Entanglement entropy and Quantum field theory

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Joint work with John Cardy

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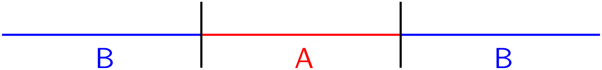
see also cond-mat/0601225, cond-mat/0512586

Entanglement Entropy: what is it?

Quantum system in the ground state $|\Psi\rangle$
The density matrix is $\rho = |\Psi\rangle\langle\Psi|$ $(\mathrm{Tr}\rho^n = 1)$
A measures a subset, B the remainder:



Reduced density matrix $\rho_A = \mathrm{Tr}_B\rho$ $(\rho_B = \mathrm{Tr}_A\rho)$
Entanglement Entropy $\equiv$ Von Neumann entropy of ρ_A:

$$S_A = -\mathrm{Tr}\,\rho_A \ln \rho_A$$

Note: $S_A = S_B$ if ρ corresponds to a pure state

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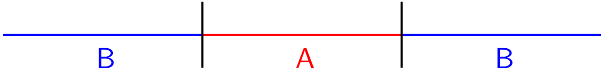

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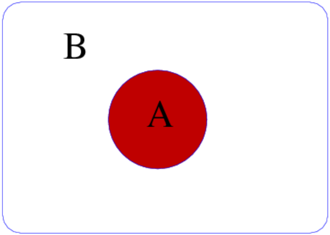
What is the meaning of S_A?

It is the amount of information that A and B are shearing
The amount of quantum correlations between A and B

- Srednicki '93: Area Law

  in a $d+1$ critical $T=0$ QFT
  $S_A \propto \mathcal{A} \Rightarrow S \propto \mathcal{A}\Lambda^{d-1}$
  and for $d=1$?
  $S \propto \ln \Lambda \Rightarrow S \propto \ln \ell\Lambda$
  Non extensive



Historical review

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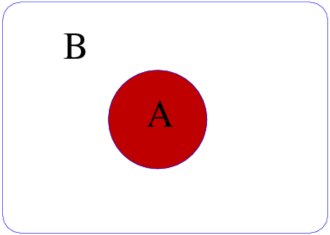

- Holzhey, Larsen, Wilczek '94: In a 1+1D $T=0$ CFT

$$S_A = \frac{c}{3} \ln \frac{\ell}{a}$$

- Black holes?

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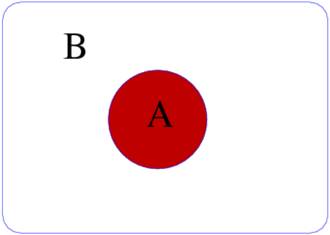

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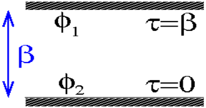
- Black holes?
- Vidal, Latorre, Rico, Kitaev '03: QI perspective

Entanglement Entropy and path integral

Lattice QFT in 1+1 dimensions: $\{\hat{\phi}(x)\}$ a set of fundamental fields with eigenvalues $\{\phi(x)\}$ and eigenstates $\otimes_x |\{\phi(x)\}\rangle$
The density matrix at temperature β^{-1} is ($Z = \mathrm{Tr}\, e^{-\beta \hat{H}}$)

$$\rho(\{\phi_1(x)\}|\{\phi_2(x)\}) = Z^{-1}\langle\{\phi_2(x)\}|e^{-\beta\hat{H}}|\{\phi_1(x)\}\rangle$$

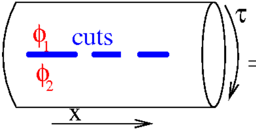
Euclidean path integral:

$\rho =$ 

$$= \int \frac{[d\phi(x,\tau)]}{Z} \prod_x \delta(\phi(x,0)-\phi_1(x)) \prod_x \delta(\phi(x,\beta)-\phi_2(x))\, e^{-S_E}$$

$S_E = \int_0^\beta L_E \, d\tau$, with L_E the Euclidean Lagrangian

The trace sews together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β.
$A = (u_1, v_1), \ldots, (u_N, v_N)$: ρ_A sewing together only those points x which are not in A, leaving open cuts for (u_j, v_j) along the the line $\tau = 0$.

