CHAPTER 1

# Rigid and balanced categories

## 1. Rigid categories

Let us start with two examples.

Example 1.1. *Let $k$ be a field. The category of finite dimensional vector spaces over $k$ is a rigid monoidal category. Every finite dimensional vector space $V$ has its dual vector space $V^*$, the space of all $k$-valued linear functions on $V$. There are canonical mappings:*

- *The embedding $in_V : k \to V \otimes V^*$ mapping $1 \to \sum_i e_i \otimes e^i$ where $e_i$ is a basis in $V$ and $e^i$ is a basis in $V^*$. Clearly $in_V(1)$ does not depend on the choice of basis.*
- *Similarly we have a mapping $in'_V : k \to V^* \otimes V$.*
- *The evaluation mapping $ev_V : V^* \otimes V \to k$ acts as $ev(l \otimes x) = l(x)$ where $l(x)$ is the value of the linear functional $l$ on the vector $x$.*
- *Similarly we have the evaluation mapping $ev'_V : V \otimes V^* \to k$.*

*Notice that $k$ is a one dimensional vector space with a basis $1$. Any one dimensional vector space $\mathbb{1}$ is isomorphic to $k$. Using this isomorphism we have the following data which we can assign to each finite dimensional vector space:*

- *A triple $(V^*, i_V : \mathbb{1} \to V \otimes V^*, e_V : V^* \otimes V \to \mathbb{1}$. Such triple is unique up to an isomorphism.*
- *A triple $(V^*, i'_V : \mathbb{1} \to V^* \otimes V, e'_V : V \otimes V^* \to \mathbb{1}$. Such triple is also unique up to an isomorphism.*

*The morphisms $i_V, e_V, i'_V$, and $e'_V$ satisfy certain natural identities which will be listed below.*

Example 1.2. *Let $G$ be a group. Denote by $\underline{G - mod}$ the category of finite dimensional $G$-modules (over $\mathcal{C}$). The dual object to the $G$-module $(V, \pi_V : G \to End(V))$ is the dual vector space $V^*$ with the module map $\pi_{V*}(g) = \pi_V(g^{-1})^*$. The evaluation and injection morphisms are the same as in the category of vector spaces.*

*Now assume that $G$ is a Lie group. By taking its neighborhood of $1$ we can turn the category of $G$-modules to the category of modules over its Lie algebra $\mathfrak{g} = Lie(G)$. This gives another example of a category with duals: the category of finite dimensional modules over a Lie algebra, or equivalently, the category of modules over the corresponding universal enveloping algebra $U(\mathfrak{g})$. The left dual of $(\pi_V, V)$ in this example coincides with the right dual and is the pair $(\pi_{V^*}, V^*)$ with $\pi_{V^*}(x) = -\pi^*_V(x), \quad x \in \mathfrak{g}$.*

### 1.1. Dual objects.

Definition 1.1. *An object $A \in Ob(\mathcal{C})$ has the* right dual $A^*$ *if there are morphisms $i_A : \mathbb{1} \to A \otimes A^*$, $e_A : A^* \otimes A \to \mathbb{1}$ such that the compositions*

$$A \to \mathbb{1} \otimes A \xrightarrow{i_A \otimes id} (A \otimes A^*) \otimes A \xrightarrow{a^{-1}_{A,A^*,A}} A \otimes (A^* \otimes A) \xrightarrow{id \otimes e_A} A \otimes \mathbb{1} \to A$$

*and*

$$A^* \to A^* \otimes \mathbb{1} \xrightarrow{id \otimes i_A} A^* \otimes (A \otimes A^*) \xrightarrow{a_{A^*,A,A^*}} (A^* \otimes A) \otimes A^* \xrightarrow{e_A \otimes id} \mathbb{1} \otimes A^* \to A^*$$

*are identity morphisms.*

Lemma 1.1. *The triple $(A^*, i_A, e_A)$ is unique up to an isomorphism.*

Proof. Suppose $(\widetilde{A^*}, \widetilde{i_A}, \widetilde{e_A})$ is another dual object to $A$ and consider the morphisms

$$A^* \xrightarrow{id \otimes \widetilde{i_A}} A^* \otimes (A \otimes \widetilde{A^*}) \xrightarrow{a} (A^* \otimes A) \otimes \widetilde{A^*} \xrightarrow{e_A \otimes id} \widetilde{A^*} \tag{1}$$

$$\widetilde{A^*} \xrightarrow{id \otimes i_A} \widetilde{A^*} \otimes (A \otimes A^*) \xrightarrow{a} (\widetilde{A^*} \otimes A) \otimes A^* \xrightarrow{\widetilde{e_A} \otimes id} A^* \tag{2}$$

and let us show that they are inverse to each other.

