Probabilistic Dependence Logic

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February 1, 2009

Abstract

Given a finite model $\mathcal{M}$, it is possible to associate to every sentence ϕ of Backslash Logic and Dependence Logic the value of the Nash equilibria of the corresponding imperfect information game $H(\phi)$. Hodges' compositional semantics can then be adapted to this new logic, and the value of atomic dependence formulas in the resulting framework is seen to correspond to one of Kivinen and Mannila's measures of approximate functional dependency.

Keywords: Semantic games, Imperfect information, Nash equilibria, Probabilistic logic, Dependence logic

1 Introduction

The idea of associating to undetermined formulas the values of the corresponding semantic games has a long history in the study of logics of imperfect information.

Its earliest mention, as far as the author knows, is in [2], which reports that Miklos Ajtai suggested to use this technique to calculate the value of the Branching Quantifier expression

$$\begin{pmatrix} \forall x \\ \exists y \end{pmatrix} x = y.$$

As its intuitive interpretation would suggest, this formula then takes value $1/n$ in all finite models with n elements, and 0 in all infinite models.

Sevenster's Ph.D. thesis [12] formalizes this proposal in the framework of generalized quantifiers; moreover, the same concepts are discussed by van Benthem in [14] are [15]. More recently, Sandu and Sevenster ([11]) gave a formalization of equilibrium semantics for a version of IF-Logic, and proved some results about the expressive power and complexity of the resulting logic.

In [4], a part of the ideas and results of Sevenster and Sandu's work was independently developed; moreover, this document contains a Hodges-style compositional semantics for the resulting "Probabilistic Dependence Logic", and points out a relation between the values of Väänänen's Dependence Atomic Formulas (or, equivalently, their translations in Backslash Logic) and a g_3 measure of approximate functional dependency, introduced in [8] and used in Database Theory.

In the present work, we we will investigate the fragment of the language for which a behavioral-strategy equilibrium exists.[1]
In particular, we will prove that, if an upper bound to the size of the model is known, there is a value-preserving translation from the whole logic to this fragment, and that for this fragment the compositional semantics described in [4] is valid and can be used to compute the value of the equilibria.

2 Game Values

Given a formula ϕ of Backslash Logic[2] , a model $\mathcal{M}$ and an assignment s with $Dom(s) \supseteq FV(\phi)$, we indicate with $H_s(\phi)$ the imperfect information game associated to the tuple (ϕ, s) by this logic's Game Theoretic Semantics.

[1] In [4], it was believed that this fragment was the whole language. Later, the author was able to prove that this is not the case.

[2] Backslash Logic is the variant of IF-Logic in which *dependence*, rather than *independence*, is the primitive notion. For example, in $\exists x \exists y (\exists z \backslash y)\psi$ the choice of x must be determined by the value of y: in IF-logic, one would write $\exists x \exists y (\exists z / x)\psi$ instead.
The author finds functional dependence a more natural concept that functional independence to work with: indeed, while checking whether a quantifier Qy is independent from a quantifier Qx, one must keep in mind *with respect to what* is independence being verified, while nothing of this kind is required when checking dependence.
In any case, it is simple to find a (non-compositional) translation between IF-Logic and Backslash Logic.

In brief, the game starts from the position $(\phi, s, \mathbf{II})$, and proceeds according to the following rules:

Position	Successors
$(\psi \vee \theta, s, \alpha)$	$\{(\psi, s, \alpha), (\theta, s, \alpha)\}$
$(\exists x \psi, s, \alpha)$	$\{(\psi, s[m/x], \alpha) : m \in Dom(\mathcal{M})\}$
$(\exists x \backslash W \psi, s, \alpha)$	$\{(\psi, s[m/x], \alpha) : m \in Dom(\mathcal{M})\}$
$(\sim \psi, s, \alpha)$	$\{(\psi, s, \alpha^*)\}$

where $\alpha \in \{\mathbf{I}, \mathbf{II}\}$, α^* is α's opponent, and Player α moves in every position of the form (ψ, s, α).

Definition 1 (Play and payoff)

A play *of the game $H_s(\phi)$ is a sequence of positions $\bar{p} = p_1 \dots p_n$ such that*

- *p_1 is $(\phi, s, \mathbf{II})$;*
- *For every $i < n$, p_{i+1} is a possible successor of p_i.*

If moreover p_n corresponds to an atomic formula, we say that $\bar{p}$ is complete*; otherwise, we say that $\bar{p}$ is* partial.

Given a complete play $\bar{p} = p_1 \dots p_n$, where p_n is (ψ, s, α) and ψ is atomic, we define the payoff *for Player* **II** *of $\bar{p}$ as*

$$P(H_s(\phi); \bar{p}) = \begin{cases} 1 & \text{if } \alpha = \mathbf{II} \text{ and } s \models \psi, \text{ or if } \alpha = \mathbf{I} \text{ and } s \not\models \psi; \\ 0 & \text{otherwise.} \end{cases}$$

The payoff for player **I** *is $1 - P(H_s(\phi); \bar{p})$.*

A *local strategy* for the game $H_s(\phi)$ describes a possible behavior at a given point of the game:

Definition 2 (Local Strategy)

Given a subformula instance ψ of ϕ, a local strategy *for Player* **I** *in ψ is a function σ_ψ from partial plays $\bar{p} = p_1 \dots p_i$, where*

$$p_i = (\psi, s', \mathbf{I})$$

to $\{L, R\}$ if ψ is a disjunction, and to $Dom(\mathcal{M})$ otherwise.

A local strategy τ_ψ for Player **II** *is defined analogously.*

Given a (possibly partial) play $\bar{p} = p_1 \dots p_n$, we say that Player $\alpha \in \{\mathbf{I}, \mathbf{II}\}$ *follows* σ_ψ if and only if, for all $i \in 1 \dots n-1$ such that p_i is (ψ, s, α),

$$p_{i+1} = \begin{cases} (\theta_1, s, \alpha) & \text{if } \sigma_\psi(p_1 \dots p_i) = L; \\ (\theta_2, s, \alpha) & \text{otherwise.} \end{cases}$$

if ψ is of the form $\theta_1 \vee \theta_2$, and

$$p_{i+1} = (\theta, s[m/x], \alpha) \text{ where } m = \sigma_\psi(p_1 \dots p_i).$$

if ψ is of the form $\exists x \psi$ or $(\exists x \backslash W)\psi$.

A *pure strategy* for the game $H_s(\phi)$ is formally defined as follows:

Definition 3 (Pure Strategy)

A pure strategy σ *for Player* **I** *is a family of local strategies* σ_ψ *for Player* **I***, where* ψ *ranges over all nonatomic, negative subformula instances of* ϕ *(that is, all subformulas which are in the scope of an odd number of negations* $\sim$ *and are not atomic).*

Given a partial play $\bar{p}$*, we say that Player* **II** *follows* σ *in* $\bar{p}$ *if he follows every* σ_ψ *in* $\bar{p}$*.*

A pure strategy τ *for Player* **II** *is defined analogously.*

It is easy to see that, given a pair of pure strategies (σ, τ) for the two players, there exists a single complete play $\bar{p}$ in which Player **I** follows σ and Player **II** follows τ; we will call this play $(\sigma; \tau)$.

In particular, the payoff, for Player **II**, of the couple of strategies (σ, τ) will be given by $P(H_s(\phi); \sigma; \tau) = P(H_s(\phi); (\sigma; \tau))$.

We are especially interested in *uniform strategies*, that is, in strategies in which the players do not "cheat" by using information they should not have access to:

Definition 4 (Uniform local strategy)

A local strategy σ_ψ *for Player* α *is* uniform *if it is not the case that* ψ *is of the form* $(\exists x \backslash W)\theta$ *and that there exist two partial plays* $\bar{p} = p_1 \dots p_{i-1}, ((\exists x \backslash W)\theta, s, \alpha)$ *and* $\bar{q} = q_1 \dots q_{i-1}, ((\exists x \backslash W)\theta, s', \alpha)$ *such that* $s(y) = s'(y)$ *for all* $y \in W$ *and* $\sigma_\psi(\bar{p}) \neq \sigma_\psi(\bar{q})$*.*

A pure strategy σ for player α is *uniform* if it contains only uniform local strategies.

As usual, a *mixed strategy* is a probability distribution over pure strategies; in particular, a *uniform mixed strategy* for Player α will be a probability distribution $\mathfrak{m}$ over all uniform pure strategies σ.
The *payoff* of a pair of mixed strategies is then the weighed average of the payoffs of pairs of pure strategies, that is,

$$P(H_s(\phi); \mathfrak{m}; \mathfrak{n}) = \sum_{\sigma} \sum_{\tau} \mathfrak{m}(\sigma)\mathfrak{n}(\tau) P(H_s(\phi); \sigma; \tau).$$

The *Minimax Theorem* states that, if the model $\mathcal{M}$ (and, therefore, the game) is finite[3] there exists a pair of uniform mixed strategies ($\mathfrak{m}$, $\mathfrak{n}$) which is in equilibrium, that is, such that neither player can improve its payoff by switching to a different strategy:

Theorem 1 (Minimax Theorem)
There exists a pair of uniform mixed strategies $(\mathfrak{m}, \mathfrak{n})$ *such that, for all uniform mixed strategies* $\mathfrak{m}'$ *and* $\mathfrak{n}'$ *for Players* **I** *and* **II***,*

$$P(H_s(\phi); \mathfrak{m}'; \mathfrak{n}) \leq P(H_s(\phi); \mathfrak{m}; \mathfrak{n}) \leq P(\mathfrak{m}; \mathfrak{n}').$$

Moreover, it is well known that, in a zero-sum game G, all equilibrium pairs have the same payoff[4].
This justifies the following definition:

Definition 5 (Value of a formula)
Given a model $\mathcal{M}$*, a formula* ϕ *and an assignment* s *with* $Dom(s) \supseteq FV(\phi)$*,*

$$V_s^{\mathcal{M}}(\phi) = P(H_s(\phi); \mathfrak{m}; \mathfrak{n})$$

where $(\mathfrak{m}; \mathfrak{n})$ *is an equilibrium pair of the game* $H_s(\phi)$*.*

This is the same notion that Sandu and Sevenster study in [11].