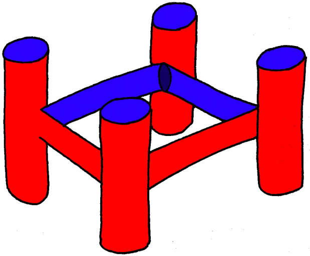
$\rho_A =$ 

$$= \int_{x\in B} [d\phi(x,0)]\delta(\phi(x,\beta) - \phi(x,0))\rho$$

Replica trick

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_A^n$$

$\mathrm{Tr}\rho_A^n$ (for integer n) is the partition function on n of the above cylinders attached to form an $n-$sheeted Riemann surface



$$= \text{``}\rho_A^{ij}\rho_A^{jk}\rho_A^{kl}\rho_A^{li}\text{''}$$

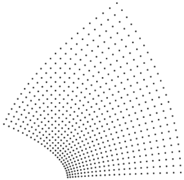
$\mathrm{Tr}\rho_A^n$ has a unique analytic continuation to $\mathrm{Re}\, n > 1$ and that its first derivative at $n = 1$ gives the required entropy:

$$S_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_n(A)}{Z^n}$$

CFT: a remind

- A physical systems at a quantum critical point is scale invariant
  $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = b^{2\Delta_\phi}\langle\phi(b\mathbf{r}_1)\phi(b\mathbf{r}_2)\rangle$ $\qquad$ $\langle\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle = |\mathbf{r}_1-\mathbf{r}_2|^{-2\Delta_\phi}$
- A Hamiltonian that is invariant under translations, rotations, and scaling transformations has usually the symmetry of the larger *conformal* group defined as the set of transformations that do not change the angles.
- In 2D the consequences are extraordinary:

all the analytic functions $f(z)$ are conformal



$$\langle\phi(z_1)\phi(z_2)\rangle = |w'(z_1)w'(z_2)|^{2\Delta_\phi}\langle\phi(w(z_1))\phi(w(z_2))\rangle$$

- Under an arbitrary transformation $x^\mu \to x^\mu + \epsilon^\mu$

$$S \to S + \delta S, \qquad \text{with} \quad \delta S = \int d^2x T^{\mu\nu}\partial_\mu\epsilon_\nu$$

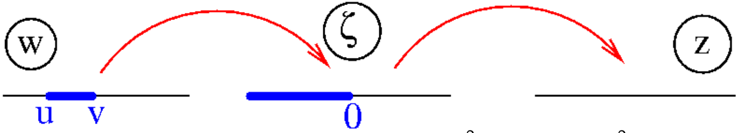
Under a conformal transformation $w \to z$

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\frac{z'''z' - 3/2z''^2}{z'^2}$$

Entropy and CFT

Single interval (u, v). We need $Z_n/Z^n = \langle 0|0\rangle_{\mathcal{R}_n} \Rightarrow$ compute $\langle T(w)\rangle_{\mathcal{R}_n}$

$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



$\langle T(z)\rangle_{\mathbb{C}} = 0 \Rightarrow$ $$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c(1-(1/n)^2)}{24}\frac{(v-u)^2}{(w-u)^2(w-v)^2}$$

To be compared with the Conformal Ward identities
$$\frac{\langle T(w)\Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbb{C}}}{\langle \Phi_n(u)\Phi_{-n}(v)\rangle_{\mathbb{C}}} = \frac{\Delta_\Phi (v-u)^2}{(w-u)^2(w-v)^2}$$

Z_n/Z^n transforms under conformal transformations as n^{th} power of the two point function of a (fake) primary field on the plane with scaling dimension

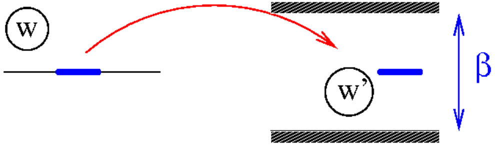
$\Delta_\Phi = \overline{\Delta}_\Phi = \frac{c}{24}\left(1-\frac{1}{n^2}\right) \Rightarrow$ $$\mathrm{Tr}\,\rho_A^n = \frac{Z_n}{Z^n} = c_n\left(\frac{v-u}{a}\right)^{-(c/6)(n-1/n)}$$

Finally with the replica trick $(v-u=\ell)$

$$S_A = \frac{c}{3}\ln\frac{\ell}{a} + c_1'$$

Generalization I

- Finite temperature: map the plane into a cylinder



$$w \to w' = \frac{\beta}{2\pi} \ln w$$

$$S_A = \frac{c}{3} \log\left(\frac{\beta}{\pi a} \sinh \frac{\pi \ell}{\beta}\right) + c_1' \simeq \begin{cases} \frac{\pi c}{3} \frac{\ell}{\beta}, & \ell \gg \beta \quad \text{classical extensive} \\ \frac{c}{3} \log \frac{\ell}{a}, & \ell \ll \beta \quad T = 0 \text{ non} - \text{extensive} \end{cases}$$

