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Convexity and the Well-formedness of Musical Objects

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Abstract

It is well known that subsets of the two-dimensional space $\mathbb{Z}^2$ can represent prominent musical and music-theoretical objects such as scales, chords and chord vocabularies. It has been noted that the major and minor diatonic scale form convex subsets in this space. This triggers the question whether *convexity* is a more widespread concept in music. This article systematically investigates the convexity for a number of musical phenomena including scales, chords and (harmonic) reduction. It is hypothesised that the notion of convexity may be a covering concept of musical phenomena and possibly reflects other mathematical properties of these musical structures. Furthermore, convexity can be used in a pitch-spelling model.

1. Introduction

Are there general principles that govern the "well-formedness" of musical items? For example, when is a sequence of notes a well-formed musical scale, chord or melody? General perceptual principles for musical structuring have been proposed, ranging from the Gestalt laws (Wertheimer, 1923) and preference rules (Lerdahl & Jackendoff, 1983) to stochastic principles (Bod, 2002). More specific research on the goodness or well-formedness of melodies has been carried out by Povel (2002). Furthermore, attempts to simulate music-theoretical prominence in terms of mathematical prominence have been made by Carey and Clampitt (1989), who developed a theory about the well-formedness of $\mathbb{Z}_n$ scales.[1] Mazzola (2002) has discussed the consonance/dissonance dichotomy, and Noll (2001) measured the "goodness" of chords in $\mathbb{Z}_{12}$ in terms of the number of transformation classes. However, the studies applying to scales and chords are limited in the sense that they do not account for 5-limit just intonation.

This article intends to make a start in finding general empirical principles for the "well-formedness" of a large number of musical items including both 3-limit and 5-limit just intonation, as well as equal temperament. The items discussed range from ancient Greek scales to Chinese Zhou scales, and from the major triad to the eleventh and thirteenth chords. Together with other music-theoretical objects such as harmonic reductions, we find that there is a highly persistent principle holding for all well-formed musical objects: if represented in a tone space, scales, chords and harmonic reductions are virtually always convex or star-convex. The convexity of harmonic reductions is due to the convexity of triads. An application of the convexity of scales can be found in a pitch-spelling model.

Although the convexity of the major and minor diatonic scale was noted earlier by Balzano (1980), Longuet-Higgins and Steedman (1987) and Chew (2000), and convexity of triads was noted by Chew (2000, 2003),[2] we are not aware of any studies that investigate the convexity of other scales, the convexity of chords other than triads, and of harmonic reductions.

In the next section, we will first explain the two topological notions of convexity and star-convexity and show how they apply to music. In Section 3, we discuss their application to 53 scales (in 5-limit just intonation). In Section 4, the convexity of 15 harmonic chords and 12 altered chords is investigated. Section 5 deals with convexity in harmonic reduction, and in Section 6, we

[1]The symbol $\mathbb{Z}$ is here used in its standard mathematical meaning of the set of integers. $\mathbb{Z}^2$ is therefore a two-dimensional lattice of points aligning with integers on each axis. $\mathbb{Z}_n$ is the set of integers modulo *n,* so $\mathbb{Z}_{12}$ is the set {0, 1, ... 11}, like the hours of the clock. $\mathbb{R}$ is the set of real numbers.

[2]Convexity has also been observed in rhythm space (Desain & Honing, 2003).

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argue how pitch-spelling can make use of convexity. In Section 7, we critically discuss our results and show that it is far from trivial for a musical item to be convex. We finish with a conclusion in Section 8.

2. Convexity in tone lattices

"Convexity", as we use the term, is a notion from mathematical topology. A set in the Euclidean space $\mathbb{R}^n$ is convex if it contains all the line segments connecting any pair of its points (see Figure 1). Or formally: a subset Y of $\mathbb{R}^n$ is said to be convex if $\alpha x + (1-\alpha)y$ is in Y whenever x and y are in Y and $0 \le \alpha \le 1$.

Star-convexity is related to convexity. A subset X of $\mathbb{R}^n$ is star convex if there exists an $x_0 \in X$ such that the line segment from x_0 to any point in X is contained in X (see Figure 2). A convex set is always star-convex, but a star-convex object is not always convex.

We will define a discrete convex set analogous to a convex set in continuous space: A discrete set is convex if, drawing lines between all points in the set, all elements which lie within the spanned area are elements of the set. Similarly, a discrete set is star-convex if there exists a point x_0 in the set such that all points lying on the line segment from x_0 to any point in the set are contained in the set (Figure 3).

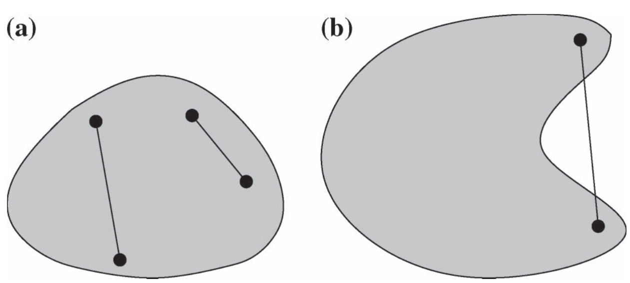

Fig. 1. (a) Convex and (b) concave set in two-dimensional space.

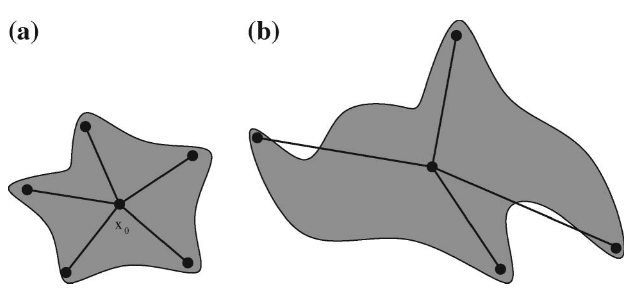

Fig. 2. (a) Star-convex and (b) non-star-convex set in two-dimensional space.