Mixed strategies are not the only way of introducing randomness in an extensive game of imperfect information. One concept which, under certain aspects, is more natural is that of *behavioral strategy*:

[3] If the model is infinite this is not the case, as can be easily be seen by examining the game $H_\emptyset(\forall x \exists y (y > x))$ in the model $\mathbb{N}$.
The rest of this work, in any case, will be concerned exclusively with finite models.

[4] The proof is very simple. In brief, let $(\mathfrak{m}, \mathfrak{n})$ and $(\mathfrak{m}', \mathfrak{n}')$ be two equilibrium pairs. Then $P(G; \mathfrak{m}, \mathfrak{n}) \leq P(G; \mathfrak{m}, \mathfrak{n}') \leq P(G; \mathfrak{m}', \mathfrak{n}')$, and the other inequality is obtained in the same way.

Definition 6 (Behavioral Strategy)
A behavioral strategy β *for Player* **I** *in the game* $H_s(\phi)$ *is a family of probability distributions* β_ψ *over local strategies* σ_ψ*, where* ψ *ranges over all nonatomic negative subformulas of* ϕ*.*
The individual β_ψ *are called* local behavioral strategies*, and are uniform if and only if they only attibute positive probabilities to uniform local pure strategies; moreover,* β *is said to be uniform if and only if every* β_ψ *is uniform.*
A behavioral strategy γ *for Player* **II** *is defined analogously.*

In what follows, all strategies will be required to be uniform unless otherwise specified.

Every behavioral strategy β corresponds to a mixed strategy β^*, defined as

$$\beta^*(\sigma) = \prod_{\psi} \beta_\psi(\sigma_\psi).$$

In particular,

$$P(H_s(\phi); \beta; \gamma) = P(H_s(\phi); \beta^*; \gamma^*).$$

Informally, the difference between mixed strategies and behavioral strategies is that, while when using a mixed strategy the player extracts a pure strategy at the beginning of the game and then commits to it, when using a behavioral strategy the player is not allowed to commit to any strategy, but has to extract a local strategy at every point of the game.

For games of perfect recall, Kuhn's Theorem [9] states that every mixed strategy is equivalent to a behavioral strategy; however, for games of imperfect recall this is not the case, as by committing to a pure strategy the player could "remember" previous moves he or she should have forgotten about.

The following example shows how this can happen in our semantic games, and that not all semantic games have an equilibrium pair of behavioral strategies:

Example 1

Let $dom(\mathcal{M}) = (\{a, b\})$ *and let*

$$\phi := \exists x(\exists y \backslash \{\}) \sim (\exists z \backslash \{\})(x \neq y \vee x = z);$$

$$\phi' := (\exists y \backslash \{\}) \sim (\exists z \backslash \{\})(x \neq y \vee x = z);$$

$$\phi'' := (\exists z \backslash \{\})(x \neq y \vee x = z).$$

Now, consider the pure strategies for Player $\mathbf{I}$ *given by* $\sigma^a = \{\sigma^a_{\phi''}, \sigma_{x \neq y \vee x = z}\}$ *and* $\sigma^b = \{\sigma^b_{\phi''}, \sigma_{x \neq y \vee x = z}\}$, *where*

$$\sigma^a_{\phi''}(p_1 p_2 p_3(\phi'', s), \mathbf{I}) = a; \qquad \sigma^b_{\phi''}(p_1 p_2 p_3(\phi'', s), \mathbf{I}) = b;$$

$$\sigma_{x \neq y \vee x = z}(p_1 p_2 p_3 p_4(x \neq y \vee x = z, s), \mathbf{I}) = \begin{cases} L & \text{if } s(x) \neq s(y); \\ R & \text{otherwise.} \end{cases}$$

In other words, if Player $\mathbf{I}$ *uses* σ^a *then he chooses* $z = a$, *and if he uses* σ^b *then he chooses* $z = b$; *then, he selects a true disjunct if one exists.*

Now, let the mixed strategy $\mathfrak{n}$ *for Player* $\mathbf{I}$ *be defined by*

$$\mathfrak{n}(\sigma^a) = \mathfrak{n}(\sigma^b) = \frac{1}{2},$$

so that Player $\mathbf{I}$ *chooses either* a *or* b *randomly, and then picks the disjunct which makes him win if one can be found.*

Analogously, let

$$\tau^a_{\phi}(p_1) = a; \qquad \tau^b_{\phi}(p_1) = b;$$

$$\tau^a_{\phi'}(p_1 p_2) = a; \quad \tau^a_{\phi'}(p_1 p_2) = b$$

and consider the two pure strategies for Player $\mathbf{II}$ $\tau^{aa} = \{\tau^a_{\phi}, \tau^a_{\phi'}\}$ *and* $\tau^{bb} = \{\tau^b_{\phi}, \tau^b_{\phi'}\}$, *and the mixed strategy*

$$\mathfrak{n}(\tau^{aa}) = \mathfrak{n}(\tau^{bb}) = \frac{1}{2},$$

According to $\mathfrak{n}$, *Player* $\mathbf{II}$ *chooses* $x = y = a$ *or* $x = y = b$ *with equal probability.*

Then we have the following facts:

1. $(\mathfrak{m}, \mathfrak{n})$ *is an equilibrium pair, and its payoff is 1/2;*

2. *The mixed strategy* $\mathfrak{n}$ *does not correspond to any behavioral strategy;*

3. *No pair of behavioral strategies* (β, γ) *is in equilibrium.*

Let us verify this:

1. *Player* **I***'s uniform pure strategies are* σ^a, σ^b, *and some clearly inefficient strategies which sometimes pick a false disjunct when a true one exists and can be discarded; analogously, Player* **II***'s uniform pure strategies are* τ^{aa}, τ^{bb}, *and the two inefficient strategies* τ^{ab} *and* τ^{ba} *which surrender the game to Player* **I**.

   If Player **I** *uses* $\mathfrak{m}$, *both* τ^{aa} *and* τ^{bb} *give a payoff of* $1/2$ *to Player* **II***: for example,*

$$P(H_s(\phi); \mathfrak{m}; \tau^{aa}) = \frac{1}{2}P(H_\emptyset(\phi); \sigma^a; \tau^{aa}) + \frac{1}{2}P(H_\emptyset(\phi); \sigma^b; \tau^{aa}) = 0 + \frac{1}{2} = \frac{1}{2}.$$

   Thus, by the definition of payoff for mixed strategies,

$$P(H_\emptyset(\phi); \mathfrak{m}; \mathfrak{n}) = \frac{1}{2}P(H_\emptyset(\phi); \mathfrak{m}; \tau^{aa}) + \frac{1}{2}P(H_\emptyset(\phi); \mathfrak{m}; \tau^{bb}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

   This also implies that Player **II** *cannot improve her payoff by switching strategies: every mixture of* τ^{aa} *and* τ^{bb} *will give her a payoff of* $1/2$, *and selecting* τ^{ab} *or* τ^{ba} *with some positive probability would only make things worse for her.*

   Analogously, if Player **II** *uses* $\mathfrak{n}$ *both* σ^a *and* σ^b *lead to a payoff of* $1/2$ *for him; thus, he cannot improve his payoff by changing strategy, as every mixture of* σ^a *and* σ^b *will lead to a payoff of* $1/2$, *and again using "unreasonable" strategies would not improve it.*

2. *Suppose that* γ *is a behavioral strategy which is equivalent to* $\mathfrak{n}$, *in the sense that - no matter what Player* **I** *does -* $\mathfrak{n}$ *and* γ *generate the same plays with the same probabilities.*

   Then, as $\mathfrak{n}$ *uses either* τ^a_ϕ *or* τ^b_ϕ *(that is,* $x = a$ *or* $x = b$*) with equal probability, we must have that*

$$\gamma_\phi(\tau^a_\phi) = \gamma_\phi(\tau^b_\phi) = \frac{1}{2}.$$

   Analogously, since $\mathfrak{n}$ *uses either* $\tau^a_{\phi'}$ *or* $\tau^b_{\phi'}$ *(that is,* $y = a$ *or* $y = b$*) with equal probability,*

$$\gamma_{\phi'}(\tau^a_{\phi'}) = \gamma_{\phi'}(\tau^b_{\phi'}) = \frac{1}{2}.$$

But this is not possible: indeed, according to $\mathfrak{n}$ *it is never the case that Player* **II** *chooses* $x = a$ *(that is,* τ^a_ϕ*) and then* $y = b$ *(that is,* $\tau^b_{\phi'}$*), whereas accoding to the* γ *defined above this occurs with probability* $1/4$*.*

