Quantum Quenches:
A new way of looking at extended systems

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Joint work with John Cardy

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Statement of the problem

Consider a quantum system in d dimensions

- Lattice of interacting spins (Spin chains)
- Optical lattices
- Quantum field theory

Prepare it at $t = 0$ in a the ground state of an Hamiltonian H_0
Let it evolve unitarily for $t > 0$ with a different Hamiltonian H

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Questions

Does the system reach (in the TD limit) any stationary state?
If yes, which state?
Not the ground state since $\langle H \rangle$ is conserved
How do the correlation functions of local operators depend on t??

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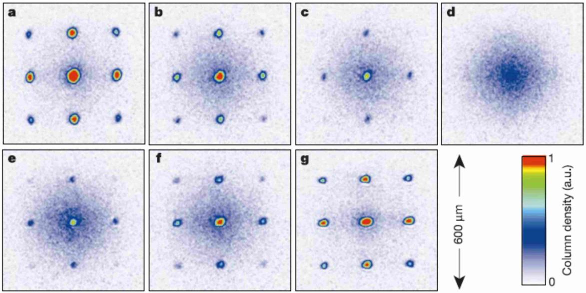
- but, up to now a rather academic question because for most of condensed matter systems the coupling to the environment is unavoidable $\Rightarrow$ decoherence, dissipation
- Today in optical lattices built with cold atoms this effect is minimized



[Greiner et al 2002]

Path integral formulation

$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$, thus

$$\langle \mathcal{O}(t, \{\mathbf{r}_i\})\rangle = \langle\psi_0| e^{iHt} \quad \mathcal{O}(\{\mathbf{r}_i\}) e^{-iHt} \quad |\psi_0\rangle$$

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where $Z = \langle\psi_0|e^{-2\epsilon H}|\psi_0\rangle$.

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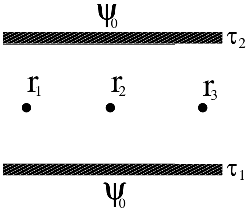
Path integral in imaginary time

$$\frac{1}{Z}\int[d\phi]\mathcal{O}(\{\mathbf{r}_i\})\,e^{-S[\phi]}\delta(\phi(\tau_2)-\psi_0)\delta(\phi(\tau_1)-\psi_0) =$$



continued to $\tau_1 = -\epsilon - it$ and $\tau_2 = \epsilon - it$
We end in a slab of width 2ϵ

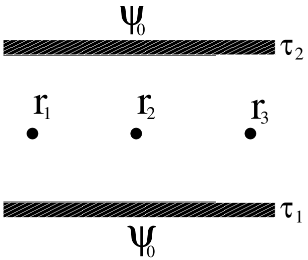
Boundary renormalization group theory

- We want to consider the asymptotic limit when t and the separation $\mathbf{r}_i - \mathbf{r}_j$ are $\gg$ microscopic length and time scale
- if H is at (or close to) a quantum critical point, bulk properties are described by a bulk RG fixed point (or some relevant perturbation thereof)
- Any translational invariant boundary condition flows to one of a number of possible boundary fixed points
- replace $|\psi_0\rangle$ with the appropriate RG-invariant boundary state $|\psi_0^*\rangle$

Problem

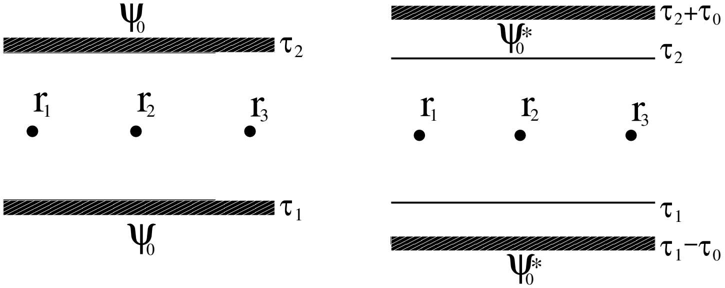
$|\psi_0^*\rangle$ is not normalizable (eg "$|\psi_0^*\rangle = \infty, 0$")

Solution



Solution

Move the goalposts



- impose RG-invariant boundary conditions at $\tau = \tau_1 - \tau_0$ and $\tau = \tau_2 + \tau_0$