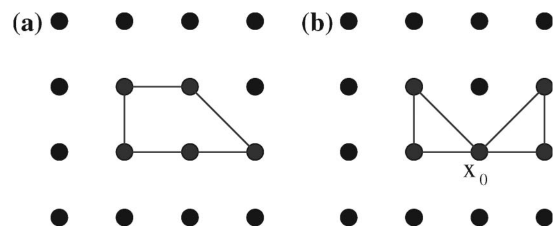

Fig. 3. (a) Convex and (b) star-convex set in discrete two-dimensional space.

These notions of convexity can be applied to music as well. To show this we first need to introduce the concept of "tone space".

Any musical interval can be expressed in terms of a frequency ratio. Tuning according to whole number ratios is referred to as just intonation (Lindley, 2005). Since any positive integer a can be written as a unique product $a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_n^{e_n}$ of positive integer powers e_i of primes $p_1 < p_2 < \cdots < p_n$, any whole number frequency ratio a/b can be expressed as

$$2^p 3^q 5^r \ldots \tag{1}$$

with $p,\ q,\ r \in \mathbb{Z}$. For example $2^{-1}3^1 (= \frac{3}{2})$ represents a perfect fifth and $2^{-2}5^1 (= \frac{5}{4})$ a major third. If the highest prime that is taken into account in describing a set of intervals is n, we refer to it as "n-limit just intonation." Focusing on 5-limit just intonation, all intervals can be described by $\{2^p 3^q 5^r |\ p,\ q,\ r \in \mathbb{Z}\}$ or, equivalently by[3]

$$\left\{ 2^p \left(\frac{5}{4}\right)^q \left(\frac{6}{5}\right)^r | p, q, r, \in \mathbb{Z} \right\}, \tag{2}$$

meaning that every interval can be built from a number of major thirds (5/4), minor thirds (6/5) and octaves (2/1) (Honingh, 2003). The intervals from 5-limit just intonation within one octave can be represented in the two-dimensional lattice $\mathbb{Z}^2$, by making a projection

$$\varphi : 2^p \left(\frac{5}{4}\right)^q \left(\frac{6}{5}\right)^r \to (q, r), \tag{3}$$

whereby the frequency ratios are chosen to lie within the span of one octave – that is for the ratio a/b: $1 < a/b < 2$. Thus this is a projection from three dimensions to two dimensions by dividing out all multiples of 2.

[3]We can write: $2^u 3^v 5^w = 2^k \left(\frac{5}{4}\right)^l \left(\frac{6}{5}\right)^m,\ u, v, w, k, l, m \in \mathbb{Z}$,with $k = u + v + 2w,\ l = v + w,\ m = v$. Or, in vector notation:

$$T \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}, \quad \text{with } T = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This map is invertible if T is invertible; T is invertible if Det(T) $\neq 0$. It is easy to calculate that Det(T) $= 1$ so T is invertible.

In Figure 4 the space constructed from major and minor thirds is shown.

This lattice representation and minor variants of it appear in numerous, discussions on tuning systems (e.g., Helmholtz, 1954 [1863]; Riemann, 1914; Fokker, 1949; Longuet-Higgins, 1962a, 1962b; for an overview on geometric representations of musical pitch, see Krumhansl, 1990). Fokker (1949) attributes this lattice representation originally to Leonhard Euler; therefore it is often called the "Euler-lattice".

We will see that the particular basis does not matter for our purposes (equation 4). In our choice to build the tone space (Figure 4), from major thirds and minor thirds, we followed Balzano (1980).

In this interval space, musical items such as scales and chords can be identified. It turns out that virtually all these items form convex sets in this space.

When we choose basis-vectors for the tone space that are different from the major and minor third, a (star-)convex set will still remain a (star-)convex set. This can be proved as follows. Consider a linear basis-transformation T. A line between two points x and y is given by: $\alpha x + (1-\alpha)y$, with $\alpha \in [0,1]$ for a continuous space, and $\alpha \in \{0, f_1, f_2, \ldots, 1\}$ for a discrete space, with f_i representing the fractions of the line between x and y where other lattice points are situated. Under a transformation T, it transforms into a line again:

$$T(\alpha x + (1-\alpha)y) = \alpha T(x) + (1-\alpha)T(y) \qquad (4)$$

r ↑

			216/125	27/25	27/20	27/16
		144/125	36/25	9/5	9/8	45/32
	192/125	48/25	6/5	3/2	15/8	75/64
128/125	32/25	8/5	**1**	5/4	25/16	125/64
128/75	16/15	4/3	5/3	25/24	125/96	
64/45	16/9	10/9	25/18	125/72		
32/27	40/27	50/27	125/108			

→ q

Fig. 4. Visualisation from the mapping of frequency ratios to the lattice $\mathbb{Z}^2$, following the mapping $2^p\left(\frac{5}{4}\right)^q\left(\frac{6}{5}\right)^r \to (q,r)$. At the origin, where $q = r = 0$, we find frequency ratio 1, the prime interval.

Therefore, a convex set transforms under a basis transformation T into a convex set again.[4] This property is important to ensure that convexity is a meaningful property and not just an artifact of the chosen basis.

The frequency ratios in the tone space can be identified with note names if a key is chosen and the prime interval 1 is identified with the tonic of that key. Therefore a projection can be made from the intervals to the note names to obtain the tone space of note names (see Figures 5a and 5b). In turn, all note names can be identified with the keys on a piano, or numbers 0 to 11 representing all 12 equal tempered semitones. A projection from the note names to these numbers can be made to obtain the tone space of $\mathbb{Z}_{12}$ pitch numbers (Figure 5c). (We left out some complex frequency ratios and note names with many accidentals; however, all three spaces are infinite in horizontal and vertical directions.) We already explained that we use octave equivalence in the interval tone space. We do the same in the note name space and in the pitch number space too.