3. *We can just consider behavioral strategies for* **I** *which attribute positive probabilities only to* $\sigma^a_{\phi'''}$*,* $\sigma^b_{\phi'''}$ *and* $\sigma_{x\neq y \vee x=z}$*: indeed, the first two are the only possible local strategies for the subformula* ϕ'''*, and all other local strategies for the subformula* $x \neq y \vee x = z$ *are dominated by* $\sigma_{x\neq y\vee x=z}$*.*
   Thus, a (not dominated) behavioral strategy β *for Player* **I** *is completely determined by the value* $r = \beta_{\phi'''}(\sigma^a_{\phi'''}) \in [0,1]$*: Player* **I** *will choose* $z = a$ *with probability* r *and* $z = b$ *with probablity* $1 - r$*, and then will always choose a winning disjunct if one exists.*
   A behavioral strategy γ *for Player* **II** *may instead choose between* τ^a_ϕ *and* τ^b_ϕ*, and between* $\tau^a_{\phi'}$ *and* $\tau^b_{\phi'}$*: thus, every* γ *is identified by the two values* $p = \gamma_\phi(\tau^a_\phi)$ *and* $q = \gamma_{\phi'}(\tau^a_{\phi'})$*, both in* $[0,1]$*: as her first move, Player* **II** *will choose* $x = a$ *with probability* p *and* $x = b$ *with probability* $1 - p$*, while as her second move she will pick* $y = a$ *with probability* q *and* $y = b$ *with probability* $1 - q$*.*
   Note that Player **II** *has no way to coordinate her first and second move, for example by choosing* $x = y = a$ *or* $x = y = b$ *with equal probability as in the previous point.*

Given a pair of behavioral strategies (β, γ)*, let* $(p, q, r) \in [0,1]^3$ *be the three values defined above: then,*

$$\begin{aligned}
P(H_\emptyset(\phi); \beta; \gamma) &= pqrP(H_\emptyset(\phi); \sigma^a; \tau^{aa}) + pq(1-r)P(H_\emptyset(\phi); \sigma^b; \tau^{aa}) + \\
&\quad + p(1-q)rP(H_\emptyset(\phi); \sigma^a; \tau^{ab}) + p(1-q)(1-r)P(H_\emptyset(\phi); \sigma^b; \tau^{ab}) + \\
&\quad + (1-p)qrP(H_\emptyset(\phi); \sigma^a; \tau^{ba}) + (1-p)q(1-r)P(H_\emptyset(\phi); \sigma^b; \tau^{ba}) + \\
&\quad + (1-p)(1-q)rP(H_\emptyset(\phi); \sigma^a; \tau^{bb}) + (1-p)(1-q)(1-r)P(H_\emptyset(\phi); \sigma^b; \tau^{bb}) = \\
&= pq(1-r) + (1-p)(1-q)r
\end{aligned}$$

Now, let us suppose that Player **I***'s strategy* β *(that is, the value of* r*) is fixed, and let us find the optimal behavioral strategy* γ *for Player* **II***:*

$$\arg\max_\gamma P(H_\emptyset(\phi); \beta; \gamma) = \arg\max_{p,q} pq(1-r) + (1-p)(1-q)r = \begin{cases} p = q = 0 & \text{if } r > 1 - r; \\ p = q = 1 & \text{otherwise.} \end{cases}$$

In other words, Player **II** *should always choose* $x = y = b$ *if she knows that the probability that Player*

II *picks* $z = a$ *is greater than the probability that he picks* $z = b$*, and she should always pick* $x = y = a$ *otherwise (if* $r = 1/2$*, that is, if Player* **I** *chooses either* a *or* b *with equal probability, then clearly all choices are equivalent for* **II***).*

Analogously, let us suppose that γ *(and, thus,* p *and* q*) are fixed, and let us find the optimal behavioral strategy for Player* **I***:*

$$\arg\min_{\beta} P(H_{\emptyset}(\phi); \beta; \gamma) = \arg\min_{r} pq(1-r) + (1-p)(1-q)r = \begin{cases} r = 0 & \textit{if } (1-p)(1-q) > pq; \\ r = 1 & \textit{otherwise.} \end{cases}$$

In other words, Player **I** *should always choose* $z = b$ *if he knows that the probability Player* **II** *picks* $x = y = b$ *is greater than the probability that she picks* $x = y = a$*, and otherwise she should always choose* $z = a$ *(again, if* $pq = (1-p)(1-q)$ *then* $p = q = 1/2$ *and the two possibilities are indifferent for Player* **I***).*

Now, suppose that (β, γ) *is an equilibrium pair of behavioral strategies, and let* p*,* q *and* r *be as above. Suppose that* $r > 1/2$*: then, as Player* **II** *cannot improve her payoff by changing strategy,* $p = q = 0$*. But then* $(1-p)(1-q) > pq$*, and therefore, as Player* **I** *also cannot improve his payoff by switching strategy,* $r = 0$*, which contradicts our hypothesis.*

Suppose now that $r < 1/2$*: then, for the same argument used before, we must have* $p = q = 1$*, which implies* $r = 1$*, contraditing our hypothesis.*

We are thus left with $r = 1/2$*. In this case Player* **II** *has two optimal behavioral strategies, corresponding to* $p = q = 0$ *and* $p = q = 1$*.*

Suppose that she picks the former: then, since r *must also be optimal and* $(1-p)(1-q) > pq$*, we have* $r = 0$*, which contradicts our premise; and if she instead picks the latter,* $(1-p)(1-q) < pq$ *and thus* $r = 1$*, which also contradicts the premise.*

In conclusion, there exists no behavioral equilibrium in $H_{\emptyset}(\phi)$*, as stated.*

3 Behavioral equilibria and mixed equilibria

As the previous example showed, it is not always the case that a behavioral-strategy equilibrium pair exists for a semantic game over a finite model.

However, the next theorem shows that behavioral equilibria are, in a certain sense, a special case of mixed equilibria:

Theorem 2

The following statements are true:

1. *If a behavioral equilibrium exists, its payoff is the same payoff of every mixed equilibrium;*

2. *Given a $n \in \mathbb{N}$ there exists a function T_n from formulas to formulas such that, for every model $\mathcal{M}$ with $|Dom(\mathcal{M})| \leq n$, for every formula ϕ*

$$V(\phi) = V(T_n(\phi)),$$

and moreover $T_n(\phi)$ has a behavioral-strategy equilibrium.

Proof:

1. Let (β^e, γ^e) be a behavioral strategy equilibrium. Then, for all behavioral strategies β and γ for Players **I** and **II**,

$$P(H_s(\phi); \beta^e; \gamma) \leq P(H_s(\phi); \beta^e; \gamma^e) \leq P(H_s(\phi); \beta; \gamma^e).$$

Since all pure strategies σ for Player **I** are also behavioral strategies, we have in particular that

$$P(H_s(\phi); \beta^e; \gamma^e) \leq P(H_s(\phi); \sigma; \gamma^e)$$

for all γ; but then, if β^{e*} and γ^{e*} are the two mixed strategies corresponding to β^e and γ^e,

$$P(H_s(\phi); \beta^{e*}; \gamma^{e*}) \leq P(H_s(\phi); \sigma; \gamma^{e*}).$$

But this implies that, for every mixed strategy $\mathfrak{m}$ for Player **I**,

$$P(H_s(\phi); \mathfrak{m}; \gamma^{e*}) = \sum_{\sigma} \mathfrak{m}(\sigma) P(H_s(\phi); \sigma; \gamma^{e*}) \geq \sum_{\sigma} \mathfrak{m}(\sigma) P(H_s(\phi); \beta^{e*}; \gamma^{e*}) = P(H_s(\phi); \beta^{e*}; \gamma^{e*}).$$

Analogously, since $P(H_s(\phi); \beta^e; \gamma^e) \geq P(H_s(\phi); \beta^e; \tau)$ for all pure strategies τ of Player **II**,

$$P(H_s(\phi); \beta^{e*}; \mathfrak{n}) = \sum_{\tau} \mathfrak{n}(\tau) P(H_s(\phi); \beta^{e*}; \tau) \leq \sum_{\tau} \mathfrak{n}(\tau) P(H_s(\phi); \beta^{e*}; \gamma^{e*}) = P(H_s(\phi); \beta^{e*}; \gamma^{e*})$$

for all mixed strategies $\mathfrak{n}$.

