CHAPTER 1

Braided categories

1. Braided categories

1.1. Definitions. A functorial system of isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$ in a category $(\mathcal{C}, \otimes, a)$ with the tensor is called a *commutativity constraint* if it satisfies the *hexagon identities*

$$\begin{aligned} c_{A\otimes B,C} &= a_{C,A,B} \circ (c_{A,C} \otimes id_B) \circ a^{-1}_{A,C,B} \circ (id_A \otimes c_{B,C}) \circ a_{A,B,C} \\ c_{A,B\otimes C} &= a^{-1}_{B,C,A} \circ (id_A \otimes c_{B,C}) \circ a_{B,A,C} \circ (c_{A,B} \otimes id_C) \circ a^{-1}_{A,B,C} \end{aligned}$$

or, the commutativity of the hexagon diagrams:

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a_{A,B,C}"] \arrow[d, "c_{A\otimes B,C}"'] & A \otimes (B \otimes C) \arrow[r, "id_A \otimes c_{B,C}"] & A \otimes (C \otimes B) \arrow[d, "a^{-1}_{A,C,B}"] \\
C \otimes (A \otimes B) & (C \otimes A) \otimes B \arrow[l, "a_{C,A,B}"'] & (A \otimes C) \otimes B \arrow[l, "c_{C,A} \otimes id_B"']
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "a^{-1}_{A,B,C}"] \arrow[d, "c_{A,B\otimes C}"'] & (A \otimes B) \otimes C) \arrow[r, "c_{A,B} \otimes id_C"] & (B \otimes A) \otimes C \arrow[d, "a_{C,A,B}"] \\
(B \otimes C) \otimes A) & B \otimes (C \otimes A) \arrow[l, "a^{-1}_{B,C,A}"'] & B \otimes (A \otimes C) \arrow[l, "id_B \otimes c_{A,C}"']
\end{tikzcd} \tag{2}$$

The functoriality means that it commutes with morphisms in $\mathcal{C}$, i.e. the diagram

$$\begin{tikzcd}
A \otimes B \arrow[r, "c_{A\otimes B}"] \arrow[d, "f"'] & B \otimes A \arrow[d, "g"] \\
C \otimes D \arrow[r, "c_{C\otimes D}"] & D \otimes C
\end{tikzcd} \tag{3}$$

is commutative for all $A, B, C, D \in Ob(\mathcal{C})$, and all $f : A \to C$ and $g : B \to D$.

A commutativity constraint on a category with tensor product defines an isomorphisms of functors:

$$\otimes \xrightarrow{c_\pm} \otimes^{op}$$

where $\otimes^{op}$ is the opposite tensor product: $A \otimes^{op} B = B \otimes A$ with $(c_+)_{A,B} = c_{A,B} : A \otimes B \to B \otimes A$ and $(c_-)_{A,B} = c^{-1}_{B,A} : A \otimes B \to B \otimes A$.

Definition 1.1 (braided monoidal category). *A monoidal category with a commutativity constraint is called a* braided monoidal category.

Using the commutativity constraint one can define the monoidal functor $\Psi : \mathcal{C} \to \mathcal{C}$, which acts as the identity functor on object and morphisms of $(\mathcal{C}, \otimes)$, but the monoidal structure given by the composition of commutativity constraints:

$$\Psi_{A,B} = c_{BA} \circ c_{AB} : A \otimes B \to A \otimes B$$

If this functor is equal to the identity, the category is called symmetric.

Definition 1.2. *A category which has a commutativity constraint satisfying* $c_{A,B}c_{B,A} = id_A$ *is called a* symmetric category*.*

As we will see in the next section, the is an important class of categories where the two functors, Ψ and id are not equal but isomorphic.

An important consequence of the definition is that in a braided monoidal category the *Yang-Baxter equation* holds:

$$\begin{aligned} &a^{-1}_{C,B,A} \circ (c_{B,C} \otimes id) \circ a_{B,C,A} \circ (id \otimes c_{A,C}) \circ a^{-1}_{B,A,C} \circ (c_{A,B} \otimes id) \\ &\qquad = (id \otimes c_{A,B}) \circ a^{-1}_{C,A,B} \circ (c_{A,C} \otimes id) \circ a_{A,C,B} \circ (id \otimes c_{B,C}) \circ a^{-1}_{A,B,C} \end{aligned}$$

This identity follows from the fucntoriality of the commutativity constraint

$$c_{B \otimes A,C} \circ (c_{A,B} \otimes id) = (id \otimes c_{A,B}) \circ c_{A \otimes B,C}$$

and from the hexagon identities.

Lemma 1.1. *If the monoidal category* $\mathcal{C}$ *is braided, so is* $\mathcal{C}_{str}$*.*

For strict monoidal categories the Yang-Baxter equation is the identity

$$(c_{B,C} \otimes id) \circ (id \otimes c_{A,C}) \circ (c_{A,B} \otimes id) = (id \otimes c_{A,B}) \circ (c_{A,C} \otimes id) \circ (id \otimes c_{B,C})$$

For a braided category the Grothendieck ring is commutative. Indeed $[U][V] = [U \otimes V] = [V \otimes U] = [V][U]$. The second equality is due to the existence of the commutativity constraint.