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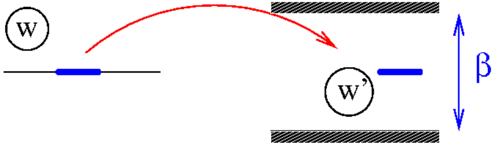

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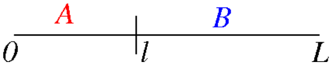
- Finite size: orient the branch cut perpendicular to the axis $\beta \rightarrow L$ and $w \rightarrow iw$

$$S_A \sim \frac{c}{3} \log \left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right) + c_1'$$

It is symmetric under $\ell \rightarrow L - \ell$. It is maximal when $\ell = L/2$

Generalization II

- Open boundaries: semi-infinite system



If $L=\infty$ and $T=0$, it is uniformised by $z=\left(\frac{w-i\ell}{w+i\ell}\right)^{1/n}$

$$\mathrm{Tr}\,\rho_A^n \simeq \tilde{c}_n \left(\frac{2\ell}{a}\right)^{(c/12)(n-1/n)} \Rightarrow S_A \simeq \frac{c}{6}\log\frac{2\ell}{a} + \tilde{c}_1'$$

and at finite temperature β^{-1} and finite size

$$S_A(\beta) \simeq \frac{c}{6}\log\left(\frac{\beta}{\pi a}\sinh\frac{2\pi\ell}{\beta}\right) + \tilde{c}_1' \qquad S_A(L) \simeq \frac{c}{6}\log\left(\frac{2L}{\pi a}\sin\frac{\pi\ell}{L}\right) + \tilde{c}_1'$$

Note: $\tilde{c}_1' - c_1' = g$ boundary entropy [Affleck, Ludwig]

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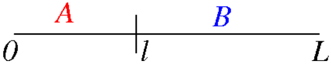

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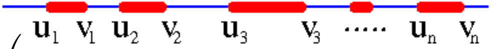
- General case



$$S_A = \frac{c}{3}\left(\sum_{j \le k} \log \frac{v_k - u_j}{a} - \sum_{j<k} \log \frac{u_k - u_j}{a} - \sum_{j<k} \log \frac{v_k - v_j}{a}\right) + N c_1'$$

A similar expression holds in the case of a boundary, with half of the w_i corresponding to the image points

Entanglement Entropy in non-critical systems

What is the scaling of S_A when the correlation length is large but finite?

Following the line of the c-theorem proof, we showed

$$S_A = \mathcal{A}\frac{c}{6}\log\frac{\xi}{a}$$

$\mathcal{A}$ is the number of boundary points between A and B (1D area).

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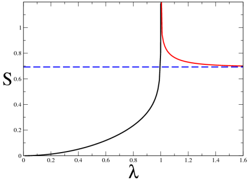
We checked this result in some cases with $\mathcal{A} = 1$

- Gaussian Massive FT
- Ising model in a transverse magnetic field $H_I = -\sum_{n=1}^{L-1}\sigma_n^x - \lambda\sum_{n=1}^{L-1}\sigma_n^z\sigma_{n+1}^z$
  corner transfer matrix ($\epsilon = \epsilon(\lambda)$)

$$S_A = \begin{cases} \epsilon\sum_{j=0}^{\infty}\frac{2j}{1+e^{2j\epsilon}} + \sum_{j=0}^{\infty}\log(1+e^{-2j\epsilon})\,, & \lambda > 1 \\ \epsilon\sum_{j=0}^{\infty}\frac{2j+1}{1+e^{(2j+1)\epsilon}} + \sum_{j=0}^{\infty}\log(1+e^{-(2j+1)\epsilon})\,, & \lambda < 1 \end{cases}$$



For $\lambda \to 1$
$S \simeq \frac{1}{12}\log\xi$

- XXZ model (close to $\Delta = -1$), similar results with $c = 1$
- In the finite slit geometry (i.e. $\mathcal{A} = 2$), it was calculated by Its et al. and Peschel for the XY chain, finding agreement!

Dynamics of Entanglement

How does entanglement evolve from a state that is *not* an eigenstate?