In conclusion, $(\beta^{e*}, \gamma^{e*})$ is a mixed strategy equilibrium, and as its payoff coincides with the payoff of the behavioral strategy equilibrium (β^{e}, γ^{e}) this concludes the first part of the proof.[5]

2. Let $|Dom(\mathcal{M})| \leq n$, and let ϕ be any formula.

   Then, let V be the set of all variables occurring in ϕ, let

$$p_n = \lceil \log_2 |\{\sigma : \sigma \text{ pure strategy for } \mathbf{I} \text{ when the model has } n \text{ elements}\}| \rceil;$$

$$q_n = \lceil \log_2 |\{\sigma : \sigma \text{ pure strategy for } \mathbf{II} \text{ when the model has } n \text{ elements}\}| \rceil;$$

   and define $T_n(\phi)$ as

$$T_n(\phi) := \forall z_1 \ldots \forall z_{p_n} (\exists w_1 \backslash \{\})(\exists w_2 \backslash \{w_1\}) \ldots (\exists w_{q_n} \backslash \{w_1 \ldots w_{q_n - 1})\phi^+$$

   where $z_1 \ldots z_{p_n}, w_1 \ldots w_{q_n}$ are variables which do not occur in ϕ, and ϕ^+ is derived from ϕ according to the following rules:

   (a) If ϕ is atomic, $\phi^+ = \phi^- = \phi$;

   (b) $(\sim\phi)^{\pm} = \sim\phi^{\mp}$;

   (c) If ϕ is of the form $\psi \vee \theta$,

$$(\psi \vee \theta)^+ = \exists x \backslash (V - \{z_1 \ldots z_{p_n}\}) \exists y \backslash (V - \{z_1 \ldots z_{p_n})(x = y \wedge \psi^+) \vee (x \neq y \wedge \theta^+);$$
$$(\psi \vee \theta)^- = (\exists x \backslash (V - \{w_1 \ldots w_{q_n}\}))(\exists y \backslash (V - \{w_1 \ldots w_{q_n}))(x = y \wedge \psi^-) \vee (x \neq y \wedge \theta^-);$$

   (d) if ϕ is of the form $\exists x \psi$,

$$(\exists x \psi)^+ = (\exists x \backslash (V - \{z_1 \ldots z_{p_n}\}))\psi^+;$$
$$(\exists x \psi)^- = (\exists x \backslash (V - \{w_1 \ldots w_{q_n}\}))\psi^-;$$

[5] The author wishes to thank Allen Mann, one private message of whom provided the main ingredient of this first part of the proof.

(e) If ϕ is of the form $(\exists x \backslash W)\psi$,

$$(\exists x \backslash W\psi)^+ = (\exists x \backslash W \cup \{w_1 \dots w_{q_n}\})\psi^+;$$

$$(\exists x \backslash W\psi)^- = (\exists x \backslash W \cup \{z_1 \dots z_{p_n}\})\psi^-;$$

In other words, ϕ' is derived by ϕ by letting all moves of Player **I** depend on the z_i but not on the w_i, and letting all moves of Player **II** depend on the w_i but not on the z_i.

Then $T_n(\phi)$ has a behavioral equilibrium in every model with at most n element, and its value coincides with the value of the mixed equilibria of ϕ.[6]

Indeed, let $(\mathfrak{m}, \mathfrak{n})$ be a mixed strategy equilibrium in $H_s(\phi)$, let e_1 and e_2 be two encodings of the pure strategies of Player **I** and Player **II** into tuples of elements of $\mathcal{M}$, and let β and γ be as follows:

- β chooses $z_1 \dots z_{p_n}$ so that $Prob(z_1 \dots z_{p_n} = e_1(\sigma)) = \mathfrak{m}(\sigma)$ for all σ, and analogously γ chooses $w_1 \dots w_{q_n}$ so that $Prob(w_1 \dots w_{q_n} = e_2(\tau)) = \mathfrak{n}(\tau)$ for all τ;
- Then, the players play according to the pure strategies encoded in $z_1 \dots z_{p_n}$ and $w_1 \dots w_{q_n}$.

Since ϕ^+ is structurally different from ϕ, and in particular a disjunction in ϕ becomes a more complex subformula in ϕ^+, the players cannot use directly these strategies; however, their translations into equivalent strategies for ϕ^+ are easy to find out.

As the probability that a certain terminal position is reached when the two players use $\mathfrak{m}$ and $\mathfrak{n}$ in $H_s(\phi)$ coincides with the probability that the same position (save for some extra variables in the assignment) is reached when β and γ are used in $H_s(T_n(\phi))$, the payoffs of $(\mathfrak{m}, \mathfrak{n})$ and (β, γ), respectively in $H_s(\phi)$ and $H_s(T_n(\phi))$, are the same.

[6] Technically, this is only the case if the model has at least two elements: if it has just one or none, the rule for the translation of the disjunction makes one of the disjuncts impossible to select.
If we are also interested in these degenerate cases, however, it suffices to take

$$\exists x \exists y (x \neq y \wedge T_n(\phi)) \vee \forall x \forall y (x = y \wedge \phi)$$

rather than just $T_n(\phi)$: indeed, it is easy to see that, in a model with less than two elements, a player never has any uncertainty about the previous moves of a play. Thus, the semantic game $H_s(\phi)$ is of perfect information, and as a consequence it already has a behavioral strategy equilibrium.
This minor point will be ignored in the following work.

Moreover, (β, γ) is a behavioral equilibrium for $T_n(\phi)$. Indeed, suppose that

$$P(H_s(T_n(\phi)); \beta; \gamma) \leq P(H_s(T_n(\phi)); \beta; \gamma')$$

for some γ', and as usual let γ'^* be the mixed strategy for $H_s(T_n(\phi))$ equivalent to γ'.

Then, let $\mathfrak{n}'$ be the mixed strategy for $H_s(\phi)$ which corresponds to γ'^* (that is, which associates to every pure strategy σ of $H_s(\phi)$ the sum of the probabilities, according to γ'^*, of all pure strategies σ' for $H_s(T_n(\phi))$ which differ from σ only in the choice of the initial string of quantifiers): we would then have that $P(H_s(\phi); \mathfrak{m}; \mathfrak{n}) \leq P(H_s(\phi); \mathfrak{m}; \mathfrak{n}')$, which is impossible.

The other part of the proof is verified in a similar way.

□

Thus, if we have an upper limit to the size of the model we can find, for every formula, an equivalent formula which has a behavioral equilibrium.

4 A compositional semantics

In [5], Hodges described a semantics for a game of imperfect information in terms of sets of assignments, which, following [13], will be called *teams*.

In particular, the first-order semantics notion of *assignment satisfying a formula* was substituted in Hodges' semantics with the notion of *team satisfying a formula*, called *trump*.

In the light of [6] and [13], it seems clear that this shift from assignments to sets of assignments is caused by the fact that logics of imperfect information can express statements about *functional dependency*, and such a concept cannot be meaningfully applied to single assignments.

In what follows, we will adapt Hodges' machinery to the computation of formula values, in the case that a behavioral strategy equilibrium exists: by the results in the previous chapter, then, once an upper bound n to the size of the model is known we can take any formula and find its value by applying the compositional semantics to $T_n(\phi)$.

Definition 7 (Probabilistic Team)

A probabilistic team *μ with domain $dom(\mu) = \{x_1 \dots x_n\}$ is a probability distribution over the set of all assignments on $\{x_1 \dots x_n\}$, that is, a function*

$$\mu : \{s : dom(s) = \{x_1 \dots x_n\}\} \to [0, 1]$$

such that

$$\sum_{dom(s)=dom(\mu)} \mu(s) = 1.$$

Then, the game $H^{\mathcal{M}}_{\mu}$ is defined as follows:

Definition 8 **($H^{\mathcal{M}}_{\mu}(\phi)$)**

Let ϕ be a formula, $\mathcal{M}$ a model, and let μ be a probabilistic team.

The game $H^{\mathcal{M}}_{\mu}(\phi)$ is then played as follows:

1. *First, an assignment s is selected randomly, according to the distribution μ;*

2. *Then, the game $H^{\mathcal{M}}_{s}(\phi)$ is played.*

The definitions of strategy, uniform strategy, behavioral strategy and uniform behavioral strategy are as usual; however, this time a play *will be determined by a triple (s, σ, τ), where s is the initial assignment (chosen according to μ) and σ, τ are pure strategies.*

Thus,

$$P(H_\mu(\phi); \beta; \gamma) = \sum_{dom(s)=dom(\mu)} \mu(s) P(H_s(\phi); \beta; \gamma).$$

It can be easily verified that $P(H_s(\phi); \beta; \gamma) = P(H_{\eta_s}; \beta; \gamma)$, where η_s is the probabilistic team which chooses s with certainty, that is, $\eta_s(s') = \delta_{s,s'}$. The following operations will be useful to characterize $\mathcal{T}$:

Definition 9 (Convex combination)

If μ_1, μ_2, and μ' are probabilistic teams with $dom(\mu_1) = dom(\mu_2) = dom(\mu')$ and $p \in [0, 1]$, it holds that

$$\mu' = p\mu_1 + (1 - p)\mu_2$$

if and only if

$$\mu'(s) = p\mu_1(s) + (1 - p)\mu_2(s), \text{ for all } s \text{ with } dom(s) = dom(\mu')$$

Definition 10 (Supplementation)

If μ is a probabilistic team, F is a function from $\{s : dom(s) = dom(\mu)\}$ to probability distributions over M, *that is, a mapping*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

where $\mathcal{D}(M) = \{f : M \to [0,1], \sum_{m \in M} f(m) = 1\}$ *and* $y \notin dom(\mu)$, *then* $\mu[F/y]$ *is defined as the probabilistic team such that*

$$\mu[F/y](s[m/y]) = \mu(s) \cdot F(s)(m)$$

for all s *such that* $dom(s) = dom(\mu)$ *and for all* $m \in M$.

In order to obtain a compositional semantics for the computation of the value, we want to be able to fix the local behavioral strategies in a "bottom-up" manner, i.e., very much like in the backwards induction method to compute subgame-perfect equilibria in perfect information games.