1.2. Rigid braided monoidal categories.

Definition 1.3. *A rigid monoidal category* $(\mathcal{C}, \otimes, a, 1)$ *with the structure of a braided monoidal category a* rigid monoidal braided category *if the commutativity constraint satisfies the idenity* $c_{A^*B^*} = (c_{BA})^*$*.*

Example 1.1. *The category of finite dimensional vector spaces is a braided monoidal category with the commutativity constraint* $c_{V,W} : V \otimes W \to W \otimes V : c_{V,W}(v \otimes w) = w \otimes v$*. It is easy to see that it is also rigid braided monoidal.*

Example 1.2. *Category of finite dimensional modules over* $U(\mathfrak{g})$ *is a braided monoidal category with the commutativity constraint* $c_{V,W} : V \otimes W \to W \otimes V : c(v \otimes w) = w \otimes v$*. Clearly this is map commutes with the diagonal action of the universal enveloping algebra. Again, it is easy to check that this category is rigid braided monoidal.*

More generally, if (H, R) *is a quasitriangular Hopf algebra, its category of finite dimensional modules is rigid braided monoidal. The rigid monoidal structure on this category was given in the previous section. The braiding is given by the mapping* $c_{V,W} : V \otimes W \to W \otimes V$

$$c_{V,W}(x \otimes y = \sigma \circ (\pi_V \otimes \pi_W)(R)(x \otimes y)$$

where $\sigma(x \otimes y) = y \otimes x$ *is the permutation operator.*

More interesting examples will be constructed below.

1.3. Double duals in a rigid braided monoidal category.

Lemma 1.2. *In section... we construct the functor from the category of colored tangles to a rigid braided monoidal category. All identities between morphisms given below are images of simple isotopies between diagram of tangles. We include the corresponding pictures in this section.*

(1) *The morphism*

$$B \xrightarrow{id\otimes i_A} B\otimes(A\otimes A^*) \xrightarrow{a^{-1}} (B\otimes A)\otimes A^* \xrightarrow{c^{\pm 1}\otimes id} (A\otimes B)\otimes A^*$$

is equal to the morphism

$$B \xrightarrow{i_A\otimes id} (A\otimes A^*)\otimes B \xrightarrow{a} A\otimes(A^*\otimes B) \xrightarrow{id\otimes c^{\mp 1}} A\otimes(B\otimes A^*) \xrightarrow{a^{-1}} (A\otimes B)\otimes A^*$$

(2) *The morphism*

$$(A^*\otimes B)\otimes A \xrightarrow{a} A^*\otimes(B\otimes A) \xrightarrow{id\otimes c^{\pm 1}} A^*\otimes(A\otimes B) \xrightarrow{a^{-1}} (A^*\otimes A)\otimes B \to B$$

is equal to

$$(A^*\otimes B)\otimes A \xrightarrow{c^{\mp 1}\otimes id} (B\otimes A^*)\otimes A \xrightarrow{a} B\otimes(A^*\otimes A) \to B$$

Proof. Follows from the compatibility of $\mathbb{1}$ and c together with the hexagon identities using the diagram

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "l_B"'] & B \arrow[d, "r_B"] \\
\mathbb{1}\otimes B \arrow[r] \arrow[d] & B\otimes\mathbb{1} \arrow[d] \\
(A\otimes A^*)\otimes B & B\otimes(A\otimes A^*) \arrow[l, "c_{B,A\otimes A^*}"'] \arrow[d, "a^{-1}"'] \\
A\otimes(A^*\otimes B) \arrow[u, "a^{-1}"] & (B\otimes A)\otimes A^* \arrow[d, "c_{B,A}\otimes id_{A^*}"'] \\
A\otimes(B\otimes A^*) \arrow[u, "id_A\otimes c_{B,A^*}"] & (A\otimes B)\otimes A^* \arrow[l, "a"']
\end{tikzcd}$$

□

Define morphism $u_A = e_A\otimes id_{A^{**}}\circ c_{A,A^*}\otimes id_{A^{**}}\circ a^{-1}\circ id_A\otimes i_{A^{**}} : A\to A$. In other words u_A is the following composition map

$$A \xrightarrow{id\otimes i_{A^*}} A\otimes(A^*\otimes A^{**}) \xrightarrow{a^{-1}} (A\otimes A^*)\otimes A^{**} \xrightarrow{c_{A,A^*}\otimes id} (A^*\otimes A)\otimes A^{**} \xrightarrow{e_A\otimes id} A^{**}$$

The corresponding diagram of a colored tangle is given on Fig. 1

Here and below we will not write the evaluation and injection morphisms in compositions when it is clear by the context where they should be. We will also drop the name of the morphism when it is clear that it should be an associator, or a commutativity morphism.

The following lemma describes some basic properties of u.

