvalues are proportional to $-(l+\frac{1}{2})^2$, wave-like solutions will satisfy $l=-\frac{1}{2}+i\lambda$. The bound states have real $l<-\frac{1}{2}$ and fall of at infinity as e^{lr}, and thus are localized around $r=0$.

It is interesting to compare this analysis with the case of the momentum states. For these modes the relevant potential is of the form

$$V(r) = (\omega^2 - \frac{1}{16}) \coth^2 \frac{r}{2} - \frac{1}{16} \tanh^2 \frac{r}{2}. \tag{4.24}$$

For non-zero ω this is a *repulsive* $1/r^2$ potential. So we find only wave-like solutions and no bound states. This indeed confirms our result that only the representations with $l=-\frac{1}{2}+i\lambda$ occur in this sector of the string spectrum.

We have to notice here a somewhat counter-intuitive effect. The discrete representations only occur for the dual tachyon $\widetilde{T}(r, \widetilde{\theta})$, and correspond to winding modes. In the target space (4.13) that describes the dynamics of the momentum modes, we can understand the existence of these winding states by an asymptotic analysis at spatial infinity. Here the target space is isomorphic to a cylinder, around which the string can wrap itself. However, the bound states are actually localized at the origin, where we do not expect to find winding states at all. So we should conclude that this particular σ-model point of view breaks down. Of course, in the dual picture where the string moves on the 'trumpet-like' target space (4.19) there is no mystery and they naturally occur as bound states.

The spectrum of bound states can also be exhibited in the asymptotic behaviour of the wave-like modes $\widetilde{T}_n^\lambda$ at spatial infinity. Since we have an unpenetrable potential every wave will scatter back from the potential and return to infinity. However, there will be in general a delay, expressed in the phase shift or scattering matrix. More concretely, from the above representation of the modes $\widetilde{T}_n^\lambda$ we find

$$\widetilde{T}_n^\lambda(r, \widetilde{\theta}) \overset{r\to\infty}{\sim} e^{-r/2+ink\widetilde{\theta}}\Big[c_{\lambda,n}e^{-i\lambda r} + c^*_{\lambda,n}e^{i\lambda r}\Big] \tag{4.25}$$

with

$$c_{\lambda,n} = \Gamma(-2l-1)\Big/\Gamma(-l-\tfrac{1}{2}nk)\Gamma(-l+\tfrac{1}{2}nk) \tag{4.26}$$

Here we substituted $l = -\frac{1}{2} + i\lambda$. It is evident that the coefficient $c_{l,n}$ only has zeroes for real $l < 0$, precisely at values where bound states occur.

We have seen that, apart from the tachyon zero-mode (4.14), no physical states — that satisfy both (4.1) and (4.2) — occur in the momentum sector. This need no longer be true for the winding states. Here we have the discrete representations available, and there exist solutions of the mass-shell condition, that with $\omega = \frac{1}{2}nk$ and $k = 9/4$ reads

$$-4l(l+1) + \frac{9}{16}n^2 = 1 \tag{4.27}$$

Of course, l and n are subject to the restrictions discussed in section 3.2. The solutions are labeled by

$$\pm n = 4N+2, \quad l = -\tfrac{3}{2}N - \tfrac{5}{4}, \qquad\qquad N \in \mathbf{Z}_{\geq 0}. \tag{4.28}$$

These special bound states, together with the zero-mode, are the *only* tachyon physical states in the Euclidean black hole string theory. Probably these modes are not integrable as marginal operators, and so do not give rise to additional moduli of the theory. Moreover we note that these states disappear if we impose the unitarity truncation (3.17).

5. Scattering and Duality in the Minkowskian Black Hole

In this section we will describe in more detail the structure of the tachyon vertex operators on the maximally extended Lorentzian black hole geometry. In general there are two interesting quantum mechanical interactions between the tachyon field and the black hole. The first effect is the scattering of a tachyon off the geometry of the black hole. We will compute the amplitude for that process in this section. The second effect is the celebrated Hawking radiation, to which we turn in section 6.

5.1. Construction of the vertex operators

In section 3 we have shown that the physical tachyon vertex operators can be expressed as $\widetilde{SL}(2,R)$ matrix elements in the continuous representations. In principle, we can obtain these matrix elements by analytic continuation $\theta \to it$. However, the coordinates r and t only parametrize part of the black hole geometry, namely region I (see fig. 1). To define the vertex operators on the maximally extended space time, we should use the global coordinates u and v in which (for large k) the metric takes the form

$$ds^2 = \frac{du\,dv}{1-uv}. \tag{5.1}$$

These coordinates correspond to the parametrization of the $SL(2,R)/U(1)$ coset as [8]

$$g(u,v) = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \qquad\qquad ab + uv = 1, \tag{5.2}$$

where a and b are redundant variables under the gauge symmetry $g \to hgh$. (Strictly speaking, the $SL(2,R)/U(1)$ coset is a double cover of the (u,v) plane, where the two sheets are distinguished by the sign of a or b. The two sheets intersect at the singularity $uv = 1$.) We can now define the vertex operators globally as

$$T^\lambda_\omega(u,v) = \langle l,\omega|g(u,v)|l,-\omega\rangle \tag{5.3}$$

where $-\infty < \omega < \infty$ denotes the H eigenvalue, and $l = -\frac{1}{2} + i\lambda$ the $SL(2,R)$ isospin. The parameter λ has the interpretation of spatial momentum, whereas ω corresponds to the energy of the string state. Since in this case the subgroup H is non-compact, the eigenvalues ω of the Killing vector are continuous. In terms of λ and ω the mass shell condition reads

$$(L_0 - 1)|T^\lambda_\omega\rangle = (4\lambda^2 - \tfrac{4}{9}\omega^2)|T^\lambda_\omega\rangle = 0 \tag{5.4}$$

Fig. 5: The contours $\mathcal{C}_i$ ($i = 1, \ldots, 4$) that feature in the integral representation (5.5) of the four different vertex operators. In passing the branch cuts, that start at $x = 0$, $x = x_1$, and $x = x_2$, the integrand is multiplied by $e^{2\pi i \nu_\pm}$ as indicated.

where we used that $k = 9/4$. As we will discuss, the two solutions $\lambda = \pm\omega/3$ describe the incoming and outgoing tachyon modes.

There are several approaches to find the explicit form of the vertex operators. As explained in the Appendix the matrix elements (5.3) in the Lorentzian model can be obtained by an integral representation that is very similar to the one that gave us the Euclidean vertex operators. We only have to deform the integration contour $\mathcal{C}$, that was given by the unit circle, to the real axis. In this way we find the representation

$$T_\omega^\lambda(u,v) = \int_{\mathcal{C}} \frac{dx}{x} x^{-2i\omega} (\sqrt{1-uv} + u/x)^{-\nu_-} (\sqrt{1-uv} - vx)^{-\nu_+} \tag{5.5}$$

where

$$\nu_\pm = \tfrac{1}{2} - i(\lambda \pm \omega). \tag{5.6}$$

In fact, there are several possible choices for the contour of integration $\mathcal{C}$ since the integrand has a number of branch cuts, at $x = 0$ and $x = \infty$, and two other ones at

$$x_1 = \sqrt{1-uv}/v, \qquad \text{and} \qquad x_2 = -u/\sqrt{1-uv}. \tag{5.7}$$

This ambiguity in the choice for $\mathcal{C}$ is directly related to the fact that one actually has two eigenstates for fixed ω in the principal continuous series of $SL(2,R)$ representations, and correspondingly four different vertex operators with the quantum numbers of T_ω^λ. This reflects the existence of *four* asymptotic regions: the null infinities and the event horizons in the 'universes' I and II. However, the conventional matrix elements are not analytic functions in the (u,v)-plane [15]. We therefore follow a slightly different approach.