This justifies the following definition:

Definition 11 **($W_\mu(\phi)$)**
Let ϕ *be any formula, let* μ *be any probabilistic team, and for every subformula instance* ψ *of* ϕ *let* $d(\phi)$ *be the depth of* ψ *in* (ϕ) *(that is,* $d(\phi) = 0$, $d(\psi) = 1$ *for the immediate subformulas of* ϕ*, and so on).*
Then let

$$Sub^{\pm}(\phi) := \{\psi : \psi \text{ is a positive [negative] subformula instance of } \phi\};$$

and, for every $d \in \mathbb{N}$, *let*

$$Sub^{\pm}_d(\phi) = \{\psi : \psi \in Sub^{\pm}(\phi) \text{ and } d(\psi) = d\}.$$

Then define

$$W_\mu(\phi) := \sup_{\gamma_\psi : \psi \in Sub^+_0(\phi)} \inf_{\beta_\psi : \psi \in Sub^-_0(\phi)} \sup_{\gamma_\psi : \psi \in Sub^+_1(\phi)} \inf_{\beta_\psi : \psi \in Sub^-_1(\phi)} \dots$$
$$\dots \sup_{\gamma_\psi : \psi \in Sub^+_k(\phi)} \inf_{\beta_\psi : \psi \in Sub^-_k(\phi)} P(H_\mu(\phi); \{\beta_\psi : \psi \in Sub^-(\phi)\}, \{\gamma_\psi : \psi \in Sub^+(\phi)\})$$

where k is the depth of ϕ.

The intuition behind this is to make the choices of the "inner" local strategies be dependent on those of the "outer" local strategies, but not viceversa: in this way, the order of computation of the optimal local strategies coincides with the syntactical structure of the formula, and we can compute the value $W_\mu(\phi)$ through a compositional semantics.

Since every "branch" of the syntactic structure of the formula ϕ contains at most one formula at every depth, it is never the case that the choices of the local strategies of Player **I** at depth d are dependent on the choices of the local strategies of Player **II** at depth d, or vice versa; therefore, it is always the case that

$$\begin{array}{l} W_\mu(\phi) = \inf_{\beta_\psi:\psi\in Sub_0^-(\phi)} \sup_{\gamma_\psi:\psi\in Sub_0^+(\phi)} \dots \inf_{\beta_\psi:\psi in Sub_k^-(\phi)} \sup_{\gamma_\psi:\psi\in Sub_k^+(\phi)} \\ \quad P(H_\mu(\phi); \{\beta_\psi : \psi \in Sub^-(\phi)\}; \{\gamma_\psi : \psi \in Sub^+(\phi)\}) \end{array} \tag{1}$$

Moreover, $Sub_0^-(\phi) = \emptyset$, since the only subformula of depth 0 of ϕ is ϕ itself, and it is positive.

The next theorem shows that $W_\mu(\phi)$ is the value of the behavioral equilibria, if they exist:

Theorem 3

If the game $H_\mu(\phi)$ has a behavioral equilibrium (β^e, γ^e) then $P(H_\mu(\phi); \beta^e; \gamma^e) = W_\mu(\phi)$.

Proof:

If (β^e, γ^e) is a behavioral strategy equilibrium, then we have that

$$\sup_\gamma \inf_\beta P(H_\mu(\phi); \beta; \gamma) = P(H_\mu(\phi); \beta^e; \gamma^e) = \inf_\beta \sup_\gamma P(H_\mu(\phi); \beta; \gamma)$$

as can be inferred by observing that

$$\begin{array}{l} \sup_\gamma \inf_\beta P(H_\mu(\phi); \beta; \gamma) \geq \inf_\beta P(H_\mu(\phi); \beta; \gamma^e) = P(H_\mu(\phi); \beta^e; \gamma^e); \\ \inf_\beta \sup_\gamma P(H_\mu(\phi); \beta; \gamma) \leq \sup_\gamma P(H_\mu(\phi); \beta^e; \gamma) = P(H_\mu(\phi); \beta^e; \gamma^e); \\ \sup_\gamma \inf_\beta P(H_\mu(\phi); \beta; \gamma) \leq \inf_\beta \sup_\gamma P(H_\mu(\phi); \beta; \gamma). \end{array}$$

Moreover, if k is the depth of ϕ then

$$P(H_\mu(\phi);\beta^e;\gamma^e) = \sup_\gamma \inf_\beta P(H_\mu(\phi);\beta;\gamma) = \sup_{\gamma_\psi:\psi\in Sub_0^+(\phi)} \ldots \sup_{\gamma_\psi:\psi\in Sub_k^+(\phi)}$$

$$\inf_{\beta_\psi:\psi\in Sub_0^-(\phi)} \ldots \inf_{\beta_\psi:\psi\in Sub_k^-(\phi)} P(H_\mu(\phi);\{\beta_\psi:\psi\in Sub^-(\phi)\};\{\gamma_\psi:\psi\in Sub^+(\phi)\}) \leq$$

$$\leq \sup_{\gamma_\psi:\psi\in Sub_0^+(\phi)} \inf_{\beta_\psi:\psi\in Sub_0^-(\phi)} \ldots \sup_{\gamma_\psi:\psi\in Sub_k^+(\phi)} \inf_{\beta_\psi:\psi\in Sub_k^-(\phi)}$$

$$P(H_\mu(\phi);\{\beta_\psi:\psi\in Sub^+(\phi)\};\{\gamma_\psi:\psi\in Sub^-(\phi)\}) = W_\mu(\phi)$$

and

$$P(H_\mu(\phi);\beta^e;\gamma^e) = \inf_\beta \sup_\gamma P(H_\mu(\phi);\beta;\gamma) = \inf_{\beta_\psi:\psi\in Sub_0^-(\phi)} \ldots \inf_{\beta_\psi:\psi\in Sub_k^-(\phi)}$$

$$\sup_{\gamma_\psi:\psi\in Sub_0^+(\phi)} \ldots \sup_{\gamma_\psi:\psi\in Sub_k^+(\phi)} P(H_\mu(\phi);\{\beta_\psi:\psi\in Sub^-(\phi)\};\{\gamma_\psi:\psi\in Sub^+(\phi)\}) \geq$$

$$\geq \sup_{\gamma_\psi:\psi\in Sub_0^+(\phi)} \inf_{\beta_\psi:\psi\in Sub_0^-(\phi)} \ldots \sup_{\gamma_\psi:\psi\in Sub_k^+(\phi)} \inf_{\beta_\psi:\psi\in Sub_k^-(\phi)}$$

$$P(H_\mu(\phi);\{\beta_\psi:\psi\in Sub^+(\phi)\};\{\gamma_\psi:\psi\in Sub^-(\phi)\}) = W_\mu(\phi)$$

Thus, $W_\mu(\phi) = P(H_\mu(\phi);\beta^e;\gamma^e)$, as required.

□

At this point, we can find out a compositional semantics for $W_\mu(\phi)$:[7]

Theorem 4

Let $\mathcal{M}$ be a fixed model, let ϕ be any formula, and let μ be any probabilistic team with $Dom(\mu) \supseteq FV(\phi)$. Then

1. *If ϕ is atomic, $W_\mu(\phi) = \sum\{\mu(s) : s \models \phi\}$.*
2. *If ϕ is $\sim \psi$, $W_\mu(\phi) = 1 - W_\mu(\psi)$.*
3. *If ϕ is $\psi_1 \vee \psi_2$, $W_\mu(\phi) = \sup\{pW_{\xi_1}(\psi_1) + (1-p)W_{\xi_2}(\psi_2) : p\xi_1 + (1-p)\xi_2 = \theta\}$.*
4. *If ϕ is $\exists x\psi$, $W_\mu(\phi) = \sup_F W_{\mu[F/x]}(\psi)$.*

[7] These rules are the same of [4]. After the publication of [4], Mann independently suggested to study what is essentially the same semantics.

5. *If* ϕ *is* $(\exists x\backslash V)\psi$, $W_\mu(\phi) = \sup_F W_{\mu[F/x]}(\psi)$, *where* F *ranges over all functions from assignments to probability distribution over the domain such that*

$$s(y) = s'(y)\forall y \in V \Rightarrow F(s) = F(s').$$

Proof:

1. If ϕ is atomic, no player has a nontrivial strategy, and the game $H_\mu(\phi)$ is played as follows:

   - An assignment s is extracted, according to μ;
   - Player **II** wins if and only if $s \models \phi$.

   Thus, $W_\mu(\phi) = P(H_\mu(\phi); \beta; \gamma) = \sum\{\mu(s) : s \models \phi\}$, where β and γ are the trivial strategies.

2. If ϕ is $\sim \psi$, the first move of Player **II** is forced, as the only possible successor of $(\sim \psi, s, \mathbf{II})$ is $(\psi, s, \mathbf{I})$. Moreover, $Sub_0^-(\phi) = \emptyset$, as there is no negative subformula of depth 0; and, since the only subformula of depth 1 is ψ and it is negative, $Sub_1^+(\phi) = \emptyset$ too.
   Thus,

$$W_\mu(\phi) = \inf_{\beta_\theta : \theta \in Sub_1^-(\phi)} \sup_{\gamma_\theta : \theta \in Sub_2^+(\phi)} \ldots \sup_{\gamma_\theta : \theta \in Sub_k^+(\phi)} \inf_{\beta_\theta : \theta \in Sub_k^-(\phi)}$$
$$P(H_\mu(\phi); \{\beta_\theta : \theta \in Sub^+(\phi)\}; \{\gamma_\theta : \theta \in Sub^-(\phi)\}).$$

   where the local behavioral strategy β_ϕ for the first move of Player **II** is presumed fixed.