Example

Ising model in a transverse field with $H(h)$:

- Prepare the system in a pure state $|\psi_0\rangle$ (ground state of $H(h_0)$)
- Let it evolve with $H(h)$ with $h \neq h_0$ (at $t = 0$ h has been quenched)

$$|\psi(t)\rangle = e^{-iH(h)t}|\psi_0\rangle \qquad \rho_A(t) = \mathrm{Tr}_B\, e^{-iH(h)t}|\psi_0\rangle\langle\psi_0|e^{iH(h)t}$$

- Clearly, the system does *not* relax to the ground state

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How can we study this problem with QFT?

$$\langle\psi''(x'')|\rho(t)|\psi'(x')\rangle = Z_1^{-1}\langle\psi''(x'')|e^{-itH} \qquad |\psi_0(x)\rangle\langle\psi_0(x)|e^{+itH} \qquad |\psi'(x')\rangle$$

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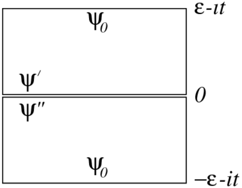
$e^{-\epsilon H}$ makes the path integral convergent!
Is it justified to send it to 0?



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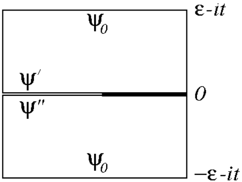

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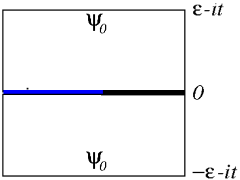

CFT result

The strip geometry of above is obtained by transforming the upper half-plane with $w = (2\epsilon/\pi)\log z$. In the upper half-plane

$$\mathrm{Tr}\,\rho_A^n = \langle \Phi_n \Phi_{-n} \rangle \sim c_n \left(\frac{|z_1 - \bar{z}_2||z_2 - \bar{z}_1|}{|z_1 - z_2||\bar{z}_1 - \bar{z}_2||z_1 - \bar{z}_1||z_2 - \bar{z}_2|} \right)^{2n\Delta_n}$$

$z_1 = \rho^{-1} e^{i\pi\tau_1/2\epsilon} = z_2^{-1}$ where $\rho = e^{\pi\ell/4\epsilon}$
Algebra $\ldots \ell/\epsilon$ and $t/\epsilon \gg 1 \ldots$

$$\mathrm{Tr}\,\rho_A^n = c_n (\pi/2\epsilon)^{4n\Delta_n} \left(\frac{e^{\pi\ell/2\epsilon} + e^{\pi t/\epsilon}}{e^{\pi\ell/2\epsilon} \cdot e^{\pi t/\epsilon}} \right)^{2n\Delta_n}$$

$$\text{Differentiating wrt } n \quad S_A(t) \sim \begin{cases} \dfrac{\pi c t}{6\epsilon} & (t < \ell/2)\,, \\[2ex] \dfrac{\pi c\, \ell}{12\epsilon} & (t > \ell/2)\,, \end{cases}$$

$S_A(t)$ increases linearly until it saturates at $t = \ell/2$.

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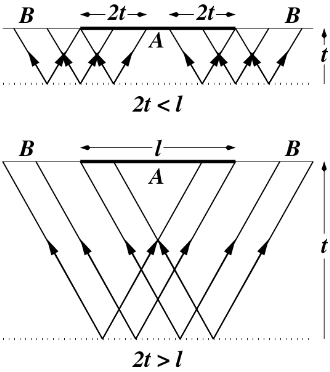
ϵ enters in an essential way:

in a continuum FT $|\psi_0^*\rangle$ has infinitely large energy, the RG invariant conditions must be imposed at $\pm\epsilon - it \pm \tau_0$ (τ_0 extrapolation time)

Physical Interpretation

- $|\psi_0\rangle$ has a very high energy relative to the ground state
- it acts as a source of (quasi-)particles at $t = 0$ that for $t > 0$ move straightly at velocity $\pm v$
- particles emitted from regions size $\sim \tau_0$ are entangled
- particles from far points are incoherent
- The point x in A is entangled with a point $x' \in B$ if a left (right) moving particle arriving at x is entangled with a right (left) moving particle arriving at x'. This can happen only if $x \pm t \sim x' \mp t$



$S_\ell(t)$ is proportional to the length of the interval in x for which this is true
$\Rightarrow S_\ell(t) \propto t$ for $t < \ell/2$ and $S_\ell(t) \propto \ell$ for $t > \ell/2$

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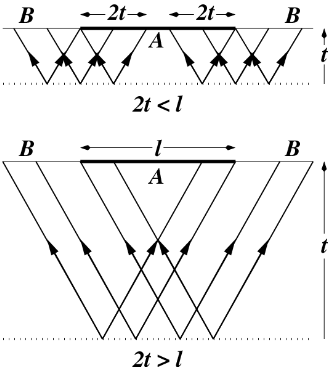