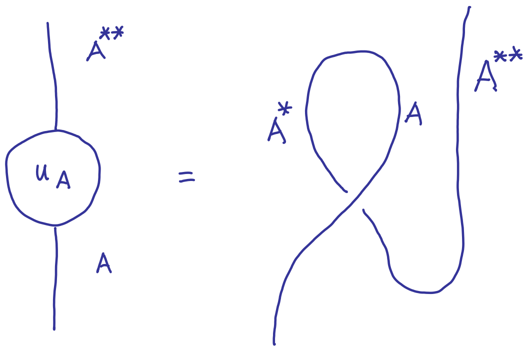

FIGURE 1. Morphisms u_A

LEMMA 1.3. (1) *u_A equals the following morphisms, see also Fig. 2 and Fig. 3:*

(a) $(e_A \otimes id_{A^{**}}) \circ \left(id_{A^*} \otimes c^{-1}_{A,A^{**}}\right) \circ a_{A^*,A^{**},A} \circ (i_{A^*} \otimes id_A)$:

$$A \to (A^* \otimes A^{**}) \otimes A \to A^* \otimes (A^{**} \otimes A) \xrightarrow{id\otimes c^{-1}_{A,A^{**}}} A^* \otimes (A \otimes A^{**}) \to (A^* \otimes A) \otimes A^{**} \to A^{**}$$

(b) $(id_{A^{**}} \otimes e_A) \circ a_{A^{**},A^*,A} \circ (c_{A^*,A^{**}} \otimes id_A) \circ (i_{A^*} \otimes id_A)$

$$A \to (A^* \otimes A^{**}) \otimes A \xrightarrow{c_{A^*,A^{**}}\otimes id} (A^{**} \otimes A^*) \otimes A \to A^{**} \otimes (A^* \otimes A) \to A^{**}$$

(2) *We also have the equalities*

(a)

$$(u_A \otimes id) \circ i_A = c_{A^*,A^{**}} \circ i_{A^*} : \mathbb{1} \to A^{**} \otimes A^*$$

(b)

$$e_{A^*} \circ (u_A \otimes id) = e_A \circ c_{A,A^*} : A \otimes A^* \to \mathbb{1}$$

The corresponding diagrams are given on Fig. 4

PROOF. The proof of the first part follows from lemma 1.2. The identity (a) from the second part can be proven by the following simple computation.

$$\begin{aligned}(u_A \otimes id) \circ i_A &= (id_{A^{**}} \otimes e_A \otimes id_{A^*}) \circ (c_{A^*,A^{**}} \otimes id_A \otimes id_{A^*}) \circ (i_{A^*} \otimes i_A) \\ &= (id_{A^{**}} \otimes e_A \otimes id_{A^*}) \circ (id_{A^{**}} \otimes id_{A^*} \otimes i_A) \circ c_{A^*,A^{**}} . i_{A^*} \\ &= (id_{A^{**}} \otimes id_{A^*}) \circ c_{A^*,A^{**}} \circ i_{A^*}\end{aligned}$$

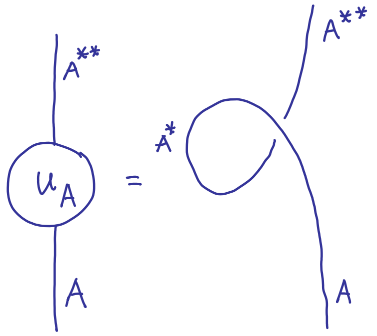

Figure 2. Lemma 1.3 identity 1a

Similar computation proves the identity (b) from the second part. □

Proposition 1.4. *u_A is an isomorphism for each object A.*

Proof. As it was mentioned above, we will suppress brackets in the tensor products when they can be reconstructed in an obvious way.

Let us prove that the inverse map to u is given as

$$A^{**} \to A^{**} \otimes (A \otimes A^*) \to (A^{**} \otimes A) \otimes A^* \xrightarrow{a^{-1}\circ(c\otimes id)} A \otimes (A^{**} \otimes A^*) \to A$$

The diagram representing u^{-1} is given on Fig. 5.

We can write the composition $u_A^{-1} \circ u_A$ as

$$\begin{aligned} A \to A^* \otimes A^{**} \otimes A \to A^* \otimes A^{**} \otimes A \otimes A \otimes A^* \\ \xrightarrow{id\otimes c^{-1}\otimes id\otimes id} A^* \otimes A \otimes A^{**} \otimes A \otimes A^* \\ \xrightarrow{id\otimes id\otimes c\otimes id} A^* \otimes A \otimes A \otimes A^{**} \otimes A^* \to A \end{aligned}$$