   Now, by definition $P(H_\mu(\sim \psi); \beta; \gamma) = 1 - P(H_\mu(\psi); \gamma; \beta)$, as $H_\mu(\sim \psi)$ is nothing more than $H_\mu(\psi)$ with the roles of the players switched; thus,

$$W_\mu(\phi) = 1 - \sup_{\beta_\theta : \psi \in Sub_1^-(\phi)} \inf_{\gamma_\theta : \theta \in Sub_2^+(\phi)} \ldots \inf_{\gamma_\theta : \theta \in Sub_k^+(\phi)} \sup_{\beta_\theta : \theta \in Sub_k^-(\phi)}$$
$$P(H_\mu(\phi); \{\gamma_\theta : \theta \in Sub^+(\phi)\}; \{\beta_\theta : \theta \in Sub^-(\phi)\}) =$$
$$= 1 - \sup_{\beta_\theta : \psi \in Sub_0^+(\psi)} \inf_{\gamma_\theta : \theta \in Sub_1^-(\phi)} \sup_{\beta_\theta : \theta \in Sub_1^+(\phi)} \ldots \inf_{\gamma_\theta : \theta \in Sub_k^-(\psi)} \sup_{\beta_\theta : \theta \in Sub_k^+(\psi)}$$
$$P(H_\mu(\psi); \{\gamma_\theta : \theta \in Sub^-(\psi)\}; \{\beta_\theta : \theta \in Sub^+(\psi)\}) = 1 - W_\mu(\psi)$$

where for the last equivalence we used the fact that $Sub_0^-(\psi) = \emptyset$ and equation (1).

3. If ϕ is $\psi_1 \vee \psi_2$, the unique formula in $Sub_0^+(\phi)$ is ϕ itself, and as usual $Sub_0^-(\phi) = \emptyset$.

   Moreover, by definition, a local behavioral strategy γ_ϕ is a probability distribution over pure local strategies τ_ϕ, that is, over functions from assignments to $\{L, R\}$; and since it is still Player **II**'s turn in the second move, $Sub_1^-(\phi)$ is also $\emptyset$ and $Sub_1^+(\phi) = \{\psi_1, \psi_2\}$.

   Therefore, let

$$p = Prob(\mathbf{II} \text{ chooses } L) = \sum\{\mu(s) \cdot \gamma_\phi(\tau_\phi) : Dom(s) = Dom(\mu) \text{ and } \tau_\phi((\phi, s, \mathbf{II})) = L\};$$

$$\xi_1(s) = Prob(s \text{ was extracted } | \mathbf{II} \text{ chose } L) = \frac{\mu(s) \cdot \sum\{\gamma_\phi(\tau_\phi) : \tau_\phi((\phi, s, \mathbf{II})) = L\}}{p};$$

$$\xi_2(s) = Prob(s \text{ was extracted } | \mathbf{II} \text{ chose } R) = \frac{\mu(s) \cdot \sum\{\gamma_\phi(\tau_\phi) : \tau_\phi((\phi, s, \mathbf{II})) = R\}}{1-p}.$$

   Clearly, $\mu = p\xi_1 + (1-p)\xi_2$; moreover, it can be verified that every decomposition of μ in $p\xi_1 + (1-p)\xi_2$ can be obtained for an opportune choice of γ_1.

   Let us then compute $W_\mu(\phi)$: as usual,

$$W_\mu(\phi) = \sup_{\gamma_\theta : \theta \in Sub_0^+(\phi)} \inf_{\beta_\theta : \theta \in Sub_0^-(\phi)} \sup_{\gamma_\theta : \theta \in Sub_1^+(\phi)} \inf_{\beta_\theta : \theta \in Sub_1^-(\phi)} \ldots$$
$$\ldots \sup_{\gamma_\theta : \theta \in Sub_k^+} \inf_{\beta_\theta : \theta \in Sub_k^-} P(H_\mu(\phi); \{\beta_\psi : \psi \in Sub^+(\phi)\}; \{\gamma_\psi : \psi \in Sub^-(\phi)\})$$

   The first supremum, for the above reasons, defines a decomposition of μ as $p\xi_1 + (1-p)\xi_2$; as for the other local strategies, the ones corresponding to subformulas of ψ_1 will only have an effect when Player **II** chooses L, and the ones corresponding to subformulas of ψ_2 will only matter when Player **II** chooses R.

   In other words, the subgame $H(\psi_1)$ will be chosen with probability p, and the distribution of the assignments at the start of this subgame will be ξ_1, the subgame $H(\psi_2)$ will be instead chosen with probability $1-p$, and the assignment distribution at the beginning of this subgame will be ξ_2.

Thus,

$$W_\mu(\phi) = \sup\{pW_{\xi_1}(\psi_1) + (1-p)W_{\xi_2}(\psi_2) : p\xi_1 + (1-p)\xi_2 = \mu\}.$$

as required.

4. Suppose that ϕ is $\exists x\psi$: then the first local behavioral strategy γ_ϕ of Player **I** defines a probability distribution over local pure strategies τ_ϕ, that is, over functions from assignments to elements of the model.

   Equivalently, γ_ϕ can be seen as describing a function F from assignments to probability distributions over the model, that is,

   $$F(s)(m) = Prob(\mathbf{II} \text{ chooses } m \mid s \text{ is extracted}) = \sum\{\gamma_\phi(\tau_\phi) : \tau_\phi((\phi, s, \mathbf{II})) = m\}$$

   Then the probability distribution over the assignments after Player **II**'s first move is $\mu[F/x]$; moreover, every F as above can be obtained for some choice of γ_ϕ, and as usual $Sub_0^-(\phi) = \emptyset$.

   Therefore,

   $$\begin{aligned} W_\mu(\phi) = \sup_F \sup_{\gamma_\theta:\theta\in Sub_1^+(\phi)} \inf_{\beta_\theta:\theta\in Sub_1^-(\phi)} \ldots \sup_{\gamma_\theta:\theta\in Sub_k^+(\phi)} \inf_{\beta_\theta:\theta\in Sub_k^-(\phi)} \\ P(H_{\mu[F/x]}(\psi); \{\beta_\theta : \theta \in Sub^-(\phi)\}; \{\gamma_\theta : \theta \in Sub^+(\phi)\}) = \\ = \sup_F \sup_{\gamma_\theta:\theta\in Sub_0^+(\psi)} \inf_{\beta_\theta:\theta\in Sub_0^-(\psi)} \ldots \sup_{\gamma_\theta:\theta\in Sub_{k-1}^+(\psi)} \inf_{\beta_\theta:\theta\in Sub_{k-1}^-(\psi)} \\ P(H_{\mu[F/x]}(\psi); \{\beta_\theta : \theta \in Sub_{k-1}^-(\psi)\}; \{\gamma_\theta : \theta \in Sub^+(\psi)\}) = \\ = \sup_F W_{\mu[F/x]}(\psi). \end{aligned}$$

5. If ϕ is $(\exists x \backslash W)\phi$, the proof is exactly as in the previous case, with the added condition that γ_ϕ may only attribute positive probabilities to local strategies which satisfy the dependency condition, and therefore

   $$s(y) = s'(y) \forall y \in W \Rightarrow F(s) = F(s').$$

□

5 The range of the value function

Any finite model $\mathcal{M}$ induces now a function $\phi \mapsto V(\phi)$.

In this section, it will be attempted to obtain some results about the range of this mapping.

The following theorem was proved independently by Sevenster and Sandu ([11]):

Theorem 5 *If the domain of $\mathcal{M}$ is finite and contains at least two elements,*

$$\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} = \mathbb{Q} \cap [0, 1].$$

Proof:

- $\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} \supseteq \mathbb{Q} \cap [0, 1]$:

  Let $r = p/q$, where $p < q$, and let $s = \lceil \log_2(q) \rceil$.

  Then, let ϕ be the following sentence:

$$\phi \equiv \exists x_0 \exists x_1 ((x_0 \neq x_1) \wedge (\exists y_{1,1} \exists y_{1,2} \dots \exists y_{1,s})(\exists y_{2,1} \exists y_{2,2} \dots \exists y_{2,s}) \dots (\exists y_{q,1} \exists y_{q,2} \dots \exists y_{q,s})$$
$$\left(\bigwedge_{i=1}^{q} \bigwedge_{k=1}^{s} y_{i,k} = x_0 \vee y_{i,k} = x_1\right) \wedge \left(\bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{q} \bigvee_{k=1}^{s} y_{i,k} \neq y_{j,k}\right) \wedge \forall z_1 \forall z_2 \dots \forall z_s \left(\bigwedge_{i=1}^{q} \bigvee_{k=1}^{s} (z_k \neq y_{i,k}) \vee\right.$$
$$\vee (\exists w_{1,1}/\{z_1 \dots z_s\}) \dots (\exists w_{1,s}/\{z_1 \dots z_s\})(\exists w_{2,1}/\{z_1 \dots z_s\}) \dots (\exists w_{2,s}/\{z_1 \dots z_s\}) \dots (\exists w_{p,1}/\{z_1 \dots z_s\}) \dots$$
$$\left.\dots (\exists w_{p,s}/\{z_1 \dots z_s\}) \left(\bigvee_{i=1}^{p} \bigvee_{j=1}^{q} \bigwedge_{k=1}^{s} w_{i,k} = z_{j,k}\right)\right))$$

  where $\exists w/\{z_1 \dots z_s\}$ is the *slashed quantifier* of IF-logic, which requires the choice of w to be *independent* from the choice of $z_1 \dots z_s$, and can be easily translated in terms of the usual backslashed quantifier.