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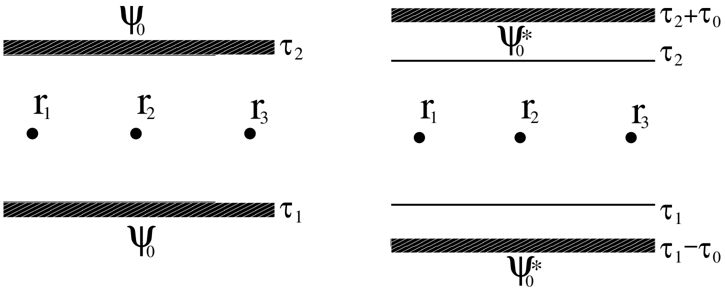

- impose RG-invariant boundary conditions at $\tau = \tau_1 - \tau_0$ and $\tau = \tau_2 + \tau_0$
- $\tau_0 \sim \xi_0$ is the *extrapolation length*: characterizes the distance of the actual boundary state from the RG-invariant one
- The total effect is to replace $\epsilon \to \epsilon + \tau_0$
- The limit $\epsilon \to 0^+$ can now be taken, so width of the slab is $2\tau_0$

Equivalently we consider $0 < \tau < 2\tau_0$, with $\mathcal{O}$ at $\tau = \tau_0 + it$

CFT in 1+1 dimensions

Le us consider the case of a 1D critical system
$\Rightarrow$ the scaling limit of H is described by a $1+1$ CFT
The strip is mapped into the half-plane by

$$w = \frac{2\tau_0}{\pi} \log z$$

In the case of $\mathcal{O}$ product of primary operators

$$\langle \prod_i \Phi_i(w_i) \rangle_{\rm strip} = \prod_i |w'(z_i)|^{-x_i} \langle \prod_i \Phi_i(z_i(w_i)) \rangle_{\rm UHP}$$

x_i is the bulk scaling dimension of Φ_i.

One-point functions

When $\langle\psi_0|\Phi(r)|\psi_0\rangle \neq 0$ (ie if Φ is not vanishing at $t = 0$)

$$\text{in half}-\text{plane} \qquad \langle\Phi(z)\rangle_{\rm UHP} = A_b^\Phi (2{\rm Im}z)^{-x_\Phi}$$

$$\text{in strip} \qquad \langle\Phi(w)\rangle_{\rm strip} = A_b^\Phi \left[\frac{\pi}{4\tau_0}\frac{1}{\sin(\pi\tau/(2\tau_0))}\right]^{x_\Phi}$$

Continuing to $\tau \to \tau_0 + it$

$$\langle\Phi(t)\rangle = A_b^\Phi \left[\frac{\pi}{4\tau_0}\frac{1}{\cosh(\pi t/(2\tau_0))}\right]^{x_\Phi} \sim e^{-\pi x_\Phi t/2\tau_0}$$

Conclusions:

- Exponential relaxation to the ground state value!
- Lifetime $\propto \tau_0/x_\Phi$ (not universal)
- Ratio of lifetimes = ratio of scaling dimensions $\Rightarrow$ universal!!
- Does not apply to the energy density since it is not primary

Two-point functions

$$\langle\Phi(z_1)\Phi(z_2)\rangle_{\rm UHP} = \left(\frac{z_{1\bar2}z_{2\bar1}}{z_{12}z_{\bar1\bar2}z_{1\bar1}z_{2\bar2}}\right)^{x_\Phi} F(\eta) \quad \text{with} \quad \eta = \frac{z_{1\bar1}z_{2\bar2}}{z_{1\bar2}z_{2\bar1}}$$

F is universal but depends in detail on the particular BCFT
Mapping the strip and continuing as before

$$\langle\Phi(r,t)\Phi(0,t)\rangle \sim \left(\frac{e^{\pi r/2\tau_0}+e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0}\cdot e^{\pi t/\tau_0}}\right)^{x_\Phi} F(\eta) \quad \text{now} \quad \eta \sim \frac{e^{\pi t/\tau_0}}{e^{\pi r/2\tau_0}+e^{\pi t/\tau_0}}\,.$$

For $2t-r \gg \tau_0$, $\eta\sim 1$. For $r-2t\gg\tau_0, \eta\sim e^{\pi(t-r/2)/\tau_0}$ and $F(\eta)\sim(A_b^\Phi)^2\eta^{x_b}$

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Conclusions:

- $\langle\Phi(r,t)\Phi(0,t)\rangle \sim \begin{cases} e^{-\pi x_\Phi r/\tau_0} & \text{for } t > r/2 \\ (A_b^\Phi)^2 e^{-\pi x_\Phi t/\tau_0} \times e^{-\pi x_b(r/2-t)/\tau_0} & \text{for } t < r/2 \end{cases}$

  note that if $\langle\Phi\rangle \neq 0$, $x_b = 0$ and the second is just $\langle\Phi(t)\rangle^2$
- If $\langle\Phi\rangle \neq 0$, connected correlations vanish for $t < r/2$
- Correlations saturate to t-independent forms for $t > r/2$
- The decay in r is exponential and not a power-law