A number of subtleties arise when we try to use the integral representation (5.5) to define the vertex operators across the horizon, since we have to give a precise prescription for the choice of contour. So, for definiteness, let us first consider the operators in the region I in which $u > 0$, $v < 0$. In order to make the physical interpretation more transparent we will make use of both the (u, v) and the (r, t) coordinates, related by

$$u = \sinh\frac{r}{2}e^{t}, \qquad v = -\sinh\frac{r}{2}e^{-t}. \tag{5.8}$$

In this case the two branch points x_1 and x_2 are located on the negative real axis as indicated in figure 5. Together with the branch point at $x = 0$ they divide up the real axis in four segments $\mathcal{C}_i$ $(i = 1, \ldots, 4)$, each of which can be taken as our contour $\mathcal{C}$. In this way we find the following four types of vertex operators with the quantum numbers λ, ω

$$\begin{aligned} R^{\lambda}_{\omega}(u,v) &: \quad \text{for} \quad \mathcal{C}_1 = [0, \infty[, \\ U^{\lambda}_{\omega}(u,v) &: \quad \text{for} \quad \mathcal{C}_2 = [x_2, 0], \\ L^{\lambda}_{\omega}(u,v) &: \quad \text{for} \quad \mathcal{C}_3 = [x_1, x_2], \\ V^{\lambda}_{\omega}(u,v) &: \quad \text{for} \quad \mathcal{C}_4 =]-\infty, x_1]. \end{aligned} \tag{5.9}$$

All the different contour integrals are of standard type, and are related to the gaussian hypergeometric function $F(\alpha, \beta; \gamma; z)$. In fact, at this point it is convenient to introduce an alternative radial coordinate y defined by

$$y = uv = -\sinh^2\frac{r}{2}. \tag{5.10}$$

The coordinates (y, t) provide a faithful parametrization of the regions I, III, and V. The horizon corresponds to $y = 0$ and the singularity occurs at $y = 1$. In region I we have $-\infty < y < 0$.

The relevant hypergeometric functions can all be written in terms of the function

$$F^{\lambda}_{\omega}(y) = (-y)^{-i\omega}B(\nu_+, \overline{\nu}_-)\, F(\nu_+, \overline{\nu}_-; 1 - 2i\omega; y) \tag{5.11}$$

where $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ denotes the Euler Beta function and $\nu_{\pm}$ are defined in (5.6). We now have the following realizations of the vertex operators U^{λ}_{ω} and V^{λ}_{ω} in (y, t) coordinates

$$\begin{aligned} U^{\lambda}_{\omega}(y,t) &= e^{-2i\omega t}F^{\lambda}_{\omega}(y), \\ V^{\lambda}_{\omega}(y,t) &= e^{-2i\omega t}F^{\lambda}_{-\omega}(y). \end{aligned} \tag{5.12}$$

Fig. 6: The four null infinities of region I (see fig. 1) of the black hole space-time depicted in a Penrose diagram: the past and future horizons $\mathcal{H}^-$ and $\mathcal{H}^+$, and the past and future light-like infinities $\mathcal{I}^-$ and $\mathcal{I}^+$.

The operators L_ω^λ and R_ω^λ have similar expressions

$$\begin{aligned} L_\omega^\lambda(y,t) &= e^{-2i\omega t} y^{-\frac{1}{2}} F^\omega_{-\lambda}(1/y), \\ R_\omega^\lambda(y,t) &= e^{-2i\omega t} y^{-\frac{1}{2}} F^\omega_{\lambda}(1/y). \end{aligned} \tag{5.13}$$

Notice that in (5.13) the role of λ and ω are interchanged in comparison with (5.12). It is easily checked that in region I these modes have the following properties under C, P and T

$$L_\omega^\lambda = (R_{-\omega}^{-\lambda})^* = R_\omega^{-\lambda}, \qquad U_\omega^\lambda = (U_{-\omega}^\lambda)^* = U_\omega^{-\lambda}, \tag{5.14}$$

with similar relations for R_ω^λ and V_ω^λ.

To find the physical interpretation of the four different vertex operators that we constructed we will now investigate the behaviour of these modes at spatial infinity ($r = \infty$) and at the horizon ($r = 0$). In our discussion we will assume that the mass-shell condition (5.4) is satisfied, and that the energy ω is positive.

The asymptotic behaviour of the operators can be read off immediately from the representations in terms of hypergeometric functions, or more directly from the integral representation. From these expressions one finds that the asymptotic forms of the modes L_ω^λ and R_ω^λ represent respectively pure left-moving and right-moving tachyon waves at spatial infinity. More precisely, in the limit $r \to \infty$ we have in (r,t) coordinates

$$\begin{aligned} L_\omega^\lambda &\sim B(\overline{\nu}_+, \overline{\nu}_-) \cdot e^{-\frac{1}{2}r} e^{-i\lambda r - 2i\omega t} \\ R_\omega^\lambda &\sim B(\nu_+, \nu_-) \cdot e^{-\frac{1}{2}r} e^{i\lambda r - 2i\omega t} \end{aligned} \tag{5.15}$$

A similar analysis shows that the modes U_ω^λ and V_ω^λ describe pure left or right-moving plane waves in the neighbourhood of the horizon $r = 0$. Here it is more convenient to use

Fig. 7: The behaviour of the four modes R^λ_ω, L^λ_ω, U^λ_ω, and V^λ_ω (defined in the text) at the infinities of region I. Each mode vanishes at one particular null boundary.

the coordinates (u, v), in terms of which we find for $r \to 0$

$$\begin{aligned} U^\lambda_\omega &\sim B(\nu_+, \overline{\nu}_-) \cdot u^{-2i\omega} \\ V^\lambda_\omega &\sim B(\overline{\nu}_+, \nu_-) \cdot (-v)^{2i\omega} \end{aligned} \tag{5.16}$$

In this fashion the modes U and V are very similar to the Rindler modes for an accelerating observer. These modes become infinitely blue-shifted when they approach the past resp. future horizon. The vertex operator U^λ_ω describes a string falling into the future horizon, that is, a string being absorbed by the black hole. The U modes can be trivially extended to region III. On the other hand the operator V^λ_ω describes the process where a particle is emitted by the 'white hole' and crosses the past horizon. These modes are naturally defined in regions I and IV. We will return to the issue of the global properties of the vertex operators in section 5.3.

In this way we arrive at a picture where the four different modes are associated to the four different light-like asymptotic regions of region I. (See fig. 6). These null infinities, that are conventionally denoted as $\mathcal{I}^+$, $\mathcal{I}^-$, $\mathcal{H}^+$, and $\mathcal{H}^-$, correspond to $(r, t) = (\infty, \pm\infty)$ and $(0, \pm\infty)$ respectively. A precise characterization of the relation between these regions and the four modes proceeds as follows. When we consider wave packets constructed just out of one of the four possible modes, these packets will *vanish* at one particular null infinity. For instance, wave packets constructed out of the R modes vanish at $\mathcal{I}^-$. The behaviour of all four modes is illustrated in fig. 7.

5.2. Scattering off the black hole

Related to the above discussion is the following fact. When we restrict ourselves to region I there will be linear relations among the four different modes, since it suffices to specify boundary conditions at two of the four null infinities, *e.g.* at $\mathcal{I}^+$ and $\mathcal{I}^-$, or at $\mathcal{H}^+$ and $\mathcal{H}^-$. So one can take for example the sets $\{L^\lambda_\omega, R^\lambda_\omega\}$ or $\{U^\lambda_\omega, V^\lambda_\omega\}$ as a basis for the physical states.