It should be evident that a convex object in the intervals space is also convex in the note name space and the $\mathbb{Z}_{12}$ pitch number space.[5] The other way round, however, mapping the figures from right to left (in Figure 5) is not necessarily true. Mapping Figure 5c onto 5b there is, for example, the problem as to which note name (D, E♭♭, *C*♯♯, …) should be assigned to number "2", and mapping Figure 5b onto 5a, we see that a *D* can refer to ratio 9/8 as well as to 10/9. We will come back to this problem in Section 6, when we treat it in the context of pitch-spelling algorithms.

3. Convexity of scales

The major scale in 5-limit just intonation is defined as the scale in which each of the major triads *I*, *IV* and *V* is taken to have frequency ratios 4 : 5 : 6. From this it can be calculated that the ratios of the scale are given by: 1/1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/8. These are indicated in Figure 6a with solid lines.

In this figure, the (neutral) minor scale is indicated with dotted lines. These scales both turn out to form convex regions. Considering all intervals internal to the major scale (which are the same intervals as the ones internal in the minor scale), and connecting these points

[4] A basis-transformation is by definition a bijective map (one-to-one correspondence). For all possible basis-transformations, the determinant of transformation matrix should equal 1. For the proof, see Honingh (2003).

[5] From here on, when we write convex sex in the note name space or pitch number space, we mean the set that is obtained by projecting the corresponding (convex) set in the interval space to the note name or pitch number space.

(a)

			216/125	27/25	27/20	27/16
		144/125	36/25	9/5	9/8	45/32
	192/125	48/25	6/5	3/2	15/8	75/64
128/125	32/25	8/5	**1**	5/4	25/16	125/64
128/75	16/15	4/3	5/3	25/24	125/96	
64/45	16/9	10/9	25/18	125/72		
32/27	40/27	50/27	125/108			

(b)

			Bbb	Db	F	A
		Ebb	Gb	Bb	D	F#
	Abb	Cb	Eb	G	B	D#
Dbb	Fb	Ab	C	E	G#	B#
Bb	Db	F	A	C#	E#	
Gb	Bb	D	F#	A#		
Eb	G	B	D#			

(c)

1	5	9	1	5	9
10	2	6	10	2	6
7	11	3	7	11	3
4	8	0	4	8	0
1	5	9	1	5	9
10	2	6	10	2	6
7	11	3	7	11	3

Fig. 5. Three representations of tone space: intervals space, note name space and space of pitch numbers. Here, C is chosen to be the key. (a) Frequency ratios; (b) note names; (c) pitch numbers.

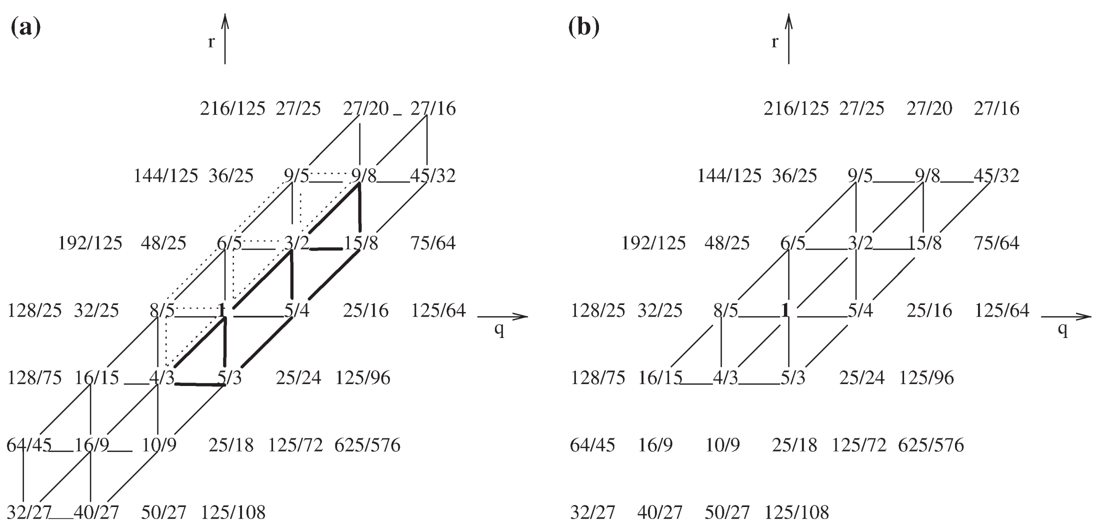

Fig. 6. Tone space of frequency ratios: (a) major scale is connected by thick lines, minor scale indicated by dashed lines, all internal intervals from major or minor scale indicated by normal lines; (b) chromatic scale.

in the tone space, again a convex region is found (see Figure 6a). The chromatic scale as defined by Vogel (1975) can also be found as a convex set in the tone space (Figure 6b).

This result triggered us to investigate more 5-limit just intonation scales and to check whether they form a convex set in the interval tone space as well. From the Scala home page (www.xs4all.nl/hugyensf/scala/), a large collection of over 3,000 scale files is available for download. From this collection, we chose the 5-limit just intonation scales and determined whether they were convex in the tone space by plotting the scales onto the lattice. The scales that are investigated are listed in Table 1, and beside the number of notes, it is indicated whether these scales are convex and star-convex respectively.

Notice that all scales but four are convex. Differentiating between "original" scales like the "Ancient Greek Aeolic" and the "Indian shrutti scale", and "constructed" scales, we observe that all original scales are convex. Moreover, all scales are *star*-convex. The two scales with the highest number of notes (53 and 81) are not convex. Several of the scales from Table 1 are symmetric around the prime interval 1/1 as well, meaning that both an interval and its inverse are present in the scale. Two scales from Table 1 are represented in Figure 7 to give an idea of the typical shape of these scales. The scales that are not convex still have a similar shape – that is, a coherent object shaped around the diagonal (from bottom left to top right) of the lattice. The non-convex scales have only a few intervals not belonging to the scale that make the scale non-convex.