- 2-time correlation function

$$\langle \Phi(r,t)\Phi(0,s)\rangle \sim \begin{cases} e^{-\pi x(t+s)/4\tau_0} & \text{for } r > t+s \\ e^{-\pi x r/4\tau_0} & \text{for } t-s < r < t+s \\ e^{-\pi |t-s|/4\tau_0} & \text{for } r < |t-s| \end{cases}$$

  Note that the system is not aging

- 1-point function in a semi-infinite chain

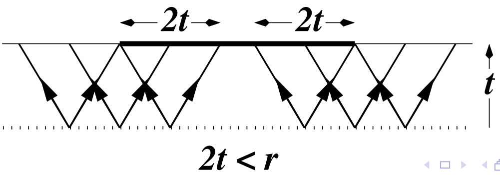
$$\langle \Phi(r,t)\rangle \sim \begin{cases} e^{-\pi x t/2\tau_0} & \text{for } t < r \\ e^{-\pi x r/2\tau_0} & \text{for } t > r \end{cases}$$

  Note $t^* = r$

- Many other, if you wish . . .

Physical interpretation (I)

- $|\psi_0\rangle$ has extensively higher energy than the ground state
- it acts as a source of (quasi-)particles at $t = 0$ that for $t > 0$ move straightly at velocity $\pm v$
- particles emitted from regions size $\sim \tau_0$ are entangled
- incoherent particles arriving at r from separated initial points cause relaxation of local observables to the ground-state value
- light-cone effect: local observables with separation r become correlated when left- and right-moving particles originating from the same spatial region $\sim \tau_0$ can first reach them
- if all the particles moves at speed v, correlations are then frozen for $t > r/2v$



Moving to 1D solvable models

Above arguments rested on two main assumptions

1. analytically continuing to real time the asymptotic CFT results in imaginary time
2. the extrapolation length

Important to check these assumptions in solvable models:

1. Chain of harmonic oscillators (free bosonic theory)
2. Ising-XY model in a transverse field (free fermion)

Taking into account both the effects of the lattice and of a finite gap

Chain of harmonic oscillators

$$H = \frac{1}{2} \sum_r \left(\pi_r^2 + m^2 \phi_r^2 + \sum_j \omega_j^2 (\phi_{r+j} - \phi_r)^2 \right) = \sum_k \Omega_k a_k^\dagger a_k$$

and similarly H_0 with Ω_{0k}

Integrating Heisenberg equation of motions

$$\langle \phi_r(t) \phi_0(t) \rangle - \langle \phi_r(0) \phi_0(0) \rangle = \int_{\mathrm{BZ}} e^{ikr} \frac{(\Omega_{0k}^2 - \Omega_k^2)(1 - \cos(2\Omega_k t))}{\Omega_k^2 \Omega_{0k}} dk$$

In the massless case $\Omega_k \sim v|k|$ for $k \to 0$ and for $m_0 \to \infty$

$$\langle \phi_r(t) \phi_0(t) \rangle = \begin{cases} 0 & \text{for } t < r/2v \\ m_0(t - r/2v) & \text{for } t > r/2v \end{cases}$$

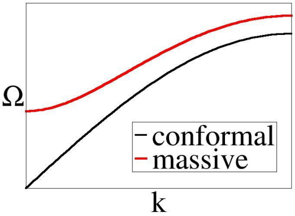
Taking $e^{iq\phi}$ (that is the primary field)

$$\langle e^{iq\phi(r,t)} e^{-iq\phi(0,t)} \rangle = \begin{cases} e^{-m_0 q^2 t} & \text{for } t < r/2 \\ e^{-m_0 q^2 r/2} & \text{for } t > r/2 \end{cases}$$

that confirms the CFT results with $\tau_0 \sim m_0^{-1}$ and $x_\Phi \propto q^2$

General dispersion relation and Physical interpretation (II)

This simple model allows us to understand the effect of general dispersion relations



- large r and t behavior given by the stationary phase approximation $2r/t = d\Omega_k/dk = v_k$
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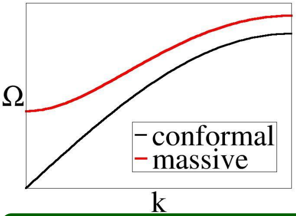

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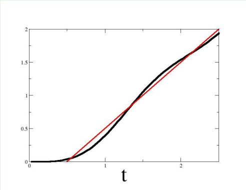
Example ($\Omega_k^2 = m^2 + 2(1 - \cos k)$)