This linear dependence can be understood immediately from the definitions in terms of contour integrals. The integrand does not possess poles or branch points in the complex x plane except at 0, x_1, x_2 and ∞. So if we put all the branch cuts in the upper half-plane (as in fig. 5), and take the contour $\mathcal{C}$ in (5.5) to be the whole real axis, *i.e.*, $\mathcal{C} = \mathcal{C}_1 + \ldots + \mathcal{C}_4$, the result vanishes. This gives us a linear relation between the four modes. A similar, but different, relation is obtained if we place the branch cuts in the lower half-plane. With a suitable prescription for the contour integrals that define our modes, the two relations can be written as

$$\begin{aligned} ie^{\pi\lambda}U^\lambda_\omega - ie^{-\pi\lambda}V^\lambda_\omega + e^{-\pi\omega}L^\lambda_\omega + e^{\pi\omega}R^\lambda_\omega &= 0, \\ -ie^{-\pi\lambda}U^\lambda_\omega + ie^{\pi\lambda}V^\lambda_\omega + e^{\pi\omega}L^\lambda_\omega + e^{-\pi\omega}R^\lambda_\omega &= 0. \end{aligned} \tag{5.17}$$

This relation between the U–V and L–R bases can be written in matrix form as

$$\begin{pmatrix} U^\lambda_\omega \\ V^\lambda_\omega \end{pmatrix} = \frac{i}{\sinh 2\pi\lambda} \begin{pmatrix} \cosh \pi(\lambda+\omega) & \cosh \pi(\lambda-\omega) \\ \cosh \pi(\lambda-\omega) & \cosh \pi(\lambda+\omega) \end{pmatrix} \begin{pmatrix} L^\lambda_\omega \\ R^\lambda_\omega \end{pmatrix} \tag{5.18}$$

As we will now show, this result can be given the simple interpretation of scattering off the black hole.

Indeed, the plane waves L^λ_ω and R^λ_ω traveling from infinity will no longer be pure incoming or outgoing when they reach the area close to the horizon. Instead, relation (5.18) tells us that they will be expressed as some linear combinations of the U and V modes. That is, they are some mixture of left and right-movers. This effect is actually describing the back-scattering of the tachyon off the black hole geometry. This scattering is familiar from the four-dimensional case, where it just describes the gravitational interaction by which all objects, including massless particles, are deflected by the black hole. In this case a classical particle, that moves in one-dimensional space with coordinate r, would simply fall into the black hole. This can be seen from the effective potential that describes the scattering. This potential can be found analogously to (4.23), and is for fixed energy ω of the form

$$V(r) = (-\omega^2 - \frac{1}{16}) \coth^2 \frac{r}{2} - \frac{1}{16} \tanh^2 \frac{r}{2}. \tag{5.19}$$

It describes a $-1/r^2$ sink in which classical particles disappear behind the horizon $r = 0$. However, quantum mechanically there is the possibility that a wave scatters back off such an attractive potential. In order to calculate this effect we first have to describe the appropriate boundary conditions at $r=0$. Here we can use a simple physical argument. We want to describe the physical process where the tachyon comes in from spatial infinity and is either absorbed by the black hole or is reflected. So we need a mode that represents only absorption and no emission of particles by the black hole, and which therefore vanishes at the past horizon $\mathcal{H}^-$. From the given asymptotic behaviour we see that the correct mode is U^λ_ω, which has no components corresponding to particles coming out of the horizon, as is illustrated in fig. 7.

The reflection coefficient $S(\lambda,\omega)$ can now be determined by decomposing U^λ_ω in plane waves at $r\to\infty$

$$U^\lambda_\omega \sim e^{-\frac{1}{2}r-2i\omega t}(e^{-i\lambda r}+S(\lambda,\omega)e^{i\lambda r}) \tag{5.20}$$

This decomposition follows directly from the equations (5.18), which expresses U in the modes L, R, and the asymptotic formulas (5.15). Combining the different ingredients we find

$$S(\lambda,\omega)=\frac{\cosh\pi(\lambda-\omega)B(\nu_+,\nu_-)}{\cosh\pi(\lambda+\omega)B(\overline{\nu}_+,\overline{\nu}_-)} \tag{5.21}$$

If we insert the mass-shell condition $\omega=3\lambda>0$, the reflection coefficient can be expressed in terms of Γ-functions as

$$S(\lambda)=\frac{\Gamma(1+2i\lambda)\Gamma^2(\frac{1}{2}-4i\lambda)}{\Gamma(1-2i\lambda)\Gamma^2(\frac{1}{2}-2i\lambda)} \tag{5.22}$$

We note that the reflection coefficient is not a phase factor, but

$$|S(\lambda)|=\frac{\cosh 2\pi\lambda}{\cosh 4\pi\lambda} \tag{5.23}$$

This expresses that only part of the tachyon wave gets reflected, the other part will enter the horizon and will be absorbed by the black hole. Finally we note that the scattering amplitude (5.22) can also be obtained by analytic continuation of the corresponding Euclidean quantity, that expresses the reflection of tachyon momentum modes on the 'cigar.'

5.3. Global properties: duality and the singularity

Up to now we discussed the vertex operators in the region I outside the horizon. We will now comment on the global properties of the modes on the fully extended space-time. The first obstacle we will have to overcome is the horizon. That is, we have to define

Fig. 8: Penrose diagram of the fully extended black hole. Here we also included the regions V and VI behind the singularity. These describe geometries with a naked, negative mass, singularity. The 'time' flow $\partial_t = u\partial_u + v\partial_v$ is indicated for all six regions. Notice that it is actually space-like for regions III and IV.

the operators in the regions II, III and IV. We will also address the behaviour at the singularity.

In this section we will further take serious the regions V and VI behind the singularities (see fig. 8). The metric in these regions describes a negative mass black hole with a naked singularity, and is in fact the analytic continuation of the trumpet which featured in the previous section. This leads to a remarkable interpretation of the duality transformation that interchanges the momentum and winding modes of the Euclidean theory. When analytically continued this transformation maps the Minkowskian black hole onto itself via the identification

$$y \to 1 - y, \qquad\qquad t \to t, \tag{5.24}$$

where y is the radial coordinate $y = uv$. Under duality the regions I and V are interchanged, whereas region III is mapped onto itself. This interchanges in particular the horizon ($y = 0$) with the singularity ($y = 1$). A similar result was obtained independently in [20]. We note that region III can be seen as the analytic continuation of a metric of the form (again in the $k \to \infty$ limit)

$$ds^2 = dr^2 + \tan^2 r \, d\theta^2, \tag{5.25}$$

which in terms of CFT can be regarded as a $SU(2)/U(1)$ coset, that is a parafermionic model, albeit at fractional level. According to our definition of duality this model can be seen to be 'self-dual.'

This self-duality of the black hole string theory can be understood directly in terms of the conformal field theory formulation. As explained in the previous section, duality relates the group actions $g \to hgh$ and $g \to hgh^{-1}$. Both group action play a role in this model. One corresponds to the gauge symmetry, while the other represents the isometry generated by the Killing vector $\partial/\partial t$. For the non-compact choice of h both actions have fixed points, namely $SL(2,R)$ elements g with either $uv = 0$ or $ab = 1 - uv = 0$, where we made use of the parametrization (5.2). The fixed point of the gauge symmetry is the singularity, and the horizon is invariant under flow of the Killing vector. Self-duality is the statement that the actions $g \to hgh$ and $g \to hgh^{-1}$ are equivalent, after the transformation $y \to 1 - y$. This latter transformation can be seen to act on the group elements $g(z, \bar{z})$ as

$$g \to \epsilon \cdot g, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.26}$$

In terms of matrix elements the duality relation reads

$${}_{\pm}\langle\omega|g(y,t)| - \omega\rangle_{\pm} = {}_{\pm}\langle\omega|g(1-y,t)|\omega\rangle_{\mp} \tag{5.27}$$

where for completeness we distinguished the two eigenstates $|\omega\rangle_{\pm}$, as discussed in the Appendix. Here the reader clearly recognizes our definition of the momentum modes on the left-hand side and that of the winding modes on the right-hand side, as discussed in section 4. So, roughly speaking, the vertex operators in region I are the analytic continuations of the Euclidean *momentum* modes, whereas the operators in region V can be seen as analytic continuations of the Euclidean *winding* modes. We stress that, as it stands, duality does *not* give identifications between the modes in the physical region I, and region V. It simply expresses the equivalence of two, at first sight different, formulations of the black hole CFT.