In his book, Barbour (2004 [1951]) gives several examples of 5-limit just intonation 12-note systems. They are listed in Table 2. Among the 26 scales, 23 are convex. Again, all of them are star-convex. The Scala archive together with Barbour's book give a balanced

Table 1. List of 5-limit just intonation scales from Scala archive

Name	Description	Number of notes	Convex	Star-convex
aeolic.scl	Ancient Greek Aeolic	7	yes	yes
chin_5.scl	Chinese pentatonic from Zhou period	5	yes	yes
cifariello.scl	F. Cifariello Ciardi, ICMC 86 Proc. 15-tone 5-limit tuning	15	yes	yes
cluster.scl	13-tone 5-limit Tritriadic Cluster	13	yes	yes
cons_5.scl	Set of consonant 5-limit intervals within the octave	8	yes	yes
coul_13.scl	Symmetrical 13-tone 5-limit just system	13	no	yes
coul_27.scl	Symmetrical 27-tone 5-limit just system	27	yes	yes
danielou5_53.scl	Danilou's Harmonic Division in 5-limit, symmetrised	53	no	yes
darreg.scl	set of 19 ratios in 5-limit JI is for his megalyra family	19	no	yes
fokker-h.scl	Fokker-H 5-limit per.bl. synt.comma small & diesis, KNAW B71, 1968	19	yes	yes
fokker-k.scl	Fokker-K 5-limit per.bl. of 225/224 & 81/80 & 10976/10935, KNAW B71, 1968	19	yes	yes
harrison_5.scl	From Lou Harrison, a pelog-style pentatonic	5	yes	yes
harrison_min.scl	From Lou Harrison, a symmetrical pentatonic with minor thirds	5	yes	yes
hirajoshi2.scl	Japanese pentatonic koto scale	5	yes	yes
indian_12.scl	North Indian Gamut, modern Hindustani gamut out of 22 or more shrutis	12	yes	yes
indian.scl	Indian shruti scale	22	yes	yes
ionic.scl	Ancient greek Ionic	7	yes	yes
ji_13.scl	5-limit 12-tone symmetrical scale with two tritones	13	yes	yes
ji_19.scl	5-limit 19-tone scale	19	yes	yes
ji_22.scl	5-limit 22-tone scale	22	yes	yes
ji_31b.scl	A just 5-limit 31-tone scale	31	yes	yes
johnston_81.scl	Johnston 81-note 5-limit scale of Sonata for Microtonal Piano	81	no	yes
kayolonian.scl	19-tone 5-limit scale of the Kayenian Imperium on Kayolonia (reeks van Sjauriek)	19	yes	yes
kring1.scl	Double-tie circular mirroring of 4 : 5 : 6 and Partch's 5-limit tonality Diamond	7	yes	yes
lumma5.scl	Carl Lumma's 5-limit version of lumma7, also Fokker 12-tone just	12	yes	yes
mandelbaum5.scl	Mandelbaum's 5-limit 19-tone scale	19	yes	yes
monzo-sym-5.scl	Monzo symmetrical system: 5-limit	13	yes	yes
pipedum_15.scl	126/125, 128/125 and 875/864, 5-limit, Paul Erlich, 2001	15	yes	yes
turkish.scl	Turkish, 5-limit from Palmer on a Turkish music record, harmonic minor inverse	7	yes	yes
wilson5.scl	Wilson's 22-tone 5-limit scale	22	yes	yes
wilson_17.scl	Wilson's 17-tone 5-limit scale	17	yes	yes

distribution of both traditionally and "recently" constructed scales and therefore provide a versatile and well-documented test set.

Thus (star)-convexity seems to be a highly persistent property for scales, and we conjecture that it may even serve as a condition for the well-formedness of scales. In Section 7, we will show that it is a non-trivial property for an item to form a convex set and elaborate on the meaning of convexity.

4. Convexity of chords

Now that we have investigated a number of scales in the tone space, we will look at smaller musical items, like chords. In the area of Neo-Riemannian theory, chords in the "Tonnetz" (a space closely related to our tone space) have been studied (see, e.g., Cohn, 1998), although not with respect to the property of convexity.

Considering different kinds of chords, a distinction can be made between chords that are built from harmonic notes, which are notes that are present in the scale of the specific key, and chords that contain non-harmonic notes, the so-called "altered chords". It turns out that all chords built from harmonic notes discussed by Piston and DeVoto (1989) are convex (and therefore also star-convex) in the note-name space. This is a special property of these chords. The chords are listed in Table 3.

Altered chords are difficult to study since it is possible, through the process of chromatic alteration, to create a