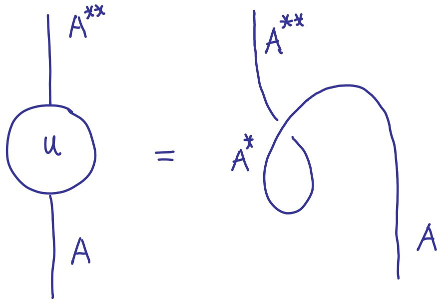

FIGURE 3. Lemma 1.3 identity 1b

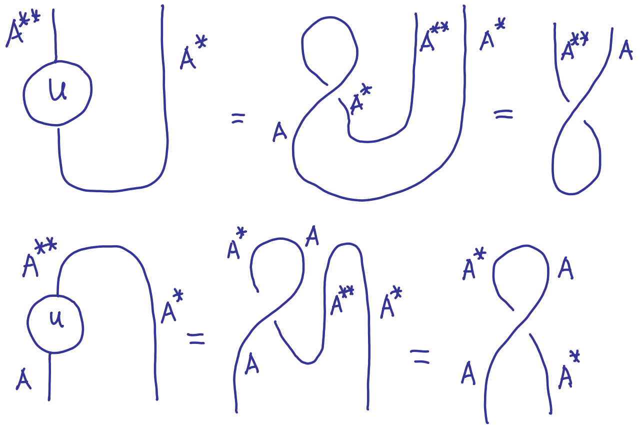

FIGURE 4. Lemma 1.3 identities 2a and 2b

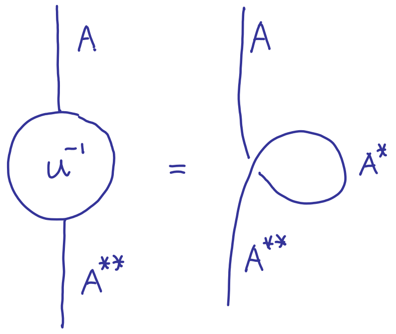

Figure 5. Morphisms u^{-1}

Introducing a cancelling pair c, c^{-1} we get

$$
\begin{aligned}
A \to A^* \otimes A^{**} \otimes A \otimes A \otimes A^* \\
\xrightarrow{id\otimes c^{-1}\otimes id\otimes id} A^* \otimes A \otimes A^{**} \otimes A \otimes A^* \\
\xrightarrow{id\otimes id\otimes c\otimes id} A^* \otimes A \otimes A \otimes A^{**} \otimes A^* \\
\xrightarrow{id\otimes c\otimes id\otimes id} A^* \otimes A \otimes A \otimes A^{**} \otimes A^* \\
\xrightarrow{id\otimes c^{-1}\otimes id\otimes id} A^* \otimes A \otimes A \otimes A^{**} \otimes A^* \to A
\end{aligned}
$$

which allows us to apply the Yang-Baxter equation to get

$$
\begin{aligned}
A \to A^* \otimes A^{**} \otimes A \otimes A \otimes A^* \\
\xrightarrow{id\otimes id\otimes c\otimes id} A^* \otimes A^{**} \otimes A \otimes A \otimes A^* \\
\xrightarrow{id\otimes c\otimes id\otimes id} A^* \otimes A \otimes A^{**} \otimes A \otimes A^* \\
\xrightarrow{id\otimes id\otimes c^{-1}\otimes id} A^* \otimes A \otimes A \otimes A^{**} \otimes A^* \\
\xrightarrow{id\otimes c^{-1}\otimes id\otimes id} A^* \otimes A \otimes A \otimes A^{**} \otimes A^* \to A
\end{aligned}
$$

Using lemma 1.2 for i_A and for e_{A^*}we get

$$\begin{aligned} A \to A^* \otimes A^{**} \otimes A \otimes A^* \otimes A & \\ \xrightarrow{id\otimes id\otimes id\otimes c^{-1}} & A^* \otimes A^{**} \otimes A \otimes A \otimes A^* \\ \xrightarrow{id\otimes c\otimes id\otimes id} & A^* \otimes A \otimes A^{**} \otimes A \otimes A^* \\ \xrightarrow{id\otimes id\otimes id\otimes c} & A^* \otimes A \otimes A^{**} \otimes A^* \otimes A \\ \to & A^* \otimes A \otimes A \\ & \xrightarrow{id\otimes c^{-1}} A^* \otimes A \otimes A \to A \end{aligned}$$

where we used

$$\begin{aligned} &(e_A \otimes id \otimes e_{A^*}) \circ (id \otimes c^{-1} \otimes id \otimes id) \circ ((id \otimes id \otimes c^{-1} \otimes id) \\ &= (e_A \otimes id).(id \otimes c^{-1}) \circ ((id \otimes id) \otimes ((id \otimes e_{A^*}) \circ (c^{-1} \otimes id))) \\ &= (e_A \otimes id) \circ (id \otimes c^{-1}) \circ ((id \otimes id) \otimes ((e_{A^*} \otimes id) \circ (id \otimes c))) \end{aligned}$$

So we have

$$\begin{aligned} A \to A \otimes A^* \otimes A \to A^* \otimes A^{**} \otimes A \otimes A^* \otimes A & \\ \xrightarrow{id\otimes c\otimes id\otimes id} & A^* \otimes A \otimes A^{**} \otimes A^* \otimes A \\ & \to A^* \otimes A \otimes A \xrightarrow{id\otimes c^{-1}} A^* \otimes A \otimes A \to A \end{aligned}$$