We will now investigate the behaviour of the vertex operators around the two interesting geometrical features: the horizon and the singularity.

The horizon

Let us first consider the horizon. Here the U, V modes are the most convenient to work with. As we remarked these modes are analogous to the standard Rindler modes, and their analysis proceeds in a similar fashion, see e.g. [22]. Since they have only been properly defined in region I, we will sometimes indicate this by writing ${}^{I}U$ and ${}^{I}V$. From the asymptotic formulas of section 5.1 we see that these modes behave near the horizon as ${}^{I}U^{\lambda}_{\omega} \sim u^{-2i\omega}$ and ${}^{I}V^{\lambda}_{\omega} \sim (-v)^{2i\omega}$, and consequently have branch cuts at the past and future horizon respectively. We see that ${}^{I}U$ can be naturally extended to region III, whereas ${}^{I}V$ is well-defined in IV.

Similarly, one can consider modes in region II defined as

$$^{II}U(u,v) \equiv {}^{I}U(-u,-v), \qquad\qquad {}^{II}V(u,v) \equiv {}^{I}V(-u,-v). \tag{5.28}$$

These operators can be related to the modes in region I through analytic continuation. For convenience we restrict our attention here to the U modes. In fact, there are *two* possible analytic continuations, corresponding to placing the branch cut in the upper or lower half of the u-plane. Since one is by definition the complex conjugate of the other, we will denote these vertex operators as ^{I}U and $^{I}U^{*}$. Of course, by (5.14) these two are related in region I. For an inertial observer at the horizon the modes ^{I}U and $^{I}U^{*}$ represent the positive and negative energy modes. One easily verifies that now

$$^{I}U^{\lambda}_{\omega} = e^{-2\pi\omega}\; {}^{II}U^{\lambda}_{\omega} \tag{5.29}$$

The appearance of the Boltzmann factor $e^{-2\pi\omega}$ has important physical consequences. In particular, it leads to Hawking radiation [21, 22]. We will return to this in the next section.

The singularity

We now have a good definition of our modes in the regions I–IV. To continue this definition into the regions V and VI we have to face the singularity. A generic mode will have a logarithmic singularity $\log(1-y)$ at the space-time singularity $y=1$. This can, for instance, be seen directly from the explicit expressions we have given for the U and V modes as hypergeometric functions in the variable y. In general hypergeometric functions are analytic in the complex y-plane, with the exception with possible branch points or poles at $y = 0, 1, \infty$. Thus, the vertex operators can be defined globally by going 'around' the singularity in a complexified target space. For our particular choice of parameters we have branch points at $y=0$ and logarithmic singularities at $y=1$. However, there is one linear combination, for fixed ω and λ, that is regular at $y=1$. The existence of this mode can be understood as follows.

In region V we can define a natural mode

$$W^{\lambda}_{\omega}(y,t) = {}_{+}\langle l,\omega|g(y,t)|l,-\omega\rangle_{+} \tag{5.30}$$

which can be seen to be the analytic continuation of the Euclidean winding modes. These modes were regular at $y=1$ in the Euclidean model, being matrix elements of discrete eigenstates, and will remain regular after the continuation. W^{λ}_{ω} is again simply expressed in a hypergeometric function

$$W^{\lambda}_{\omega}(y,t) = e^{-2i\omega t} y^{-i\omega} F(\nu_{+}, \overline{\nu}_{-}; 1; 1-y) \tag{5.31}$$

Fig. 9: The mode W_ω^λ describes the scattering of a tachyon off the naked singularity in region V of fig. 8.

However, we know that there should also be singular solutions. This is related that W_ω^λ is actually real, and that the second solution to its defining hypergeometric differential equation should be defined with a limiting procedure which gives rise to logarithmic singularities. This is also clear from the definition of vertex operators as matrix elements

$$Z_\omega^\lambda(y,t) = {}_+\langle l,\omega|g(y,t)|l,-\omega\rangle_- \tag{5.32}$$

We can see from (5.27) that at $y=1$ this mode develops a logarithmic singularity, since $g(1)|l,-\omega\rangle_- = |l,\omega\rangle_+$. The modes W_ω^λ and Z_ω^λ are a natural basis in region V. The W mode is not bothered by the singularity and can be trivially continued to region III, where it will be some linear combination of U and V modes. This can be worked out in more detail, with the result

$$W_\omega^\lambda = \alpha U_\omega^\lambda + \overline{\alpha} V_\omega^\lambda, \qquad \alpha = \frac{B(\nu_+,\overline{\nu}_-)}{B(\overline{\nu}_+,\nu_-)}. \tag{5.33}$$

Conversely, the right-hand side defines a combination of vertex operators that is regular at the singularity.

Similarly to our discussion in section 5.2 one can consider scattering of tachyons off the naked singularity in region V. Hereto we have to impose boundary conditions at $y=1$. A natural choice is suggested by the analytic continuation of the corresponding Euclidean problem, which describes the scattering of the winding modes on the geometry (4.20). That is, the natural mode to consider is W_ω^λ which is regular at the singularity, and has both left-moving and right-moving contributions at spatial infinity, as illustrated in figure 9. The resulting on-shell reflection coefficient reads

$$S(\lambda) = 2^{-8i\lambda} \frac{\Gamma(1+4i\lambda)\Gamma(\frac{1}{2}-4i\lambda)}{\Gamma(1-4i\lambda)\Gamma(\frac{1}{2}+4i\lambda)}. \tag{5.34}$$

Note that here we have unitary scattering: $|S| = 1$. This shows that the naked singularity, due to the regularity condition, acts as a perfect reflector.

6. The String Propagator and Hawking Radiation

The calculation of string amplitudes in the Minkowskian black hole geometry is obviously a rather delicate matter. In particular we have to deal with the fact that the world-sheet conformal field theory is described by an indefinite action, and therefore does not automatically have an unambiguous definition. For instance, if we want to use the functional integral to define the model, we will have to prescribe an integration contour which renders the integral well-defined. It is at this point that we should expect to encounter exactly the same subtleties found in studies of ordinary field theory in a black hole background, which led to the discovery of Hawking radiation [21, 22]. In the latter context it turns out that there are in fact several physical boundary conditions one can impose, each of which are associated to a different choice of the physical vacuum state of the second quantized field theory. It seems reasonable that the same freedom arises in defining the Minkowskian string theory, in particular since in string theory in 1+1 dimensions shares many features with ordinary QFT. Nonetheless, it may be instructive to address some of these issues in this new setting.

To illustrate some of these points, we will consider in this section the 'string propagator' in the black hole geometry. Quite generally, in string theory we may define the propagator via the Polyakov path-integral on the cylinder with two open boundaries C_1 and C_2 and consider this quantity as a functional of the boundary values of the string. Unfortunately, the actual construction of this propagator is an involved problem, since one needs to consider all states in the string spectrum [23]. In the 1+1 dimensional situation, however, it seems appropriate to simplify things by restricting ones attention to the center of mass of the string, since there are no transverse oscillations. So let us consider the propagator defined by imposing as boundary condition that the string coordinate takes a constant value x_i on each of the two boundaries C_i of the cylinder

$$\mathcal{G}(x_1, x_2) = \Big\langle\, \delta(x_{|C_1} - x_1)\, \delta(x_{|C_2} - x_2) \Big\rangle \tag{6.1}$$

Here the expectation value is taken in the world sheet CFT, and integrated over the complex modulus q of the cylinder. Although in string theory the central role of this propagator is perhaps less evident than in the point particle case, it does serve as a good model for illustrating the physical consequences (Hawking radiation) associated with the choice of boundary conditions near the black hole.