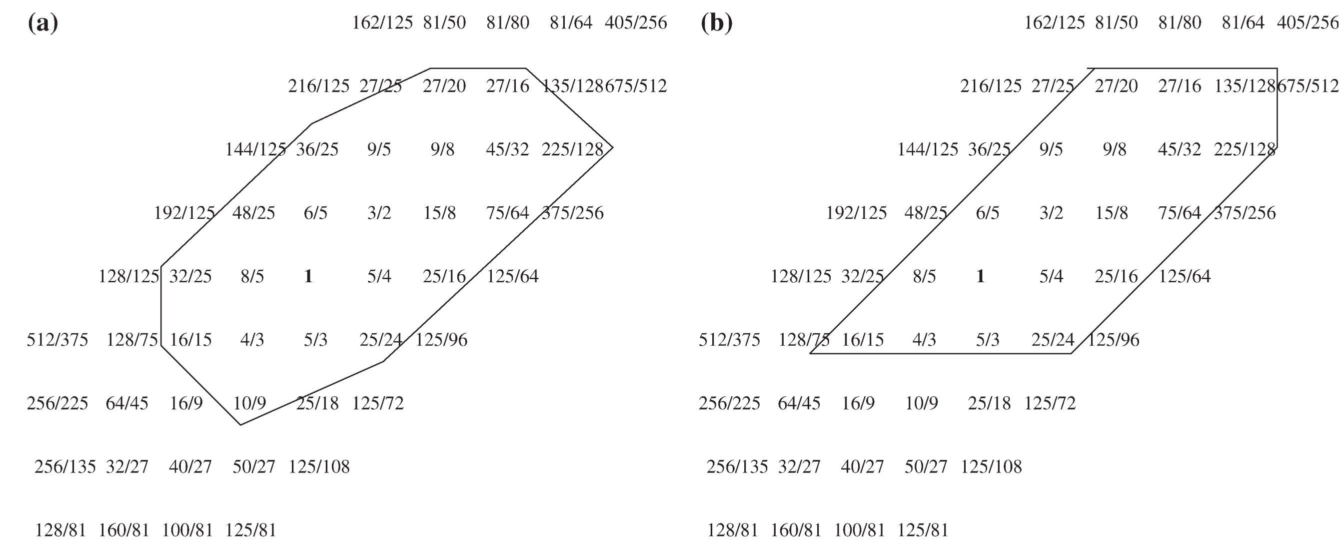

Fig. 7. Two examples of convex scales. (a) Wilson's 22-tone scale; (b) just intonation 19-tone scale.

Table 2. List of 5-limit just intonation scales from Barbour

Scale	Convex	Star-convex
Ramis' Monochord	yes	yes
Erlangen Monochord	yes	yes
Erlangen Monochord revised	yes	yes
Fogliano's Monochord no. 1	yes	yes
Fogliano's Monochord no. 2	yes	yes
Agricola's Monochord	yes	yes
De Caus's Monochord	yes	yes
Kepler's Monochord no. 1	yes	yes
Kepler's Monochord no. 2	yes	yes
Mersenne's Spinet Tuning no. 1	yes	yes
Mersenne's Spinet Tuning no. 2	no	yes
Mersenne's Lute Tuning no. 1	no	yes
Mersenne's Lute Tuning no. 2	yes	yes
Marpurg's Monochord no. 1	yes	yes
Marpurg's Monochord no. 3	no	yes
Marpurg's Monochord no. 4	yes	yes
Malcolm's Monochord	yes	yes
Euler's Monochord	yes	yes
Montvallon's Monochord	yes	yes
Romieu's Monochord	yes	yes
Kinberger I	yes	yes
Rousseau's Monochord	yes	yes

very large number of them. Therefore, we reduce the number of these chords to the ones discussed by Piston and DeVoto (1989). In Table 4, these altered chords are listed and it is indicated whether these chords are (star-) convex.

Remarkably, most altered chords are not convex, and some altered chords are not even star-convex. Precisely two chords are not star-convex: the French augmented sixth chord and the dominant chord with lowered fifth and minor seventh. These chords in fact consist of the same notes (but have a different function in harmony) and therefore describe the same shape in the tone space. Thus convexity roughly distinguishes between harmonic and altered chords, where the harmonic chords are all convex and the altered ones are almost never convex and sometimes not even star-convex.

The reason that we considered the convexity of the chords in the note name space rather than in the frequency ratio space is because it is difficult to say something *a priori* about the intonation of the chords as there is no established theory about the intonation of all chords.[6] Again, it is a non-trivial property for chords to be convex, as we will analyse in the discussion in Section 7.

5. Convexity of harmonic reduction

Harmonic reductions of music are known to be useful for discovering the harmonic structure of a piece allowing for an easier analysis. In this process, a score can be reduced to chords and ultimately to triads (Schenker, 1906; Salzer, 1962). Important theories of chord progressions include *Traité de l'harmonie* by Rameau (1722) and Hugo Riemann's theory of Tone images (*Tonvorstellungen*) (Riemann, 1914). Rameau postulated that harmonic progression is governed by the fifth and thirds connections of roots of triads. Riemann analysed harmonic

[6]E.g., the dominant seventh chord can (in 5-limit JI) be tuned as $36:45:54:64:=4:5:6|27:32$ or as $20:25:30:36=4:5:6|5:6$ (and there are more possibilities), but there is no theory that tells us which tuning is preferred. See, e.g., "The Alternate Tunings Mailing List" for an ongoing discussion about tuning (http://launch.groups.yahoo.com/group/tuning/).

Table 3. Chords built from harmonic notes

Harmonic chords	Number of notes	Convex	Star-convex
major triad	3	yes	yes
minor triad	3	yes	yes
diminished triad	3	yes	yes
augmented triad	3	yes	yes
dominant seventh chord	4	yes	yes
major seventh chord	4	yes	yes
minor seventh chord	4	yes	yes
half-diminished seventh chord	4	yes	yes
major-minor seventh chord	4	yes	yes
augmented seventh chord	4	yes	yes
diminished seventh chord	4	yes	yes
triad with added sixth	4	yes	yes
complete dominant ninth chord	5	yes	yes
tonic/dominant eleventh chord	6	yes	yes
tonic/dominant thirteenth chord	7	yes	yes

Table 4. Chords containing non-harmonic notes

Altered chords	Number of notes	Convex	Star-convex
non-dominant diminished seventh chord	4	yes	yes
Neapolitan sixth	3	yes	yes
augmented sixth (Italian)	3	no	yes
augmented six-five-three (German)	4	no	yes
augmented six-four-three (French)	4	no	no
doubly augmented fourth	4	no	yes
chords with raised fifth			
- major	3	yes	yes
- minor	3	no	yes
- with minor seventh	4	no	yes
dominant chord with lowered fifth	3	no	yes
- with seventh	4	no	no
dominant chord with lowered and raised fifth	4	no	yes

progressions in terms of chains of triads through his tone net (which is equivalent under a basis transformation to our tone space). This illustrates that the reduction of music into triads has a long tradition in analysing music.