  The game $H_\mu(\phi)$ is of *perfect recall*, as all existential quantifiers are dependent on all previous existential quantifiers and the same holds for all universal quantifiers: therefore, by Kuhn's Theorem all mixed strategies are behavioral strategies and, as a consequence, there exists a behavioral strategy equilibrium for this game.

  The value of this equilibrium is $V(\phi) = p/q$: indeed, the game $H(\phi)$ can be described as follows:

1. Player **II** selects two distinct elements $x_0, x_1 \in M$;
2. Player **II** selects q distinct strings $y_1, \dots y_q$ in $\{x_0, x_1\}^s$;
3. Player **I** selects a string $z \in \{y_1, \dots y_q\}$;
4. Player **II** selects p strings $w_1 \dots w_p$, without knowing z, and wins if and only if $w_i = z$ for some $i = 1 \dots p$.

Now, let γ be the following strategy for Player **II**:

1. Select two fixed distinct elements x_0 and x_1.
2. Select q fixed distinct strings $y_1, \dots, y_q \in \{x_0, x_1\}^s$;
3. Extract p strings $w_1, \dots w_p$ from $\{y_1 \dots y_q\}$, with uniform probability and without repetition - that is, w_1 can be each y_i with probability $1/q$, w_2 can be each remaining element with probability $1/(q-1)$, and so on.

Now, consider any strategy σ for Player **I**: by definition, σ selects an element $z \in \{y_1 \dots y_q\}$, and Player **II** wins if it is one of $\{w_1 \dots w_p\}$.

When **II** uses the behavioral strategy γ,

$$P(H(\phi); \sigma; \gamma) = Prob(w_i = z \text{ for some } i) = Prob(w_1 = z) + Prob(w_1 \neq z \;\&\; w_2 = z) + \dots +$$
$$+ prob(w_1 \neq z \;\&\; w_2 \neq z \;\&\; \dots \;\&\; w_{p-1} \neq z \;\&\; w_p = z) =$$
$$= 1/q + (q-1)/q \cdot 1/(q-1) + \dots + (q-1)/q \cdot (q-2)/(q-1) \cdot 1/(q-p+1) = p/q.$$

Since this holds for any σ, $V(\phi) = \sup_\gamma \inf_\sigma P(H(\phi); \sigma; \gamma) \geq p/q$.

On the other hand, consider the behavioral strategy β for Player **I** which selects the value of z among $y_1 \dots y_q$ with uniform probability, and let τ be any pure strategy for Player **II**.

Then, τ fixes, independently from **I**'s choice of z, some values of $w_1 \dots w_p \in \{y_1 \dots y_q\}$, and

$$P(H(\phi); \beta; \tau) = Prob(z \in \{w_1 \dots w_p\}) = \sum_{i=1}^{p} Prob(z = w_i) = p/q.$$

Thus, $\exists \beta \forall \tau P(H(\phi); \beta; \tau) \leq p/q$ and therefore, by the Minimax Theorem, $\forall \gamma \exists \sigma P(H(\phi); \sigma; \beta) \leq p/q$; but then $V(\phi) \leq p/q$ and, in conclusion, $V(\phi) = p/q$.

- $\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} \subseteq \mathbb{Q} \cap [0, 1]$:

Since the model is finite, in $H^{\mathcal{M}}(\phi)$ there exists a finite set $\{\sigma_1, \sigma_2, \dots \sigma_k\}$ of all pure strategies for

Player **I**, and a finite set $\{\tau_1, \tau_2, \ldots \tau_t\}$ of all pure strategies for Player **II**.

The value $V(\phi)$ is then the result of the following linear programming problem:

$$\begin{array}{ll} \text{maximize} & v, \text{ with respect to the variables } (v, \lambda_1, \ldots \lambda_t), \\ \text{and under the conditions} & \begin{cases} \sum_{i=1}^{t} \lambda_i = 1; & \\ \sum_{i=1}^{t} \lambda_i P(H(\phi); \sigma_j; \tau_i) \geq v, & \text{for all } j = 1 \ldots k; \\ \lambda_i \geq 0, & \text{for all } i = 1 \ldots t. \end{cases} \end{array}$$

where the tuple $(\lambda_1, \ldots \lambda_t)$ represents the probability distribution over pure strategies induced by a uniform behavioral strategy γ.

In other words, the problem of calculating $V(\phi)$ is equivalent to the problem of finding the maximum of the linear function in a $t+1$-dimensional polytope described by the above linear inequalities and equalities with rational coefficients.

It is then clear that the maximum is always reached at one of the vertices of the polytope[8]; but since the linear inequalities have rational coefficients, the coordinates of these vertices are also rational, and thus the value of our target function z at this point will also be rational.

Moreover, the value function always assumes values between 0 and 1, and this concludes the proof.

This for finite models; instead, ([4], §3.5) shows a sentence ϕ such that, for every $r \in \mathbb{R} \cap [0, 1]$ there exists an infinite model $\mathcal{M}_r$ in which $V_\mu(\phi) = r$.

The idea is to build ϕ so that the game $H(\phi)$ is as follows:

1. Player **II** chooses two points a and b on the unit circumference such that the arc $\widehat{ab}$ (obtained starting from a, and moving clockwise until b is reached) has length $r/2\pi$;

2. Player **I** chooses a point c on the circumference, without knowing a and b;

3. Player **II** wins the game if and only if c lies in the arc $\widehat{ab}$; otherwise, Player **I** wins.

It is not difficult to write ϕ and $\mathcal{M}_r$ explicitly, and to verify that the value of ϕ in $\mathcal{M}_r$ is precisely r.

□

[8]This is also the basis of the *simplex method* for solving linear optimization problems.

6 The values of first-order formulas

Let ϕ be a first-order formula. What can one say, in general, about $V_\mu(\phi)$?

First of all, if ϕ is first order then $H(\phi)$ is a game of perfect information, and thus by Kuhn's Theorem it has a behavioral strategy equilibrium.

Then the next theorem shows that the value of ϕ is the relative size of the biggest subteam of μ which satisfies ϕ:

Theorem 6

Let ϕ be a first-order formula with $FV(\phi) = \{x_1 \ldots x_n\}$, let $\mathcal{M}$ be a finite model and let μ be a probabilistic team with $dom(\mu) = FV(\phi)$.

Then

$$V_\mu^{\mathcal{M}}(\phi) = \sum_{s \models_{FO} \phi} \mu(s)$$

that is, the value of ϕ is the probability, under the distribution μ, that a random assignment satisfies classically ϕ.

Proof:

The proof is by structural induction on ϕ:

1. If ϕ is atomic, the result has already been proved.

2. $\phi = \sim\psi$: As the law of the excluded middle holds in first-order logic,

$$V_\mu(\sim \psi) = 1 - V_\mu(\psi) = 1 - \sum_{s \models_{FO} \psi} \mu(s) = \sum_{s \models_{FO} \neg\psi} \mu(s).$$

3. $\phi = \psi \vee \theta$: In this case,

$$V_\mu(\phi) = \sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\} =$$
$$= \sup\left\{p \sum_{s \models_{FO} \psi} \xi_1(s) + (1-p) \sum_{s \models_{FO} \theta} \xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu\right\}.$$

For every assignment s, let λ_s be the fraction of the weight $\mu(s)$ which is assigned to ξ_1, that is,

$$\lambda_s = \frac{p\xi_1(s)}{\mu(s)}.$$

Then, it is easy to verify that

$$p = \sum_s \mu(s)\lambda_s$$

and that

$$\xi_1(s) = \frac{\lambda_s \mu(s)}{p}; \quad \xi_2(s) = \frac{(1-\lambda_s)\mu(s)}{1-p}.$$

Then, every decomposition of μ in $p\xi_1 + (1-p)\xi_2$ is determined by the values of the λ_s; and moreover, every family of values $\lambda_s \in [0,1]$ corresponds to an unique convex decomposition of μ.

Thus,

$$V_\mu(\psi \vee \theta) = \sup \left\{ \sum_{s \models_{FO} \psi} p\xi_1(s) + \sum_{s \models_{FO} \theta} (1-p)\xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu \right\} =$$

$$= \sup \left\{ \sum_{s \models_{FO} \psi} \lambda_s \mu(s) + \sum_{s \models_{FO} \theta} (1-\lambda_s)\mu(s) : \lambda_s \in [0,1] \text{ for all } s \right\}.$$

The supremum is then obtained by letting $\lambda_s = 1$ for all s such that $s \models_{FO} \psi$ and $\lambda_s = 0$ for all s such that $s \not\models_{FO} \psi$ but $s \models_{FO} \theta$; the choice of λ_s for the remaining s does not make any difference, and

$$V_\mu(\psi \vee \theta) = \sum_{s \models_{FO} \psi \vee \theta} \mu(s_i)$$

as required.

4. $\phi = \exists x \psi$: According to the compositional semantics,

$$V_\mu(\exists x \psi) = \sup_F V_{\mu[F/x]}(\psi) = \sup_F \sum_{s[m/x] \models_{FO} \psi} \mu(s) \cdot F(s)(m).$$

The supremum is reached as follows: for every s, if there exists a $c \in M$ such that $s[c/x] \models_{FO} \psi$ then let $F(s)(m) = \delta_{m,c}$.

If this is not the case, the choice of $F(s_i)$ is again of no consequence, and

$$V_\mu(\exists x \psi) = \sum_{s \models_{FO} \exists x \psi} \mu(s).$$

This concludes the proof.