If we are interested in constructing the amplitudes in the Minkowski black hole theory, we are forced to specify an appropriate $i\epsilon$ prescription. The most practical approach seems to make use of the string analogue of the Euclidean postulate and define the Minkowski amplitudes via analytic continuation of the Euclidean correlation functions. This has the advantage that all the CFT technology remains at our disposal, since a large portion of it makes essential use of the fact that we are working on ordinary Riemann surfaces.*

So let us first consider the expectation value (6.1) in the Euclidean theory. After some rather standard manipulations, which we omit here, it can be recast in the more familiar form of a Green function

$$\begin{aligned} \mathcal{G}_{\scriptscriptstyle E}(x_1;x_2) &= \int_{|q|>1} d^2q|q|^{-2}\langle x_1|q^{L_0}\overline{q}^{L_0}|x_2\rangle \\ &= \langle x_1|\frac{1}{L_0+\overline{L}_0-2}|x_2\rangle \end{aligned} \tag{6.2}$$

where $|x_i\rangle$ denotes the position eigenstate of the string. Hence $\mathcal{G}_{\scriptscriptstyle E}(x_1,x_2)$ coincides with the standard Euclidean propagator of the tachyon field described by the target-space action (4.9), with G and Φ as in (4.12).

We can expand the states $|x_i\rangle$ in terms of the primary states $|T^\lambda_m\rangle$ by applying the Plancherel formula

$$|x\rangle = \sum_{m\in\frac{1}{2}Z}\int_0^\infty d\lambda\,\rho(\lambda,m)\,T^\lambda_m(x)^*\,|T^\lambda_m\rangle, \tag{6.3}$$

where $\rho(\lambda,m)$ denotes the Plancherel measure

$$\rho(\lambda,m) = \pi\lambda Re\ \tanh\pi(\lambda+im), \tag{6.4}$$

and rewrite the Green function (6.2) as

$$\mathcal{G}_{\scriptscriptstyle E}(x_1,x_2) = \sum_{m\in\frac{1}{2}Z}\int_0^\infty d\lambda\,\frac{\rho(\lambda,m)}{\lambda^2+m^2/9}\,T^\lambda_m(x_1)T^\lambda_m(x_2)^* \tag{6.5}$$

The discrete sum over the Euclidean energies m reflects the fact that the θ coordinate is periodic, with period 2π.

*Dealing directly with the Minkowski theory requires that the world-sheet metric is also Lorentzian.

To obtain the Minkowski propagator, we will now follow the prescription of Hartle and Hawking, and define it to be obtained from the above Euclidean Green function by analytically continuing the Euclidean time θ to the Lorentzian time coordinate $t = -i\theta$. A small subtlety one has to deal with is that, in continuing the expressions for the modes $T^\lambda_m(r,\theta)$, the cases $m > 0$ and $m < 0$ must be treated separately. For each case one obtains a different analytic continuation, that one can denote as ${}^+T^\lambda_{i\omega}(r,it)$ and ${}^-T^\lambda_{i\omega}(r,it)$, respectively. The relation of these two analytic continuations with the Minkowskian modes U^λ_ω and V^λ_ω introduced in the previous section is the following

$$U^\lambda_\omega = \frac{\pi}{\cosh\pi(\lambda-\omega)}\,{}^+T^\lambda_{i\omega}, \qquad V^\lambda_\omega = \frac{\pi}{\cosh\pi(\lambda+\omega)}\,{}^-T^\lambda_{i\omega}. \tag{6.6}$$

The rest of the procedure is standard. One replaces the energy sum over m by an appropriate contour integral, and deforms the contour, to show that in general

$$\sum_{m\in\frac{1}{2}\mathbf{Z}} \frac{1}{\lambda^2+m^2/9} f(m) = \int_0^\infty d\omega \left[\frac{1}{\lambda^2-\omega^2/9+i\epsilon} + \frac{\delta(\lambda^2-\omega^2/9)}{e^{4\pi\omega}-1}\right](f(\omega)+f(-\omega)) \tag{6.7}$$

Combining all ingredients and performing the integral over ω one finds the following result for the Minkowski propagator (for $t_1 > t_2$)

$$\mathcal{G}_H(x_1,x_2) = \int d\lambda\rho(\lambda)\Big[(1+N_\omega)U_\lambda(x_1)U^*_\lambda(x_2) + N_\omega U^*_\lambda(x_1)U_\lambda(x_2)\Big], \tag{6.8}$$

where

$$\begin{aligned} U_\lambda(x) &= U^\lambda_\omega(x), & \omega = -3\lambda > 0, \\ U_\lambda(x) &= V^\lambda_\omega(x), & \omega = 3\lambda > 0, \end{aligned} \tag{6.9}$$

and where N_ω is the Boltzmann distribution function

$$N_\omega = \frac{1}{e^{4\pi\omega}-1}. \tag{6.10}$$

The measure $\rho(\lambda)$ given by

$$\rho(\lambda) = \frac{\pi}{6}|\tanh 4\pi\lambda|. \tag{6.11}$$

The result (6.8) has the form of a thermal propagator. It describes the physical situation in which the string moves in a thermal bath at the Hawking temperature,

$$\beta_H = 6\pi\sqrt{\alpha'} \tag{6.12}$$

in thermal equilibrium with the black hole. Notice that this situation is static[†]: although the black hole emits thermal radiation, it does not loose any mass, since it absorbs an equal amount of matter from the heat bath outside the horizon.

What led to the conclusion that the black hole has a temperature and that there is a heat bath outside the horizon? The main assumption made was that the black hole event horizon describes a non-singular part of space-time. In other words, an inertial observer falling into the hole should notice nothing special at the moment he or she crosses the horizon. Indeed, the Hartle-Hawking propagator (6.8) satisfies the boundary condition that, in a local inertial frame near the event horizon, it becomes of pure positive frequency at the future horizon and of pure negative frequency near the past horizon, *i.e.* in this inertial frame there are no particles going in or out. However, a (non-inertial) observer at constant r *will* see a constant flux of thermal particles.

These implications of this regularity requirement at the horizon are perhaps more evident via its manifestation in the Euclidean theory. Namely, the condition that the point $r = 0$ is a regular point of the Euclidean black hole geometry clearly forces the coordinate θ to be periodic, and this indeed directly corresponds to having a finite temperature in the Minkowski theory. In this correspondence each interaction of a particle with the heat bath is represented by a winding of the world line of the particle around the compact Euclidean time direction.

A peculiarity of string theory is that duality tells us that this periodicity also produces new 'winding states'. The interpretation of these states in the Minkowski theory is that they are produced via the interaction of the string with the heat bath, although their precise meaning is still somewhat mysterious. One physical consequence of the winding modes is that they could lead to a Kosterlitz-Thouless type instability when the temperature exceeds a certain critical value [24, 2]. Comparison with compactified $c = 1$ string theory shows that the Hawking temperature is in fact above this KT-transition. This raises the question whether the black hole theory is perhaps unstable. We leave this issue for future study.

To end this section, let us note that one can of course impose alternative boundary conditions. In particular one may require that the theory at large distances from the black hole reduces to the conventional flat space theory. The effect of this boundary condition on the propagator is that all thermal factors N_ω disappear, and one is left with

$$\mathcal{G}_S(x_1, x_2) = \int d\lambda \rho(\lambda) U_\lambda(x_1) U^*_\lambda(x_2), \qquad t_1 > t_2. \tag{6.13}$$

This propagator equals two-point function of the tachyon field in the so-called Schwarzschild vacuum $|0_S\rangle$, which is the unique vacuum state that reduces to the usual vacuum at large

[†]This is of course no surprise since we started with a time translation invariant theory.

distances. In this vacuum, an observer falling into the black hole will notice an infinite flux of particles the moment he crosses the horizon, but no flux if he stays at constant r. The Euclidean black hole geometry corresponding to this situation is obtained from the previous one by taking out the origin $r = 0$ of the 'cigar', and going to the universal cover, so that the coordinate θ is no longer periodic. As discussed in section 4.2 the corresponding conformal field theory is related to the $SL(2,R)/U(1)$ coset theory via an appropriate orbifold construction.