We will now investigate the progression of triads in our tone space.[7] We saw in Section 4 that all triads are convex. In the major scale, only major and minor triads and one diminished triad are naturally present. It is easy to see from Figure 8a that every two triads following each other can form a convex set.[8]

Since the major scale contains only seven notes, sequences of three or more triads are necessarily also convex. Thus, whatever segmentation of music in a major key used (segmentation per chord, bar, phrase, etc.), the reduction of the music to triads is always a convex set in the tone space. For music in a minor key, the situation is somewhat more complicated. From Figure 8b, one can see that the neutral minor and the harmonic minor scale form convex regions in the tone space, the harmonic minor scale, the seventh note of the scale, raised by a half tone, such that the *III*, *V* and *VII* triad are changed. In the ascending melodic minor scale, in the sixth and the seventh tone of the scale are raised. Consequently, the triads on *II*, *IV* and *VI* are adjusted as well. Therefore, music in a minor key and the harmonic reduction thereof takes into account many more triads than music in a major key. Considering the harmonic progressions for the minor mode as given by Piston and DeVoto (1989) (see Table 5), all progressions[9] form convex sets except for the progression: *VII*(diminished)-*I*.

At least, this progression is not a convex set (but it does form a star-convex set) if the triads *VII*(dim) and *I* are chosen such that the triads themselves are convex. However, the notes of the triads (in *C* minor: $B-D-F$, $C-E\flat-G$) can also be chosen in the note name space, such that this progression does form a convex set (choosing the *F* as a perfect fifth under the *C*). The way the location of the notes is chosen depends on the intonation of the notes as the mapping in Figure 5 indicates. In just intonation, intervals are tuned to simple number ratios. The fact that intervals can be harmonic as well as melodic can cause some problems. It is not always possible to ensure that two adjacent chords are tuned to lowest number ratios in harmonic as well as melodic

[7]Since we are using octave equivalence, we are not taking into account the different inversions of a chord. Schenker (1906) and Salzer (1962) state that it depends on the bass whether a chord progression is a harmonic progression. Here we treat all triads having an harmonic function.

[8]The dotted lines indicate how the scales proceed to both sides, so that there is more than one possibility of choosing the region (convex or not convex) of two adjacent chords. E.g., for triads *IV* and *V*, it is not directly clear that they can form a convex set, but if *V* is chosen as indicated by the dotted lines, they do. Again, it depends on the intonation of the triad which one is preferred.

[9]Leaving out the progressions involving *III*(harmonic) and *VI*(ascending melodic), since these are rarely used according to Piston and DeVoto (1989).

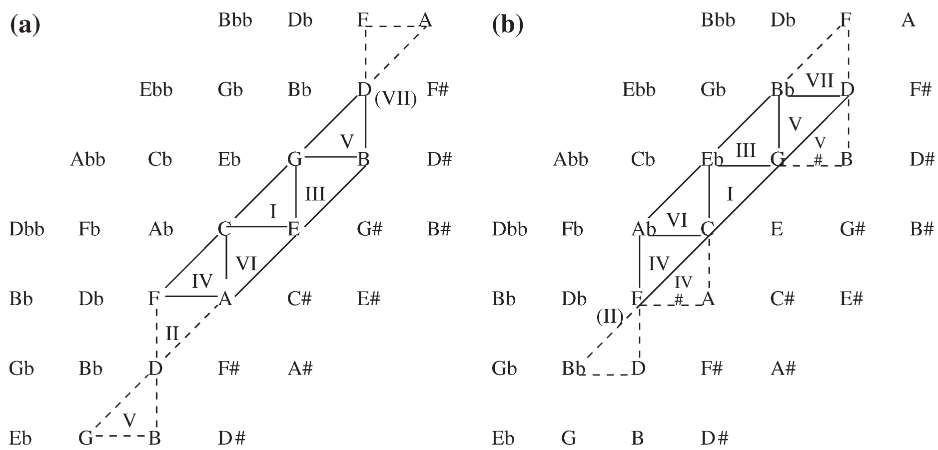

Fig. 8. (a) Triads from the major scale situated in the note-name space, *VII* is the diminished triad $B-D-F$; (b) triads from the minor scale, *II* is the diminished triad $D-F-A\flat$.

Table 5. Chord progression in minor mode, "ma" indicates major chord, "dim" indicates diminished chord

Chord	is followed by	less often by
I	IV	V, VI
II	V	IV, VI
III	VI	IV
III (ma)	VII	
IV	V	I, II
V	I	IV, VI
VI	II, V	III, IV
VII (ma)	III	
VII (dim)	I	

form. In the case of the *VII* (dim)-*I* progression, we could argue that the melodic just intonation overrules the harmonic intonation of the *VII* chord, and that therefore the *F* of the *VII* chord should not be tuned as 27/20, but as 4/3. In that case, the $VII-I$ progression does form a convex set. Since the overall shape of notes present in the (neutral, harmonic and melodic) minor scale does not represent a convex structure unlike the major scale (but does represent a star-convex structure), it cannot be stated that all reductions of music in a minor key are convex. However, above we argued that all important progressions in the minor mode form convex regions, which means that at least a segmentation per one or two harmonic entities forms a convex region.

When music is harmonically analysed, harmonic functions are assigned to groups of notes in accordance with people's perception. The music can be reduced to the triads corresponding to the harmonic functions. Therefore, it is believed that reduction represents the perception of a piece of music. Since we just argued that the reduction of a piece of music constitutes a convex body, we conjecture that this notion of convexity contributes to the perception of harmonic functions.