Again using lemma 1.2, now for $i_{A^*}$and e_A, we arrive at

$$\begin{aligned} A \to A \otimes A^* \otimes A \to A \otimes A^* \otimes A^{**} \otimes A^* \otimes A & \\ \xrightarrow{c^{-1}\otimes id\otimes id\otimes id} & A^* \otimes A \otimes A^{**} \otimes A^* \otimes A \\ & \to A^* \otimes A \otimes A \xrightarrow{c\otimes id} A \otimes A^* \otimes A \to A \end{aligned}$$

which is the identity on A, analogously $u \circ u^{-1} = id_{A^{**}}$. Diagrams on Fig. 6 and Fig. 7 are illustrating this proof. □

Let us define morphisms

$$u_{+,A} = (e_A \otimes id_{A^{**}}) \circ (id_{A^*} \otimes c_{A^{**},A}) \circ (i_{A^*} \otimes id_A)$$

To keep the dichotomy between u and u_+ we will use notation $u_- = u$. A diagram representing u_+ is given on Fig. 8. When the category $\mathcal{C}$ is symmetric, $u_+ = u_-$.

It is not difficult to modify the proofs above and to prove the following lemma.

LEMMA 1.5. *The morphism u_+ is invertible with inverse map*

$$u_{+,A}^{-1} = (e_{A^*} \otimes id_A) \circ (id_{A^{**}} \otimes c_{A,A^*}) \circ (id_{A^{**}} \otimes id_A)$$

The diagram representing this morphism is givem on Fig. 8

The proof follows from isotopies on Fig. 9

COROLLARY 1.6. (1) *For $f : A \to B$ we get $u_B \circ f = f^{**} \circ u_A$.*

(2) *The family of isomorphisms $u = \{u_A\}$ determines an isomorphism of functors $u : id \to **$.*

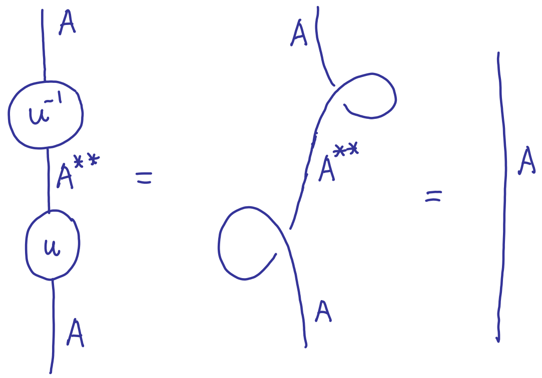

FIGURE 6. The identity $u_A^{-1}u_A = id_A$

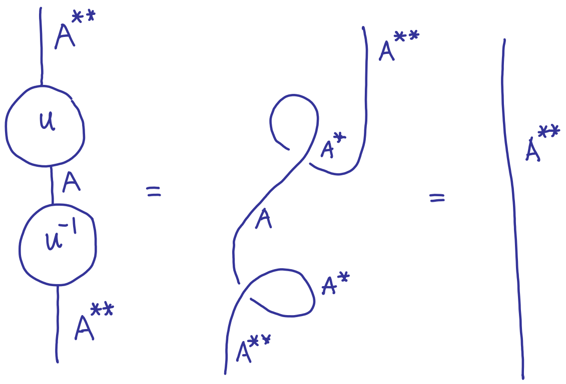

FIGURE 7. The identity $u_Au_A^{-1} = id_{A^{**}}$

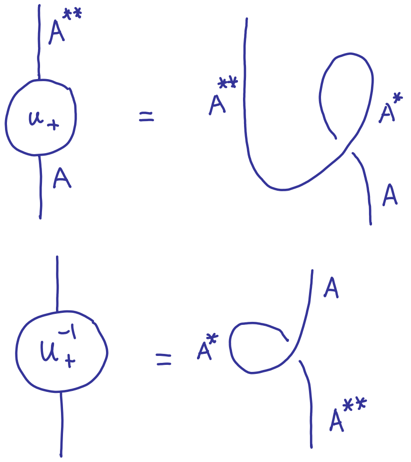

Figure 8. Morphisms u_+

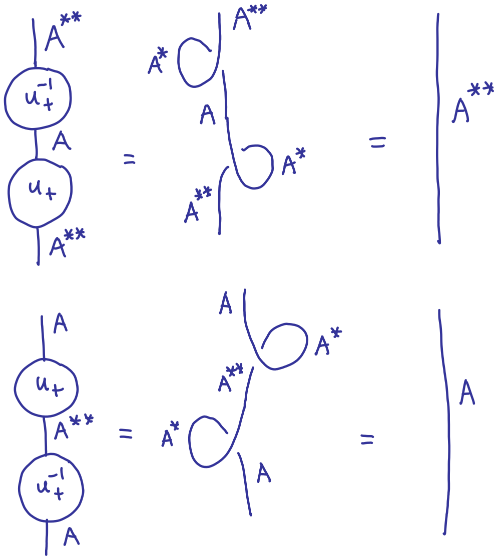

Figure 9. The proof of invertibility of u_+

PROOF. (1) The first statement follows from the equality of the following morphisms:

(a)

$$A \to A \otimes A^* \otimes A^{**} \to A^* \otimes A \otimes A^{**} \to A^{**} \xrightarrow{f^{**}} B^{**}$$

(b)

$$A \to A \otimes A^* \otimes A^{**} \xrightarrow{f^{**}} A \otimes A^* \otimes B^{**} \to A^* \otimes A \otimes B^{**} \to B^{**}$$

(c)

$$A \to A \otimes B^* \otimes B^{**} \xrightarrow{f^*} A \otimes A^* \otimes B^{**} \to A^* \otimes A \otimes B^{**} \to B^{**}$$

(d)

$$A \to A \otimes B^* \otimes B^{**} \to B^* \otimes A \otimes B^{**} \xrightarrow{f^*} A^* \otimes A \otimes B^{**} \to B^{**}$$

(e)

$$A \to A \otimes B^* \otimes B^{**} \to B^* \otimes A \otimes B^{**} \xrightarrow{f} B^* \otimes B \otimes B^{**} \to B^{**}$$

(f)

$$A \xrightarrow{f} B \to B \otimes B^* \otimes B^{**} \to B^* \otimes B \otimes B^{**} \to B^{**}$$

These equalities are represented as a sequence of isotopies on the Fig. **??**.

(2) The second part follows from the functoriality of u (all maps in its definition are functorial) and from the first part.

□

PROPOSITION 1.7.

$$u_{\mathbb{1}} = id_{\mathbb{1}}$$
$$(u_{\pm,A})^* = \left(u_{\mp,A^*}\right)^{-1}$$
$$u_{\pm,A\otimes B} = (u_{\pm,A} \otimes u_{\pm,B})(c_{BA}c_{AB})^{\pm 1}$$

PROOF. The first statement is easy. first identity ?

The second statement for u_- can be proven by the following equalities of morphisms:

$$\begin{aligned}
u^*_{-,A} &= (e_{A^{**}} \otimes id_{A^*}) \circ (id_{A^{***}} \otimes u_{-,A} \otimes id_{A^*}) \circ (id_{A^{***}} \otimes i_A) \\
&= (e_{A^{**}} \otimes id_{A^*}) \circ (id_{A^{***}} \otimes id_{A^{**}} \otimes e_A \otimes id_{A^*}) \\
&\quad \circ (id_{A^{***}} \otimes c_{A^*,A^{**}} \otimes id_A \otimes id_{A^*}) \\
&\quad \circ (id_{A^{***}} \otimes i_{A^*} \otimes id_A \otimes id_{A^*}) \circ (id_{A^{***}} \otimes i_A) \\
&= (e_{A^{**}} \otimes id_{A^*}) \circ (id_{A^{***}} \otimes c_{A^*,A^{**}}) \circ (id_{A^{***}} \otimes i_{A^*}) \\
&= u^{-1}_{+,A^*}
\end{aligned}$$

The proof for u_+ is similar.

A graphical representation of the proof is given in Fig. 10.

Now let us prove the third statement for u_-. We can identify $(A \otimes B)^{**}$ with $A^{**} \otimes B^{**}$. We proceed further having in mind this identification.

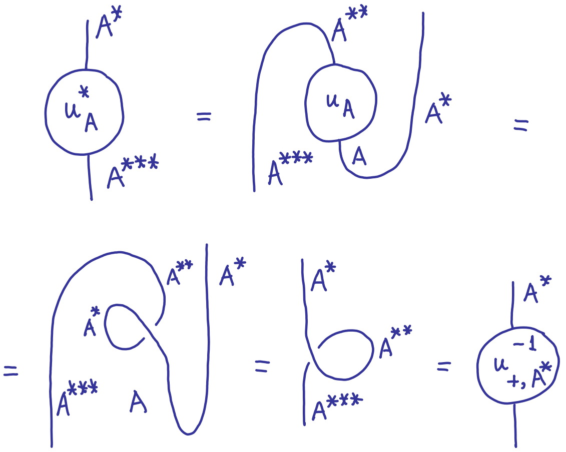

FIGURE 10. The proof of $(u_{\pm,A})^* = \left(u_{\mp,A^*}\right)^{-1}$

The following morphisms are equal:

$$A \otimes B \to ((A\otimes B)^* \otimes (A\otimes B)^{**}) \otimes (A \otimes B)$$
$$\xrightarrow{a \circ c^{-1}_{(A\otimes B)^{**},A\otimes B} \cdot a^{-1}} ((A\otimes B)^* \otimes (A \otimes B)) \otimes (A\otimes B)^{**} \to (A \otimes B)^{**}$$

$$\begin{aligned} (4) \quad A \otimes B &\to (B^* \otimes A^*) \otimes ((A^{**} \otimes B^{**}) \otimes (A \otimes B)) \\ &\xrightarrow{f} (B^* \otimes A^*) \otimes ((A \otimes B) \otimes (A^{**} \otimes B^{**})) \\ &\to A^{**} \otimes B^{**} \end{aligned}$$

where the map f is equal to $id \otimes id \otimes g$ with

$$g = (id \otimes c^{-1}_{B,A^{**}} \otimes id) \circ (c^{-1}_{A,A^{**}} \otimes id \otimes id) \circ (id \otimes id \otimes c^{-1}_{B,B^{**}}) \circ (id \otimes c^{-1}_{A,B^{**}} \otimes id)$$