Physically, perhaps the most interesting boundary condition is that in which there is no heat bath outside the horizon, while the black hole itself still has a finite temperature. This situation would in fact arise if we imagine the 2-d black hole to have been produced by collapsing matter. In four dimensions the corresponding boundary condition is described by the so-called Unruh vacuum and describes the situation in which inertial observers see no particle flux coming in from infinite r in the past, and no flux going into the future horizon. At large times and distance, however, there is a constant flux of thermal radiation leaving the black hole, which is therefore steadily loosing its mass. These boundary conditions therefore break the time translation invariance. In ordinary field theory the corresponding propagator is a 'heterotic' combination of the two propagators discussed above

$$\mathcal{G}_{_U}(x_1, x_2) = \mathcal{G}_{_S}^{(L)}(x_1, x_2) + \mathcal{G}_{_H}^{(R)}(x_1, x_2), \tag{6.14}$$

where the superscripts (L) and (R) denote the truncation to target-space left- and right-movers respectively. At this point it is not clear to us whether there exists a Euclidean conformal field theory describing this situation of an evaporating black hole. As seen from (6.14), it would have to treat the target-space left and right-movers asymmetrically. This suggests that it perhaps has to be described by a heterotic theory on the world sheet, obtained via some asymmetric orbifold construction of one of the two other theories.

7. $C = 1$ Non-Critical String Theory

It is natural to make a comparison between the black hole theory at $k = 9/4$ and the standard $c = 1$ non-critical string theory [1]-[6], since both models have an interpretation as two-dimensional critical string theories. In the case of the $c = 1$ theory the second dimension is provided by the Liouville field ϕ, which takes over the role of the radial target-space coordinate r. Indeed, by interpreting ϕ as r and the $c = 1$ string coordinate as a time t, the physical picture of scattering in the $c = 1$ string [3, 4] is very analogous to that just found in the black hole model. In this section we will try to make this correspondence even more direct, by showing that some of the recent results on scattering in non-critical string theory can be obtained in a natural way from the formulation of Liouville theory as a constrained $SL(2,R)$ WZW-model [25, 26]. By following the same steps as done before in the $SL(2,R)/U(1)$ coset CFT, we will obtain the exact form of the Liouville vertex operators, which are again given by certain $SL(2,R)$ matrix elements. Moreover, we will derive the differential equation imposing the mass-shell condition $L_0 - 1$, and find that our result coincides exactly with the Wheeler-DeWitt equation obtained recently in [27].

7.1. Liouville theory as a constrained SL(2,R) WZW-model

Motivated by Polyakov's analysis of two-dimensional quantum gravity in the light-cone gauge [25], it was noted in [26], that Liouville theory can be represented as a constrained $SL(2,R)$ WZW-model, where the constraint is given by

$$J^-(z) = \sqrt{\mu}. \tag{7.1}$$

The constant μ will later be identified with the two-dimensional cosmological constant. This constraint amounts to gauging the so-called parabolic subgroup of $SL(2,R)$. This model is most conveniently analyzed by parametrizing the group element g via the Gauss decomposition

$$g = \begin{pmatrix} 1 & 0 \\ \theta_L & 1 \end{pmatrix} \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \begin{pmatrix} 1 & \theta_R \\ 0 & 1 \end{pmatrix} \tag{7.2}$$

In these coordinates the gauged WZW-action takes the following form

$$S = S_{wzw}[g] + \frac{k}{2\pi}\int d^2z \left[\overline{A}(e^r\partial\theta_L - \sqrt{\mu}) + A(e^r\overline{\partial}\theta_R - \sqrt{\mu}) + \overline{A}Ae^r\right] \tag{7.3}$$

$$S_{wzw}[g] = \frac{k}{4\pi}\int d^2z\,(\overline{\partial}r\partial r + e^r\overline{\partial}\theta_L\partial\theta_R) \tag{7.4}$$

This action is invariant under $\theta_{L,R} \to \theta_{L,R} + \alpha$, $A \to A + d\alpha$.

The gauge-invariant content of this model is uncovered by formally integrating out the gauge field A. One finds that the remaining action is independent of both θ_L and θ_R, and has the form of the Liouville action

$$S[r] = \frac{k'}{4\pi}\int d^2z(\overline{\partial}r\partial r + QRr + \mu e^{-r}) \tag{7.5}$$

Here the dilaton term QRr is generated via the determinant factor $\prod_z e^{r(z)}$ coming from performing the gaussian integral over A [19]. The precise values of the renormalized central extension $k' = k - 2$ and the coefficient $Q = (k-1)/(k-2)$ can be determined via the β-function equations, or equivalently à la DDK [28].

We notice an important difference with the $SL(2,R)/U(1)$ CFT namely that the constraint eliminates two degrees of freedom instead of one. The reason for this is that in the present case the constraint (7.1) on the parabolic generator J^- is in fact a first class constraint, whereas the condition $J^3(z) = 0$ is second class due to the central extension.* Correspondingly, in the present case the gauge field A is a non-dynamical Lagrange multiplier. To make this more evident, let us, instead of integrating out A, use the alternative procedure of parametrizing the gauge field as $A = (\partial\phi_{_L}, \overline{\partial}\phi_{_R})$. We then find that upon the redefinition $\theta_{_{L,R}} \to \theta_{_{L,R}} + \phi_{_{L,R}}$, both $\phi_{_L}$ and $\phi_{_R}$ drop out of the action, and we are left with

$$S = S_{wzw}[g] + S_{gh}[b,c], \tag{7.6}$$

where the ghost action is the same as before. The constraints are implemented by restricting the Hilbert space to the physical subspace given by the cohomology classes of the BRST-charge

$$Q = \oint c(J^- - \sqrt{\mu}). \tag{7.7}$$

Finally, we recall that the WZW-theory (7.6) has to be somewhat modified in order to make the constraint (7.1) consistent with conformal invariance. Concretely, the stress-tensor cannot have the conventional Sugawara form, but has an additional 'improvement term'

$$T(z) = \frac{1}{k-2}\eta_{ab}J^aJ^b + \partial J^3 \tag{7.8}$$

ensuring that J^- has conformal dimension zero. The resulting relation between the central charge of the Liouville theory and the $SL(2,R)$ central extension reads [25]

$$c_{liouv} = \frac{3k}{k-2} + 6k - 2. \tag{7.9}$$

The equivalence of this second formulation with the first procedure of directly integrating out A is supported by the exact agreement between the results of [25] and [28] regarding the critical exponents 2-d quantum gravity in the light-cone and conformal gauge. An important advantage, however, of the above reformulation of Liouville theory (7.5) as the constrained $SL(2,R)$ WZW-theory (7.6) is that the underlying $SL(2,R)$ Kac-Moody symmetry gives important restrictions on the possible renormalizations that can occur in the quantum theory.

*We like to thank S.Shatashvili for a discussion on this point.