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$\Rightarrow S_\ell(t) \propto t$ for $t < \ell/2$ and $S_\ell(t) \propto \ell$ for $t > \ell/2$
Generalizable to the case when A consists of several disjoint intervals. $S_A(t)$ is not always non-decreasing:

EG, $A =$ regular array of intervals S_ℓ oscillates in a saw-tooth fashion

Lattice calculation

Transverse Ising chain $H_I(h) = -\frac{1}{2}\sum_j [\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z]$

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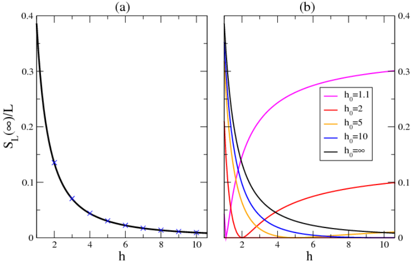
$t \to \infty$, Analytic calculations ...

$$S_\ell = \frac{\ell}{2\pi}\int_0^{2\pi} d\varphi H\left(\frac{1-\cos\varphi(h+h_0)+hh_0}{\Omega_h\Omega_{h_0}}\right)$$

$H(x) = -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2}$

$\Omega_h = \sqrt{h^2+1-2h\cos\varphi}$



always linear in ℓ, not only at the critical point

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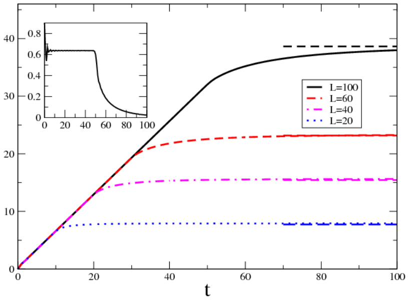
Finite time: numerical calculations

$h_0 = \infty,\ h = 1$:



Linear for $t < \ell/2$!!

But does not saturate at $\ell/2$ (??)

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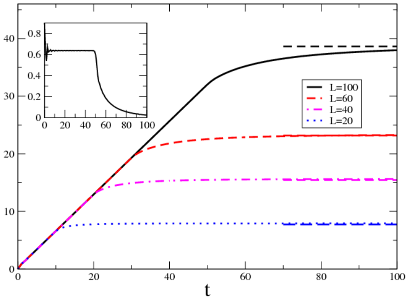

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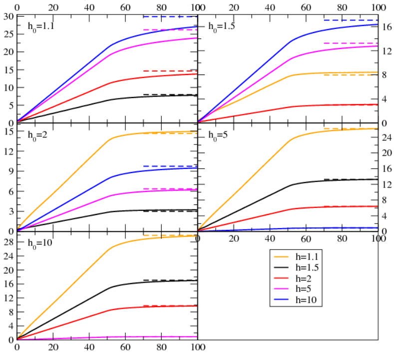
Finite time: numerical calculations
General h, h_0:



Crossover always at $t^* = \ell/2$!!

Discussion

$S_\ell(t)$ increases linearly with time up to $t^* = \ell/2$, but

$$R \equiv \frac{(\partial S_A/\partial t)_{t<t^*}}{2(\partial S_A/\partial \ell)_{t\gg t^*}} \neq 1$$

How we can match these different results?

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How we can match these different results?

There are excitations traveling with speed $v < 1$

Dispersion relation $E(p) \Rightarrow v_p = dE/dp \leq 1$
Let's call $f(p', p'')$ the rate of production of pair of particles

$$\begin{aligned} S_A(t) &\approx \int_{x'\in A} dx' \int_{x''\in B} dx'' \int_{-\infty}^{\infty} dx \int f(p',p'')dp'dp''\delta(x'-x-v_{p'}t)\delta(x''-x-v_{p''}t) \\ &\approx t\int \theta(\ell-(v_{p'}+v_{p''})t)dp'dp''f(p',p'')(v_{p'}+v_{p''}) + \ell\int \theta((v_{p'}+v_{p''})t-\ell)dp'dp''f(p',p'') \end{aligned}$$

$|v_p| \leq 1 \Rightarrow$ the second term is zero if $t < \ell/2 \Rightarrow S_A(t) \propto t$
For $t \to \infty$, the first term is negligible $\Rightarrow S_A \propto \ell$
Unless $|v| = 1$ everywhere (CFT) $S_A \not\propto \ell$ for all $t > t^*$ and

$$R = \frac{\int dp'dp''f(p',p'')[v_{p'}+v_{p''}]}{2\int dp'dp''f(p',p'')} \leq 1$$