We can rewrite g as follows:

$$\begin{aligned} g = {} & (c_{B,A} \otimes id \otimes id) \circ (c^{-1}_{B,A} \otimes id \otimes id) \circ (id \otimes c^{-1}_{B,A^{**}} \otimes id) \\ & \circ (c^{-1}_{A,A^{**}} \otimes id \otimes id) \circ (id \otimes id \otimes c^{-1}_{B,B^{**}}) \circ (id \otimes c^{-1}_{A,B^{**}} \otimes id) \end{aligned}$$

which, by two applications of the Yang-Baxter-equation, is equal to

$$\begin{aligned} & (c_{B,A} \otimes id \otimes id) \circ (id \otimes c^{-1}_{A,A^{**}} \otimes id) \circ (c^{-1}_{B,A^{**}} \otimes id \otimes id) \\ & \circ (id \otimes c^{-1}_{B,A} \otimes id) \circ (id \otimes id \otimes c^{-1}_{B,B^{**}}) \circ (id \otimes c^{-1}_{A,B^{**}} \otimes id) \end{aligned}$$

and also to

$$\begin{aligned}(c_{B,A} \otimes id \otimes id) \circ (id \otimes c^{-1}_{A,A^{**}} \otimes id) \circ (c^{-1}_{B,A^{**}} \otimes id \otimes id)& \\ \circ (id \otimes id \otimes c^{-1}_{A,B^{**}}) \circ (id \otimes c^{-1}_{B,B^{**}} \otimes id) \circ (id \otimes id \otimes c^{-1}_{B,A})&\end{aligned}$$

So we get (using lemma 1.2 for i_{A^*}, e_A)

$$\begin{aligned}A \otimes B \overset{c^{-1}_{B,A}}{\to} B \otimes A \to B^* \otimes B^{**} \otimes B \otimes A& \\ \overset{c^{-1}_{B,B^{**}}}{\to} B^* \otimes B \otimes B^{**} \otimes A& \\ \overset{c^{-1}_{A,B^{**}}}{\to} B^* \otimes B \otimes A \otimes B^{**}& \\ \to B^* \otimes B \otimes A^* \otimes A^{**} \otimes A \otimes B^{**}& \\ \overset{c_{B,A^*}}{\to} B^* \otimes A^* \otimes B \otimes A^{**} \otimes A \otimes B^{**}& \\ \overset{c^{-1}_{A,A^{**}}}{\to} B^* \otimes A^* \otimes B \otimes A \otimes A^{**} \otimes B^{**}& \\ \overset{c^{-1}_{B,A^*}}{\to} B^* \otimes B \otimes A^* \otimes A \otimes A^{**} \otimes B^{**} \to A^{**} \otimes B^{**}&\end{aligned}$$

which is equal to

$$\begin{aligned}A \otimes B \overset{c^{-1}_{B,A}}{\to} B \otimes A \to B^* \otimes B^{**} \otimes B \otimes A& \\ \overset{c^{-1}_{B,B^{**}}}{\to} B^* \otimes B \otimes B^{**} \otimes A& \\ \overset{c^{-1}_{A,B^{**}}}{\to} B^* \otimes B \otimes A \otimes B^{**}& \\ \to A \otimes B^{**}& \\ \to A^* \otimes A^{**} \otimes A \otimes B^{**}& \\ \overset{c^{-1}_{A,A^{**}}}{\to} A^* \otimes A \otimes A^{**} \otimes B^{**} \to A^{**} \otimes B^{**}&\end{aligned}$$

and also to

$$\begin{aligned}A \otimes B \overset{c^{-1}_{B,A}}{\to} B \otimes A \to B^* \otimes B^{**} \otimes B \otimes A& \\ \overset{c^{-1}_{B,B^{**}}}{\to} B^* \otimes B \otimes B^{**} \otimes A& \\ \overset{c^{-1}_{A,B^{**}}}{\to} B^* \otimes B \otimes A \otimes B^{**}& \\ \overset{c^{-1}_{A,B}}{\to} B^* \otimes A \otimes B \otimes B^{**}& \\ \overset{c_{A,B}}{\to} B^* \otimes B \otimes A \otimes B^{**}& \\ \to A \otimes B^{**} \overset{u_A}{\to} A^{**} \otimes B^{**}&\end{aligned}$$