7.2. Liouville operators and $SL(2,R)$ matrix elements

We now want to use the $SL(2,R)$ formulation to construct the primary vertex operators in Liouville theory. In analogy with the previous analysis of the black hole CFT, we expect them to be related to certain $SL(2,R)$ matrix elements. The natural basis in this case is given by the eigenstates

$$J^+|\nu_{_R}, l\rangle = \nu_{_R}|\nu_{_R}, l\rangle, \qquad\qquad \langle\nu_{_L}, l|J^- = \langle\nu_{_L}, l|\nu_{_L}. \tag{7.10}$$

As shown in the Appendix, the corresponding matrix elements can be expressed by means of the modified Bessel functions

$$\langle\nu_{_L}, l|g(r,\theta_L,\theta_R)|\nu_{_R}, l\rangle = \Big(\frac{\nu_{_R}}{\nu_{_L}}\Big)^{l+\frac{1}{2}} e^{-r/2} K_{2l+1}(2\sqrt{\nu_{_L}\nu_{_R}}e^{-r/2})e^{\nu_L\theta_L+\nu_R\theta_R} \tag{7.11}$$

where $g(r,\theta_{_L},\theta_{_R})$ denotes the parametrization via the gauss decomposition (7.2). The subspace of vertex operators in the coset theory are the ones which commute with the constraints $J^- - \sqrt{\mu} = 0$ and $\overline{J}^+ - \sqrt{\mu} = 0$, and thus correspond to setting

$$\nu_{_L} = \nu_{_R} = \sqrt{\mu}, \tag{7.12}$$

and thus we are lead to consider the matrix elements

$$\langle l, \sqrt{\mu}|g(r)|\sqrt{\mu}, l\rangle = e^{-r/2}K_{2l+1}(2\sqrt{\mu}e^{-r/2}). \tag{7.13}$$

We wish to identify these with the primary operators in Liouville theory. Here, however, we meet a small subtlety. Due to the presence of the improvement term in the stress tensor (7.8), the L_0 component is not proportional to the Casimir operator on $SL(2,R)$. Instead we have

$$L_0 = \frac{\Delta}{k-2} + \frac{\partial}{\partial r} \tag{7.14}$$

where Δ is the Casimir, reduced to the constraint surface $J_- = \sqrt{\mu}$, given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} + \mu e^{-r} \tag{7.15}$$

The matrix elements (7.13) are eigen modes of Δ with eigenvalue $-l(l+1)$. The Liouville primary fields are a simple modification of (7.13)

$$V_l(r) = e^{-\frac{1}{2}(k-2)r}\langle l, \sqrt{\mu}|g(r)|\sqrt{\mu}, l\rangle \tag{7.16}$$

It is readily checked that these operators are eigen modes of (7.14) with conformal dimension

$$h_l = -\frac{l(l+1)}{k-2} + \tfrac{1}{4}k. \tag{7.17}$$

We note that, in principle, we could have expected that the formula (7.16) for the Liouville operators needs to be renormalized, since it involves the (normal ordered) product of two quantum operators. However, strong evidence that this is in fact not necessary is that result (7.16)-(7.13) precisely coincides with the recent result of [27], where $V_l(r)$ is interpreted as the mini-superspace wavefunction of the corresponding operator. Moreover, the authors of [27] show that the same expression is also produced by the matrix model, which provides a beautiful confirmation of the equivalence between the discretized and continuum approach to 2-d quantum gravity.

7.3. Scattering off the $c = 1$ wall

We are now in a position to specialize to the case of the $c = 1$ non-critical string [1]-[6] and make the comparison with the black hole string theory. The central extension in this case is $k = 3$.

We will interpret the $c = 1$ free scalar field as the target-space time coordinate, and denote it by t. The dressed primary fields in this theory are of the form

$$V_p(r,t) = V_p(r)e^{ipt} \tag{7.18}$$

where $V_p(r)$ solves the mass-shell condition

$$\Big(\frac{\partial^2}{\partial r^2} + 2\frac{\partial}{\partial r} + \mu e^{-r} + p^2 - 1\Big)V_p(r) = 0 \tag{7.19}$$

In [27] this equation was derived as the mini-superspace Wheeler-DeWitt equation, satisfied by the two-point function of the operator $V_p(r,t)$ and a macroscopic loop operator $W(\ell)$, creating a boundary of length $\ell = e^{-r/2}$. As just described, the solution can be written in terms of Bessel functions as [27]

$$W_p(r,t) = e^{-r}K_{2ip}(2\sqrt{\mu}e^{-r/2})e^{ipt} \tag{7.20}$$

The vertex operator (7.20) describes the scattering process of a tachyon off the potential μe^{-r}. It consist of both an in-going and an out-going component, with both the same amplitude since everything is being reflected. We have

$$V_p(r,t) \sim e^{-r}(e^{ip(r+t)} + S(p)e^{-ip(r-t)}) \tag{7.21}$$

where the reflection coefficient is given by

$$S(p) = \mu^{2ip}\frac{\Gamma(1+2ip)}{\Gamma(1-2ip)} \tag{7.22}$$

This reflection coefficient is a phase for both signatures of the string time coordinate t.

7.4. Conclusion

In conclusion, by analyzing the free propagation in the black hole theory and ordinary non-critical string theory, we have found that both models share many qualitative features. A direct quantitative comparison of both models, however, reveals that both theories are certainly not equivalent, since in particular the black hole reflection amplitudes (5.22) and (5.34) differ from the expression (7.22) found in the $c = 1$ string. (Here the reader should notice that the correspondence at spatial infinity gives the relation $p = 2\lambda$.) A characteristic feature that the reflection coefficient $S(p)$ does share with the black hole scattering amplitudes $S(\lambda)$ is the occurrence of poles at an infinite equidistant set of special imaginary momenta

$$p_n \equiv 2\lambda_n = \tfrac{i}{2}n, \qquad\qquad n \in \mathbf{Z}_+. \tag{7.23}$$

In the $d = 1$ theory these poles are believed to indicate the appearence of extra states at these special momentum values [3, 5]. As is clear from the discussion in sections 4, a similar interpretation applies in the $SL(2,R)/U(1)$ theory, where the poles can be seen to be related to the existence of (generally virtual) bound states of the string and the black hole.

The main tool we used to determine these scattering amplitudes is the fact that both models can be formulated as a gauged $SL(2,R)$ WZW-model, which made it possible to express the vertex operators in terms of $SL(2,R)$ matrix elements. This fact puts strong restrictions on the possible renormalizations that could in principle occur; they can essentially only affect the overall norm of the wave-function. The reflection coefficients are obviously insensitive to these renormalizations; they can for instance be written as the ratio of two two-point functions, namely one with reflected and one with un-reflected spatial momenta.

This of course open up the interesting issue which features of the non-critical $d = 1$ string amplitudes, as obtained from the matrix model [1]-[6], are universal and carry over to the theory in the black hole background. To address this issue one in particular needs to study the interactions in the black hole model.

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Appendix: $SL(2,R)$ and its matrix elements

In this Appendix we have collected some useful facts about matrix elements of $SL(2,R)$ and their relation with special functions, in particular Jacobi and Bessel functions. More details can be found *e.g.* in [15].

A.1. Euler angles

We begin with some elementary facts about the group $SL(2,R)$, consisting of matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{A.1}$$

These matrices can be parametrized by coordinates analogous to the familiar Euler angles of $SO(3)$. (Euler angles can in fact be used for $SL(2,C)$ and all its real realizations.) We introduce real coordinates r, $\theta_{_L}$, $\theta_{_R}$ and write $g \in SL(2,R)$ as

$$g = e^{\frac{i}{2}\theta_L\sigma_2} e^{\frac{1}{2}r\,\sigma_1} e^{\frac{i}{2}\theta_R\sigma_2} \tag{A.2}$$

with σ_i the Pauli matrices. All elements of $SL(2,R)$ are covered in this way, if we let the coordinates range over $0 \le r < \infty$, $0 \le \theta_{_L} < 2\pi$, $-2\pi \le \theta_{_R} < 2\pi$.

This parametrization is best understood if we use the isomorphism between $SL(2,R)$ and the group $SU(1,1)$ that consists of matrices of the form

$$g' = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1. \tag{A.3}$$

The isomorphism is

$$g' = tgt^{-1}, \qquad t = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{A.4}$$

In $SU(1,1)$ notation our parametrization corresponds to

$$\alpha = \cosh\frac{r}{2}e^{i\theta_+}, \quad \beta = \sinh\frac{r}{2}e^{i\theta_-}, \tag{A.5}$$

with

$$\theta_\pm = \tfrac{1}{2}(\theta_L \pm \theta_R). \tag{A.6}$$

For further reference we note there, that $\widetilde{SL}(2,R)$, the universal covering group of $SL(2,R)$ is parametrized by $-\infty < \theta_+ < \infty$ and $0 \leq \theta_- < 2\pi$.