- Lattice dispersions give oscillatory power-law decaying corrections CFT $+t^{-3/2}\cos(\Omega_\pi t + \pi/4)$
- For a quench to a gapped lattice H the fastest particle has $k \neq 0$ $\Rightarrow$ spatial oscillations



A quick look to the XY-Ising chain

$$H = \sum_{r=1}^{N} \Big(\frac{(1+\gamma)}{2} \sigma_r^x \sigma_{r+1}^x + \frac{(1-\gamma)}{2} \sigma_r^y \sigma_{r+1}^y - h\sigma_r^z \Big)$$

For all $\gamma \neq 0$ is in the UC of Ising model ($\gamma = 1$)

- $h_0 = \infty,\ h = 1$ [McCoy and Wu, 1971]

$$m_z(t) - 1/2 \sim t^{-3/2} \cos(4t)$$

- $h_0 = \infty, 0,\ h = 1 \Rightarrow G_r(t = \infty) = 2^{-r}$ [Sengupta et al., 2004]
- Lots of results from Igloi and Rieger 2000, eg

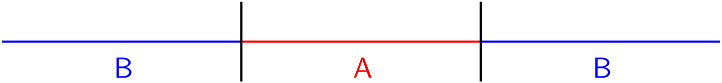
$$\langle \sigma_r^z(t) \sigma_0^z(t) \rangle \sim \begin{cases} 0 & \text{for } t < r/2 \\ r^2 & \text{for } t > r/2 \end{cases}$$

Everything agrees with our predictions

Entanglement entropy (back to the past)

[PC, J Cardy, cond-mat/0503393]

A measures a subset, B the remainder:



Reduced density matrix $\rho_A = \mathrm{Tr}_B \rho$ ($\rho = |\Psi\rangle\langle\Psi|$)
Entanglement Entropy $\equiv$ Von Neumann entropy of ρ_A:

$$S_A = -\mathrm{Tr}\, \rho_A \ln \rho_A$$

In the ground-state of a CFT $S_A = \frac{c}{3} \log \ell$.
What about time evolution?

$$S_A(t) \sim \begin{cases} \dfrac{\pi c t}{6\tau_0} & (t < \ell/2) \\[2ex] \dfrac{\pi c\, \ell}{12\tau_0} & (t > \ell/2) \end{cases}$$

Again understandable in terms of entangled particles emitted from the initial state

Effective Temperature?

Fact: All the correlation functions for long time are thermal-like

A common belief: a region of dimension r can be thermalized by the rest of the system that acts as a bath

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Intriguing idea: Generalized Gibbs ensemble [Rigol et al. 07]

$$\rho = Z^{-1} e^{-\sum_m \lambda_m I_m}, \qquad Z = \mathrm{Tr}\, e^{-\sum_m \lambda_m I_m}. \qquad \mathrm{Tr}\, I_m \rho = \langle I_m \rangle_{t=0}$$

Note: This is only an effective ensamble, the actual state is pure

When the model in diagonal in k space $\beta_{\rm eff}(k)\Omega_k = \lambda_k$
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Open question: Can we say something in general?

Higher dimensions (briefly)

Physical method and picture generalize straightforwardly, eg

- Disordered initial state, two-point function, Gaussian result

$$\langle\phi(\mathbf{r},t)\phi(0,t)\rangle \sim \begin{cases} r^{-(d-2)}e^{-\pi(t-r/2v)/\tau_0} & \text{for } t < r/2v \\ r^{-(d-2)} & \text{for } t > r/2v \end{cases}$$

- Disordered initial state, $\mathcal{O}=\phi^2$, first order ϵ expansion

$$\langle\mathcal{O}(t)\rangle - \langle\mathcal{O}(t=\infty)\rangle \sim e^{-(1-1/\nu)\pi t/2\tau_0}$$

  Similarly in large N

- Ordered initial state, magnetization evolution:
  The Gaussian profile involves elliptic functions
  $\Rightarrow$ non-decaying oscillations with period $\sim \tau_0$
  Is this recurrence modified by fluctuations?

Open questions

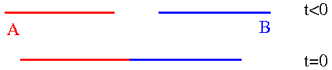
Solvable examples so far are essentially free field theory
In general, we can state that causality implies that correlations should not change before $t \sim r/2v_{\max}$, but what happens then?

1. what about integrable massive theories with non trivial scattering?
2. what about non-integrable theories?
3. what about the presence of disorder?
4. what about different initial state (eg different only locally from the ground-state)?
5. what about finite temperature?
6. what about quantum dissipation?