6. Pitch-spelling using convexity

Reading Figure 5 from *c* to *b* to *a,* the figure represents a mapping from $\mathbb{Z}_{12}$ pitch numbers to the note names (spelling of the pitches), to the preferred tuning, respectively. It it easy to map Figure 5a to 5b to 5c, but the other way round is difficult since information is missing. In studying a MIDI-input, it is not always clear what notes the composer meant to write, and in reading a score, it is up to the musician how to intone the notes. We are challenged to find algorithms following the mapping:

$$\mathbb{Z}_{12} \rightarrow \text{note names} \rightarrow \text{preferred tuning} \qquad (5)$$

Several pitch spelling algorithms have been proposed (see, e.g., Temperley, 2001; Meredith, 2003; Cambouropoulos, 2003; Chew and Chen, 2005), of which the best algorithm performs a correct spelling of the notes at 99.8 per cent. This is already very high and the question arises whether it is worth searching for a better algorithm. If, however, we manage to find a visual mapping in the tone space that maps the $\mathbb{Z}_{12}$ pitch numbers to the correctly spelled pitches, we can directly apply this visual mapping to derive a preferred tuning. Finding the right pitch-spelling is closely related to finding the right key context. As shown in Section 3, the major and minor diatonic scales are found in a specific region that can indicate the key context. If we have as input the $\mathbb{Z}_{12}$ pitch numbers from one bar of a piece of music and mark these numbers in Figure 5c, we can search for the right shape of a major or minor scale such that most of the notes of the scale are filled. For example, the first bar of Beethoven's piano Sonata Op. 109 is encoded as indicated in Figure 9.

Matching the typical shaped region of the major and minor scale and looking for a fit, the notes in the note name space in Figure 9 are the result. As equally good matches, the key with the smallest number of sharps and flats is chosen. In this case "4" major is translated into *E* major (as opposed to *F♭* major), which represents the correct spelling of the notes. This visual mapping procedure gives insight into the pitch-spelling problem and shows that the convex shapes of the major and minor scale determine the note names of the harmonic notes.

7. Discussion

The first question that may arise is: how special is it for a set to be convex? One could think that the chance to obtain a convex set from randomly chosen points in our lattice is higher than average, and that convexity is therefore an artifact of the space we use. To check this, we wrote a program in Matlab that calculates the chance that a randomly chosen set of points is convex. We started with a 5×5 lattice from which to choose our sets.

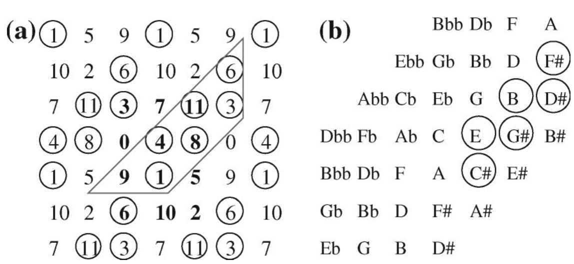

Fig. 9. Pitch-spelling method: encoding of first bar of Beethoven's piano Sonata Op. 109.

For each number of elements *n*, a large number of randomly selected sets were chosen, and for each set it was calculated whether it was convex or not. One point is always chosen in the center of the lattice, since that is our reference point (1 in the ratio-space, 0 in the $\mathbb{Z}_{12}$ pitch number space). Figure 10a shows that that the probability of convex sets is a monotonously decreasing function.

To simulate a more realistic situation, we created another figure, but now choosing sets from a 15×15 lattice. This seemed big enough (the real "tone space" lattice is infinitely big) to cover all scales. From Figure 10b, we see that the percentage of convex 2-note sets is high (65 per cent). This means that if one randomly chooses one note (the other is fixed in the center) on the lattice, the chance to obtain a convex set is 65 per cent. For 3-note sets, this percentage is around 5 per cent, and for more note sets, the chance of choosing randomly a convex set is negligible. All scales that we considered had five or more notes, and therefore it is a special property to be convex. For the chords that had three notes, there is only a very small chance that they would turn out to be convex by chance, and for the chords consisting of more than three notes, the chance on a convex randomly chosen set is approximately zero.

We also wrote a Matlab program to calculate the chance that a randomly chosen set is star-convex. The result is given in Figure 11.

Again, this is a monotonously decreasing function. The chances that randomly chosen sets appear to be star-convex are somewhat higher than in the case of convexity (which is what we expected), but still the probability of obtaining a star-convex set consisting of seven notes or more is less than 20 per cent, and for twelve notes or more, the chances to get a star-convex set are negligible.

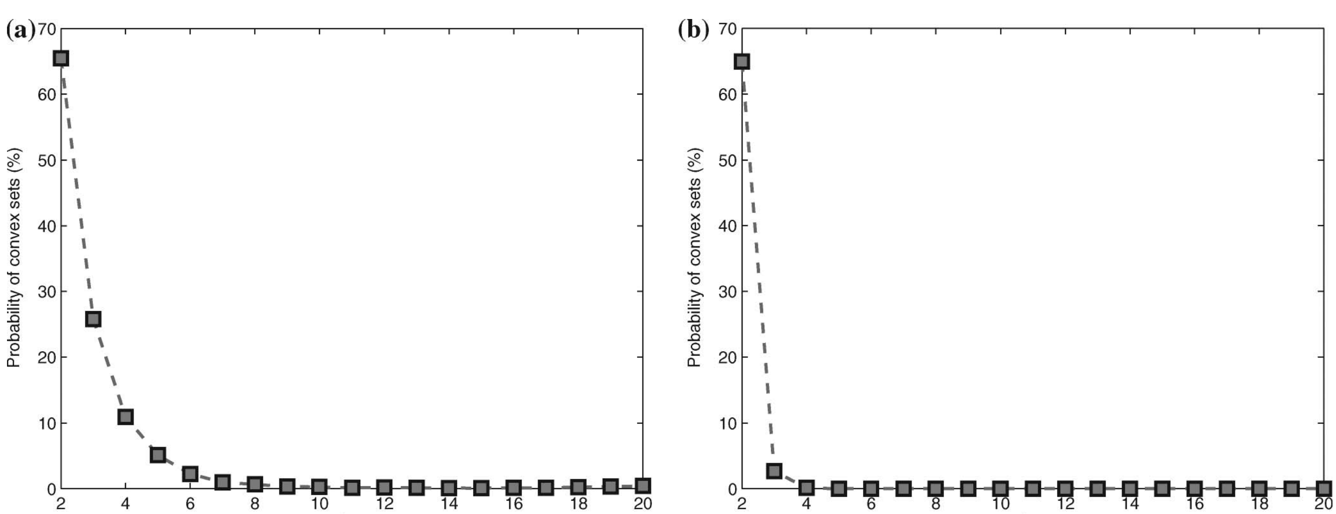

Fig. 10. Probability of convexity for n element sets. (a) 5×5 lattice; (b) 15×15 lattice.