A.2. Representations and matrix elements

Of course, the group $SL(2,R)$ acts naturally on functions $f(x,y)$ on the plane by

$$g: \ f(x,y) \to f(ax+by, cx+dy). \tag{A.7}$$

The so-called principal continuous series of representations is obtained if we demand that f scales as

$$f(ax,ay) = |a|^{2l}(\operatorname{sgn} a)^{2\epsilon}f(x,y). \tag{A.8}$$

So, a particular representation is labeled by l, a complex number that represents the $SL(2,R)$ spin, and $\epsilon = 0, \frac{1}{2}$ indicating whether f has even or odd parity:

$$f(-x,-y) = (-1)^{2\epsilon}f(x,y). \tag{A.9}$$

Through this scaling f effectively depends only on one real variable, and we can write *e.g.* $f(x) = f(x,1)$. The group $SU(1,1)$ is more natural if we use complex coordinates. If we write f as $f(z,\overline{z})$, we have

$$g': \ f(z,\overline{z}) \to f(\alpha z + \beta\overline{z}, \overline{\beta}z + \overline{\alpha}\,\overline{z}). \tag{A.10}$$

We will now construct the matrix elements of these representations. To this end we have to choose an abelian subgroup H to label our states, and here we can make some choices. In all cases we can write the functions in a particular representation as functions on one or more orbits of H. These functions can then be naturally decomposed into eigenfunctions of H.

Elliptic case: Jacobi Functions

If we choose H to be compact, *i.e.* rotations in the plane, the orbits are circles in the (x, y) plane. Here the $SU(1,1)$ notation is more convenient. H consists of the matrices

$$h' = \begin{pmatrix} e^{\frac{i}{2}\theta} & 0 \\ 0 & e^{-\frac{i}{2}\theta} \end{pmatrix}, \qquad 0 < \theta \leq 4\pi. \tag{A.11}$$

The elements of the representation (l, ϵ) are functions $f(z)$ on the unit circle $z = e^{\frac{i}{2}\theta}$ that transform as

$$g' : \ f(z) \to |\alpha z + \beta \overline{z}|^{2l} f(\frac{\alpha z + \beta \overline{z}}{\overline{\beta} z + \overline{\alpha z}}). \tag{A.12}$$

The basis of eigenfunctions of H is given by

$$f_n(z) = z^{2n}, \tag{A.13}$$

where $n = \epsilon \pmod 1$. The matrix elements are now readily calculated

$$\langle l, m|g'|l, n\rangle = \oint_C \frac{dz}{2\pi i z} f_m(z) \ (g' \cdot f_n(z)) \tag{A.14}$$

where C is the unit circle.

The Jacobi functions $\mathcal{P}^l_{mn}$ are the special matrix elements

$$\mathcal{P}^l_{mn}(\cosh r) = \langle l, m|\, g(r)\, |l, n\rangle \tag{A.15}$$

of the $SU(1,1)$ elements

$$g(r) = \begin{pmatrix} \cosh\frac{r}{2} & \sinh\frac{r}{2} \\ \sinh\frac{r}{2} & \cosh\frac{r}{2} \end{pmatrix} \tag{A.16}$$

If we substitute this in (A.14) and use (A.12), we see that, after a reparametrization $z \to z^{1/2}$, the Jacobi functions have the integral representation

$$\mathcal{P}^l_{mn}(\cosh r) = \oint_C \frac{dz}{2\pi i z} z^{m-l} \left(\cosh\frac{r}{2} + z\sinh\frac{r}{2}\right)^{l+n} \left(z\cosh\frac{r}{2} + \sinh\frac{r}{2}\right)^{l-n} \tag{A.17}$$

The Jacobi functions can be expressed in hypergeometric functions (with $m \leq n$)

$$\mathcal{P}^l_{mn}(\cosh r) = \frac{\Gamma(l+n+1)}{\Gamma(l+m+1)\Gamma(n-m+1)} \left(\sinh\frac{r}{2}\right)^{n-m} \left(\cosh\frac{r}{2}\right)^{n+m} \times F(n+l+1, n-l; n-m+1; -\sinh^2\frac{r}{2}) \tag{A.18}$$

They further satisfy the property $\mathcal{P}^l_{-m,-n} = \mathcal{P}^l_{mn}$. We notice that the Jacobi functions are *not* analytic functions of m and n.

Hyperbolic case: hypergeometric functions

In the case that H is non-compact the orbits will be hyperbolas in the (x, y) plane. If one uses $SL(2, R)$ notation, H can be chosen of the form

$$h = \begin{pmatrix} e^{\frac{1}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{pmatrix}, \qquad -\infty < t < \infty. \tag{A.19}$$

An orbit consists of four disconnected parts. A function $f(x, y)$ in a particular representation is completely determined by its values on the line $y = 1$. So, we can work with the functions $f(x) = f(x, 1)$, that transform as

$$g : f(x) \rightarrow |cx+d|^{2l} \text{sgn}^{\,2\epsilon}(cx+d) f(\frac{ax+b}{cx+d}). \tag{A.20}$$

The eigenstates of H, with eigenvalue ω are given by

$$f^{\pm}_{\omega}(x) = x^{i\omega+l}\theta(\pm x). \tag{A.21}$$

Similarly as in the compact case, matrix elements can be expressed as integrals over the real line

$$_{\pm}\langle l, \omega_L | g | l, \omega_R \rangle_{\pm} = \int \frac{dx}{2\pi i x} f^{\pm}_{\omega_L}(x) \left(g \cdot f^{\pm}_{\omega_R}(x)\right) \tag{A.22}$$

Depending on the group element g and the $\pm$ signs the integral ranges over some specific interval. The analogues of the Jacobi functions are the specific matrix elements of (A.16) now considered as group element of $SL(2, R)$. The integral expressions for these matrix elements can all be expressed by means of hypergeometric functions. For example, we have

$$_{+}\langle l, \omega_L | g(r) | l, \omega_R \rangle_{+} = \frac{\Gamma(i\omega_L - l)\Gamma(-i\omega_L - l)}{\Gamma(-2l)} \left(\sinh\frac{r}{2}\right)^{2l - i\omega_L - i\omega_R} \left(\cosh\frac{r}{2}\right)^{i\omega_L + i\omega_R} \times F(i\omega_L - l, i\omega_R - l; -2l; -1/\sinh^2\frac{r}{2}) \tag{A.23}$$

Parabolic case: Bessel functions

As our final case, we consider the action of the parabolic or triangular group elements

$$h_+ = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad -\infty < t < \infty. \tag{A.24}$$

The orbit of this subgroup consists of straight lines parallel to the x axis. We can again work with functions $f(x) = f(x, 1)$, and the eigenfunctions are given by

$$f_\mu(x) = e^{-\mu x} \tag{A.25}$$

One can similarly consider the eigenfunctions of the group elements

$$h_- = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \tag{A.26}$$

which are given by

$$f_\nu(x) = |x|^{2l} (\mathrm{sgn}\, x)^{2\epsilon} e^{-\nu/x} \tag{A.27}$$

The matrix elements between eigenstates of h_+ and h_- can be expressed in terms of Bessel functions. In particular, for the group element

$$g(r) = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \tag{A.28}$$

we find

$$\langle \nu_L, l | g(r) | \nu_R, l \rangle = \int_0^\infty dx\, x^{2l} e^{lr} \exp\Big[-\nu_L x - \nu_R \frac{e^{-r}}{x} \Big] \tag{A.29}$$

Here we recognize the modified Bessel function

$$\langle \nu_L, l | g(r) | \nu_R, l \rangle = \Big(\frac{\nu_R}{\nu_L} \Big)^{l+\frac{1}{2}} e^{-r/2} K_{2l+1}(2\sqrt{\nu_L \nu_R} e^{-r/2}) \tag{A.30}$$

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