By the Yang-Baxter equation, this is

$$\begin{aligned}
A\otimes B \overset{c^{-1}_{B,A}}{\to} B\otimes A \to B^*\otimes B^{**}\otimes B\otimes A & \\
\overset{c^{-1}_{A,B}}{\to} B^*\otimes B^{**}\otimes A\otimes B & \\
\overset{c^{-1}_{A,B^{**}}}{\to} B^*\otimes A\otimes B^{**}\otimes B & \\
\overset{c^{-1}_{B,B^{**}}}{\to} B^*\otimes A\otimes B\otimes B^{**} & \\
\overset{c_{A,B}}{\to} B^*\otimes B\otimes A\otimes B^{**} & \\
& \to A\otimes B^{**} \overset{u_A}{\to} A^{**}\otimes B^{**}
\end{aligned}$$

Using lemma 1.2 twice more we arrive at

$$\begin{aligned}
A\otimes B \overset{c^{-1}_{B,A}}{\to} B\otimes A \overset{c^{-1}_{A,B}}{\to} A\otimes B \to A\otimes B^*\otimes B^{**}\otimes B & \\
\overset{c_{A,B^*}}{\to} B^*\otimes A\otimes B^{**}\otimes B & \\
\overset{c^{-1}_{B,B^{**}}}{\to} B^*\otimes A\otimes B\otimes B^{**} & \\
\overset{c^{-1}_{A,B^*}}{\to} A\otimes B^*\otimes B\otimes B^{**} & \\
& \to A\otimes B^{**} \overset{u_A}{\to} A^{**}\otimes B^{**}
\end{aligned}$$

which is

$$\begin{aligned}
A\otimes B \overset{c^{-1}_{B,A}}{\to} B\otimes A \overset{c^{-1}_{A,B}}{\to} A\otimes B & \\
\to A\otimes B^*\otimes B^{**}\otimes B & \\
\overset{c^{-1}_{B,B^{**}}}{\to} A\otimes B^*\otimes B\otimes B^{**} & \\
& \to A\otimes B^{**} \overset{u_A}{\to} A^{**}\otimes B^{**}
\end{aligned}$$

or

$$A\otimes B \overset{c^{-1}_{B,A}}{\to} B\otimes A \overset{c^{-1}_{A,B}}{\to} A\otimes B \xrightarrow{u_A\otimes u_B} A^{**}\otimes B^{**}$$

Graphically, the proof is shown on Fig. 11. □

2. Quasitriangular quasi Hopf algebras

In addition to the quasi Hopf structure these algebras require the existence of an invertible element $R\in A\otimes A$, satisfying the following identities

$$\begin{aligned}
\Delta'(a) &= R\Delta(a)R^{-1}, \\
(id\otimes\Delta)(R) &= \Phi^{-1}_{231}R_{13}\Phi_{213}R_{12}\Phi^{-1}_{123}, \\
(\Delta\otimes id)(R) &= \Phi_{312}R_{13}\Phi^{-1}_{132}R_{23}\Phi_{123}.
\end{aligned}$$

The characteristic property of these algebras is that their categories of modules are braided monoidal categories (and rigid if modules are finite dimensional). For

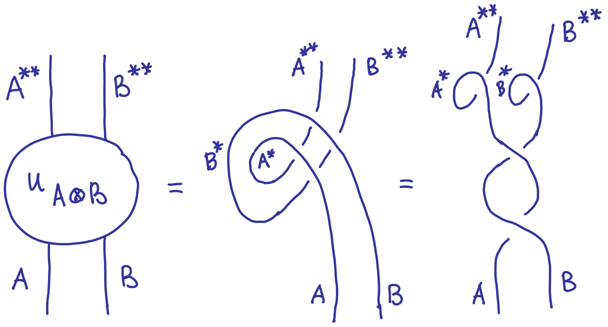

FIGURE 11. The proof of $u_{A\otimes B} = (u_A \otimes u_B)(c_{BA}c_{AB})^{-1}$

these algebras the double dual of a module is canonically isomorphic to the module itself. This isomorphism is determined by the property of the square of the antipode:

$$S^2(a) = uau^{-1}$$

where $u = \sum_{i,j} \phi_j S(\psi_j) S(\beta_i) \alpha_i \eta_j$. Here we use the notation $R = \sum_i \alpha_i \otimes \beta_i$ and $\Phi^{-1} = \sum_j \phi_j \otimes \psi_j \otimes \eta_j$.

If we drop the antipode axiom in the definition of a quasitriangular quasi Hopf algebra, we will have the definition of a quasitriangular bialgebra.

Assume we have a braided abelian k-linear monoidal category $\mathcal{C}$ and a monoidal functor between abelian categories $\mathcal{C}$ and the category of vector spaces over k. Then the algebra $End(F)$ has a natural structure of quasitriangular quasi-bialgebra with the braiding element being the image of $R = \{F(c_{A,B})\}\} \in End(\otimes)$.

EXAMPLE 2.1. *Let* $\mathfrak{g}$ *be a simple Lie algebra. There exist an element* $\Phi = 1 + \frac{h^2}{24}[t \otimes 1, 1 \otimes t] + o(h^3)$ *such that the universal enveloping algebra with the standard diagonal comultiplication together with the element* $R = exp(ht)$ *becomes a topological quasitringular quasiHopf algebra over* $\mathbb{C}[[h]]$ *. The details can be found in section .....*