□

7 The value of dependence atomic formulas

In this section, dependence atomic formulas $=(t_1 \dots t_n)$, meaning "The value of t_n is determined by the values of $t_1 \dots t_{n-1}$" ([13]), will be taken in exam.
These formulas can be defined as

$$=(t_1 \dots t_n) := \exists y_1 \dots y_{n-1} (\exists y_n \backslash \{t_1 \dots t_n\}) \bigwedge_{i=1}^{n} (y_i = t_i).$$

Then, the next theorem shows that the value of dependence formulas is the relative size of the biggest subteam of the probabilistic team μ which satisfies the dependency relation:

Theorem 7

$$V_\mu(=(t_1 \dots t_n)) = \sup_{B_i} \sum_{s \in B_i} \mu(s)$$

where B_i ranges over the maximal sets of assignments which satisfy the dependence condition, that is,

$$s, s' \in B_i, t_i\langle s \rangle = t_i\langle s' \rangle \text{ for } i = 1 \dots n-1 \Rightarrow t_n\langle s \rangle = t_n\langle s' \rangle.$$

Proof:
The optimal strategy σ for Player **I** in this game is easy to find out, and is the same no matter what Player **II**'s strategy is: if for some $i \in 1 \dots n$ it holds that $y_i \neq t_i$, Player **I** should select the corresponding conjunct and win, and otherwise he loses no matter what he does.

Thus,

$$V_\mu(\phi) = \sup_{\mathfrak{n}} P(H_\mu(\phi); \sigma; \mathfrak{n}) = \sup_{\mathfrak{n}} \sum_{\tau} \mathfrak{n}(\tau) P(H_\mu(\phi); \sigma; \tau) = \sup_{\tau} P(H_\mu(\phi); \sigma; \tau).$$

An optimal pure strategy τ for Player **II** will always choose the values $s\langle t_1 \rangle \dots s\langle t_{n-1} \rangle$ for $y_1 \dots y_n$; moreover, the value of y_n will be found by choosing a function $f : dom(\mathcal{M})^{n-1} \to dom(\mathcal{M})$ to apply to these values.

Thus,

$$V_\mu(=(t_1 \dots t_n)) = \sup_{f} \sum_{s \in B(f)} \mu(s),$$

where $B(f) = \{s : f(s\langle t_1\rangle, \dots s\langle t_{n-1}\rangle) = s\langle t_n\rangle\}$.

In order to prove the desired result, it then suffices to verify that for every choice of f the set $B(f)$ is contained in one of the maximal sets B_i and that each B_i can be written as $B(f)$ for some f.

This is trivial: as every $B(f)$ satisfies the dependency condition, it is contained in some B_i. Moreover, since every B_i satisfies the dependency condition, it is possible to define a function f_i as

$$f_i(a_1 \dots a_{n-1}) = \begin{cases} s\langle t_n\rangle & \text{if } \exists s \in B_i \text{ s.t. } s\langle t_i\rangle = a_i, i = 1 \dots n-1; \\ \text{some fixed } a_0 & \text{otherwise.} \end{cases}$$

By definition, $B_i \subseteq B(f_i)$; but B_i is maximal, and therefore $B_i = B(f_i)$, as required.

□

8 Approximate Functional Dependency in Database Theory

The concept of functional dependency is also one of the main tools of Database Theory [3], and its definition corresponds exactly to Väänänen's interpretation of the dependence atomic formulas:

Definition 12

Given a relation $r \subseteq A_1 \times \ldots \times A_k$, and two attribute sets $X, Y \subseteq \{A_1, \ldots A_k\}$, it is said that Y is functionally dependent from X if and only if, for all the tuples $u, v \in r$,

$$\pi_i(u) = \pi_i(v)\ \forall A_i \in X \Rightarrow \pi_j(u) = \pi_j(v)\ \forall A_j \in Y$$

where $\pi_i(u)$ is the i-th element of the tuple u.

In this case, one can write that

$$r \models_{DT} X \to Y.$$

Some measures of *Approximate Functional Dependency* have been introduced, one of the most commonly used ones being the g_3 measure of Kivinen and Mannila ([8], [10], [7]):

Definition 13 (g_3 measure)

Let $X \to Y$ be a functional dependency, and let r be a relation over the attribute set R.

Then $G_3(X \to Y, r)$ is the minimum number of tuples that one must remove from r in order to obtain a *relation s satisfying $X \to Y$, that is,*

$$G_3(X \to Y, r) = |r| - \max\{|r'| : r' \subseteq r, r' \models_{DT} X \to Y\}$$

Then, the g_3 measure is defined as

$$g_3(X \to Y, r) = \frac{G_3(X \to Y, r)}{|r|}.$$

This appears quite similar to the semantics of our dependence operator, as the next result illustrates:

Theorem 8

Let r be a relation over $A_1 \times \ldots \times A_n$, and let μ be the corresponding probabilistic team over $\{x_1 \ldots x_n\}$, that is,

$$\mu(s) = \begin{cases} 1/|r| & \text{if } \langle s(x_1), \ldots, s(x_n)\rangle \in r; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all functional dependencies of the form $\{A_{i_1} \ldots A_{i_{q-1}}\} \to \{A_q\}$ it holds that

$$g_3(\{A_{i_1} \ldots A_{i_{q-1}}\} \to \{A_q\}, r) = 1 - V_\mu(=\!(x_{i_1}, \ldots x_{i_q})).$$

Proof:

By Theorem 7,

$$V_\mu(=\!(x_{i_1} \ldots x_{i_q})) = \max_{B_j} \sum_{s \in B_j} \mu(s),$$

where $B_1, \ldots B_k$ are all the maximal sets of assignments which satisfy the dependency condition $=\!(x_{i_1}, \ldots x_{i_q})$. Therefore,

$$V_\mu(=\!(x_{i_1} \ldots x_{i_q})) = \max_{B_j} \sum_{s \in B_j} \mu(s) = \max_{B_j} \sum \{1/|r| : s \in B_j, \langle s(x_1), \ldots s(x_n)\rangle \in r\} =$$

$$= 1/|r| \max_{B_j} |\{s \in B_j, \langle s(x_1), \ldots s(x_n)\rangle \in r\}| = 1/|r| \max\{|r'| : r' \subseteq r, s \models_{DT} \{A_1 \ldots A_{q-1}\} \to \{A_q\}\},$$

where the last equivalence follows from the fact that every subset of r satisfying the dependence condition corresponds to a subset of some B_j.

In conclusion,

$$V_\mu(=(x_{i_1} \dots x_{i_q})) = 1 - g_3(\{A_{i_1} \dots A_{i_{q-1}}\} \to \{A_{i_q}\}, r)$$

as required.

□

9 Further work

a) Dynamic Dependence Logic

In this work, the atomic dependence formulas $=(t_1 \dots t_n)$ have been interpreted as shorthands for the corresponding DF-Logic formulas.
Because of this, it is obvious that

$$V_\mu(=(t_1 \dots t_n)) = V_\mu(\exists y_1 \dots y_{n-1}(\exists y_n \backslash \{y_1 \dots y_{n-1}\}) \bigwedge_i (y_i = t_i))$$

in all finite models $\mathcal{M}$ and for all probabilistic teams μ.

However, although in the non-probabilistic framework it is true that

$$(\exists x_n \backslash \{x_1 \dots x_{n-1}\})\psi \equiv \exists x_n(=(x_1 \dots x_n) \wedge \psi)$$

this equivalence, in general, does not carry over to probabilistic dependence logic - for example, in [4] it is shown that, for some team μ,

$$V_\mu((\exists z \backslash \{\})(=(y) \wedge x = z)) < V_\mu(\exists z(=(z) \wedge =(y) \wedge x = z))$$

The problem lies in the fact that the value of the conjunction of two dependence atomic formulas is not necessarily the measure of the biggest subteam of μ satisfying both of them, since the interpretation of $\psi \wedge \theta$ is "Player **I** decides whether to verify ψ or θ" rather than "Player **I** verifies ψ; if it turns out to be true, he verifies θ too".

Introducing this new kind of conjunction, which semantically seems to correspond to the sequential conjunction of [1], would not increase the expressive power of the logic, but would allow us to recover the above equivalence and, more importantly, to express complex patterns of dependence and independence in terms of conjunctions atomic dependence formulas.

The last part of [4] sketches how to adapt the machinery described in this article to the resulting "Dynamic Probabilistic Dependence Logic", and further investigation on this matter is underway.

b) (Probabilistic) Dependence Logic and Database Theory

The link between Dependence Logic and Database Theory runs certainly deeper than what was hinted to in this article.

For example, one of the problems in database theory and data mining is the search, given a relation r, of a minimal set of dependence relations which entail all functional dependencies occurring in r - in effect, Kivinen and Mannila introduced their measures of approximate functional dependency as a tool for searching efficiently such minimal sets [8].

This and similar questions could, in the author's opinion, benefit from a thorough investigation on the model theory of Dependence Logic and Probabilistic Dependence Logic.

c) Infinite models

One of the main drawbacks of the analysis described in this work is that it only holds for finite models. For infinite models the Minimax Theorem fails, and it may well be that no equilibrium exists; however, it is still possible to define $W(\phi)$ and other values, corresponding to other orders of priority between choice points, and examine their interaction.

10 Funding

This work was supported by the European Science Foundation Eurocores programme LogICCC [FP002 - Logic for Interaction (LINT)].

11 Acknowledgements

The author would like to thank Jouko Väänänen for his comments and corrections.

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