Local Quench I: Correlations

We physically cut a spin chain into two A and B parts

- At time $t = 0$ we join the two parts
- How do correlation functions and entanglement evolve?



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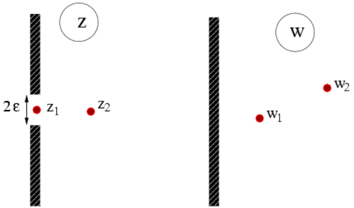
Path integral representation:

$$w = \frac{z}{\epsilon} + \sqrt{\left(\frac{z}{\epsilon}\right)^2 + 1}$$



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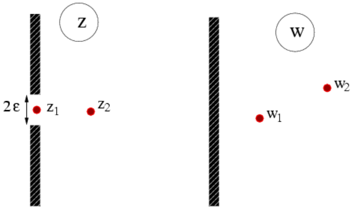


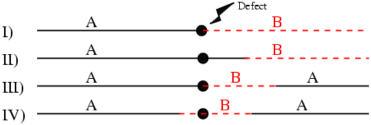
One-point function:

$$\langle \Phi(r,t) \rangle = \begin{cases} A_b^{\Phi} (2r)^{-x_\Phi} & t < r \\ A_b^{\Phi} \left(\frac{\epsilon}{2(t^2 - r^2)}\right)^{x_\Phi} & t > r \end{cases}$$

Two-point: For short and long t what expected. For $|r_2| < t < r_1$

$$\langle \Phi(r_1,t)\Phi(r_2,t) \rangle = \left[\frac{(r_1 + r_2)(r_2 + t)}{(r_1 - r_2)(r_1 - t)} \frac{\epsilon}{4 r_1 (t^2 - r_2^2)} \right]^{x_\Phi} F\left(\frac{2 r_1 (r_2 + t)}{(r_1 + r_2)(r_1 + t)} \right)$$

Bipartitions:



Local Quench II: Entanglement

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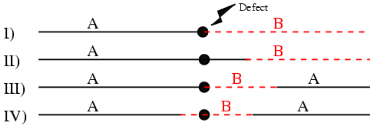

- I) $S_A = \frac{c}{6} \log \frac{t^2+\epsilon^2}{a\epsilon/2} + \tilde{c}_1' \Rightarrow \begin{cases} S_A = \frac{c}{3} \log t & t \gg \epsilon \\ \epsilon = \frac{a}{2} e^{-6\tilde{c}_1'/c} & \text{from } t = 0 \end{cases}$

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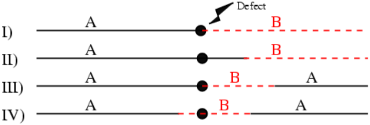

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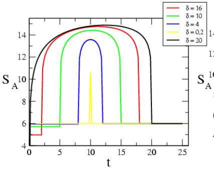
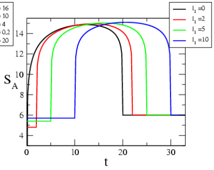
- III) & IV)

$$S_A \simeq \begin{cases} \frac{c}{3}\ln t + \frac{c}{6}\ln \ell + \frac{c}{6}\ln\frac{\ell - t}{\ell + t} & t < \ell \\ \frac{c}{3}\ln \ell & t > \ell \end{cases}$$





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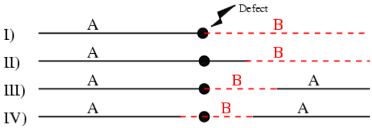
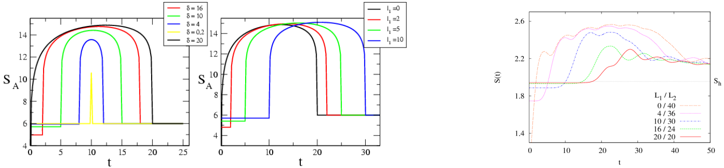


- I) $S_A = \frac{c}{6} \log \frac{t^2+\epsilon^2}{a\epsilon/2} + \tilde{c}_1' \Rightarrow \begin{cases} S_A = \frac{c}{3} \log t & t \gg \epsilon \\ \epsilon = \frac{a}{2} e^{-6\tilde{c}_1'/c} & \text{from } t = 0 \end{cases}$
- III) & IV)

$$S_A \simeq \begin{cases} \dfrac{c}{3} \ln t + \dfrac{c}{6} \ln \ell + \dfrac{c}{6} \ln \dfrac{\ell - t}{\ell + t} & t < \ell \\ \dfrac{c}{3} \ln \ell & t > \ell \end{cases}$$



- Everything agrees with Eisler and Peschel '07