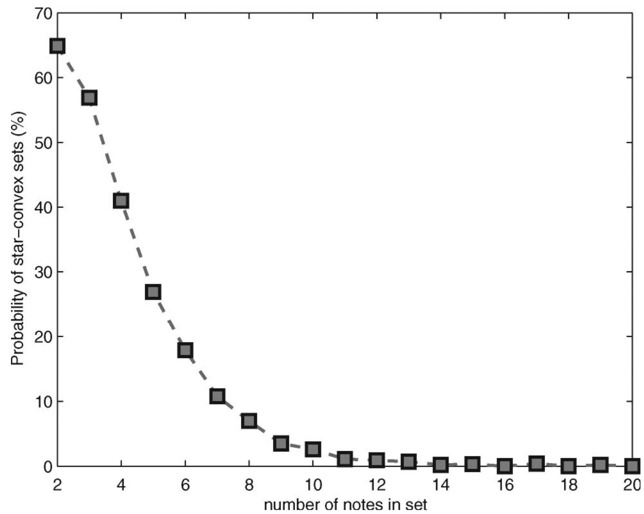

Fig. 11. Probability of star-convexity for n element sets in a 15 × 15 lattice.

A second discussion point is the discrete lattice we used. One may wonder whether it is more convincing to study convexity in a continuous space instead of in a discrete space. However, such a study is not possible. The tone space is built from points described by $\left\{2^p\left(\frac{5}{4}\right)^q\left(\frac{6}{5}\right)^r \mid p, q, r \in \mathbb{Z}\right\}$. In terms of coordinates, (0, 0) indicates the frequency ratio 1, and (1, 0) represents the frequency ratio 5/4. Between these two there is, among other things, the ratio of 6/5 ($1 < 6/5 < 5/4$), but this ratio can be found at the point with coordinates (l, 0) and thus not between (0, 0) and (0, 1). Therefore, this lattice, although it is infinite in both directions, cannot be made into a continuous space.

Instead of checking whether a scale or chord is convex or star-convex one might propose to measure the *degree* of convexity (i.e., a round object is more convex than a stretched oval object). However, due to the different bases, that are possible for the tone space, objects may change in their form. Having proved in Section 4 that a convex body is still convex in an other basis, the degree of convexity can change. Therefore the distinction between convex and star-convex can be made, but a further division is impossible in this space. Still, a measure of convexity is possible in a different way. In Table 1, the number of notes of every scale is also indicated. The more notes, the more possibilities of arranging these notes in the plane, the more special it is if these notes do form a convex set.

An important issue is what (star-)convexity actually means for music. Formally, convexity in the tone space means that all intervals lying on the line between two points are within the set. In terms of note names, it means that if two notes are in the set and the interval between these two notes can be composed from a multiple of another interval (*modulo an octave*), all other notes described by adding this interval (or a multiple thereof) to the lowest note, should be in the set. For example, the interval between a C and a $G\sharp$ is an augmented fifth. This interval can be composed from two major thirds. This means that if the C and $G\sharp$ are both present in a convex set, the note that is represented by a major third above the C (which is an E) should be in the set as well.

From the above, it appears that convexity has to do with connectivity of intervals. It is natural to strive to move by the shortest number of consonances from the tonic to any other note in the scale. Therefore convexity may be a consequence of striving for maximising connectivity (i.e., to get as many consonant intervals as possible within the notes defining the scale or chord). Star-convexity can be seen as a less strong notion: the consonance according to *one* interval is maximised. The link to well-formedness can now be made by observing that there is only a limited number of consonances and there are far more dissonances among all possible intervals. In creating a scale or a chord, one usually aims to allow for using consonant intervals. The fact that many altered chords are not convex can be related to the tension in the tension-resolution effect that is often found in music.

The convexity property can perhaps be best compared to the "Good form" principle from Gestalt theory. This principle applies to visual cognition that prefers to group shapes that are symmetrical, completed, made of clean contours, and the like. Our results suggest that this principle can now also be applied to musical cognition by stating that musical objects prefer an intervallic structure in which consonance it optimised. We conjecture that the Good form principle can be formalised by means of convexity.

8. Conclusion

In this article, we have investigated the property of (star-)convexity as a general principle or condition for the well-formedness of musical items. We noted that several musical items are (star-)convex. We discovered that all 5-limit just intonation scales and all chords built from harmonic notes are either convex or star-convex. Interesting is the non-convexity of altered chords. Our results are summarised in Table 6.

Table 6. Summary of results

Number of tested items	Convex	Star-convex
53 scales	86.8%	100%
15 harmonic chords	100%	100%
12 altered chords	25%	83.3%

We saw in the discussion that it is highly unlikely for a random chosen set to be convex or star-convex. Therefore these results are far more surprising than one may think at first glance. Our results suggest an interesting hypothesis – namely that (star-)convexity serves as a condition for the "well-formedness" of musical items. Star-convexity is a less straight notion than convexity, but it is intriguing that nearly all the items discussed here follow this property. These two notions circumscribe a certain space of good chords and scales. We discussed the meaning of convexity and saw that it may be a consequence of maximising the consonances in a musical scale or chord.

We have also shown that the tonal coherence that forms the harmonic reduction of a piece is a convex body. Prom this, we hypothesise that the notion of convexity may contribute to the perception of harmonic functions. Finally, convexity can be used as a principle in pitch spelling algorithms.

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