Using the pentagon identity, functoriality and the defining property of the maps $i$ and $e$ one can show that the following compositions of morphisms are equal

$$\begin{aligned} A^* \to A^* \otimes (A \otimes \widetilde{A^*}) \to (A^* \otimes A) \otimes \widetilde{A^*} \to \widetilde{A^*} \\ \to \widetilde{A^*} \otimes (A \otimes A^*) \to (\widetilde{A^*} \otimes A) \otimes A^* \to A^* \end{aligned}$$

$$\begin{aligned} A^* \to A^* \otimes (A \otimes \widetilde{A^*}) \to (A^* \otimes A) \otimes \widetilde{A^*} \\ \to ((A^* \otimes A) \otimes \widetilde{A^*}) \otimes (A \otimes A^*) \to (\widetilde{A^*} \otimes A) \otimes A^* \to A^* \end{aligned}$$

$$\begin{aligned} A^* \to A^* \otimes (A \otimes A^*) \to (A^* \otimes A) \otimes A^* \to (A^* \otimes ((A \otimes \widetilde{A^*}) \otimes A)) \otimes A^* \\ \to (A^* \otimes (A \otimes (\widetilde{A^*} \otimes A))) \otimes A^* \to (A^* \otimes A) \otimes A^* \to A^* \end{aligned}$$

$$A^* \to A^* \otimes (A \otimes A^*) \to (A^* \otimes A) \otimes A^* \to A^*$$

and that the last morphism is the identity

$$id : A^* \to A^*$$

Therefore the morphisms (1) are inverse to each other and therefore are isomorphisms. □

Proposition 1.2. *If $A$ and $B$ have right duals, so is $A \otimes B$ and*

$$(A \otimes B)^* \cong B^* \otimes A^*$$

*with*

$$\begin{aligned} i_{A \otimes B} &= \mathbb{1} \to A \otimes A^* \to (A \otimes (B \otimes B^*)) \otimes A^* \to (A \otimes B) \otimes (B^* \otimes A^*) \\ e_{A \otimes B} &= (B^* \otimes A^*) \otimes (A \otimes B) \to (B^* \otimes (A^*) \otimes A)) \otimes B \to B^* \otimes B \to \mathbb{1} \end{aligned}$$

PROOF. As in the proof of the lemma above we have to show that certain compositions of morphisms are equal. The morphism $(\mathrm{id}_{A\otimes B} \otimes e_{A\otimes B}) \circ a^{-1}_{A\otimes B, B^*\otimes A^*, A\otimes B} \circ (i_{A\otimes B} \otimes \mathrm{id}_{A\otimes B})$, which is the same as the composition

$$A \otimes B \to ((A \otimes B) \otimes (B^* \otimes A^*)) \otimes (A \otimes B) \to (A \otimes (B \otimes B^*)) \otimes ((A^* \otimes A) \otimes B) \to A \otimes B \tag{3}$$

Using pentagon identity, functoriality and the definition of morphisms $i$ and $e$ one can show that the composition (3) coincide with each of the following.

$$A \otimes B \to A \otimes ((A^* \otimes A) \otimes B) \to A \otimes (B \otimes (B^* \otimes ((A^* \otimes A) \otimes B))) \to A \otimes (B \otimes (B^* \otimes B)) \to A \otimes B$$

$$A \otimes B \to A \otimes (A^* \otimes A) \otimes B) \to A \otimes (1 \otimes B) \to A \otimes (B \otimes (B^* \otimes B)) \to A \otimes B$$

$$id : A \otimes B \to A \otimes (1 \otimes B) \to A \otimes B$$

$$(e \otimes id) \circ a \circ (id \otimes i) : B^* \otimes A^* \to (B^* \otimes A^*) \otimes ((A \otimes B) \otimes (B^* \otimes A^*)) \to B^* \otimes A^*$$

This proves that the triple $(B^* \otimes A^*, i_{A\otimes B}, e_{A\otimes B})$ describes an object dual to $A \otimes B$. □

**1.2. Rigid categories.**

DEFINITION 1.2. *The category $\mathcal{C}$ is* rigid *if all of its objects have right duals and the map $* : A \mapsto A^*$ is a bijection up to isomorphism.*

Thus, if $\mathcal{C}$ is rigid, every object $A$ is a dual of some object $B$ which we will call the *left dual* of $A$ and denote by ${}^*A$, so that by definition $({}^*A)^* = B^* = A$. One can also show that ${}^*(A^*) \simeq A$.

Equivalently, a rigid category is a monoidal category in which every object has left dual and right dual.

For a morphism $f \in Hom_{\mathcal{C}}(A, B)$ we define a *d*ual morphism $f^* : B^* \to A^*$ as follows:

$$f^* : B^* \stackrel{id\otimes i_A}{\longrightarrow} B^* \otimes (A \otimes A^*) \stackrel{a\circ(id\otimes f\otimes id)}{\longrightarrow} (B^* \otimes B) \otimes A^* \stackrel{e_B\otimes id}{\longrightarrow} A^*$$

LEMMA 1.3. *If $\mathcal{C}$ is rigid, so is $\mathcal{C}^{str}$ with $(A_1, \dots, A_n)^* = (A_n^*, \dots, A_1^*)$.*

PROOF. This follows from proposition 1.2. □

LEMMA 1.4.

(1) $id_A^* = id_{A^*}$

(2) $\mathbb{1}^* = \mathbb{1}$ *with* $i_{\mathbb{1}} = i_{\mathbb{1}^*} = u^{-1}$ *and* $e_{\mathbb{1}} = e_{\mathbb{1}^*} = u$ *where* $u = l_{\mathbb{1}} = r_{\mathbb{1}}$.

(3) *For $f : A \to B$ we have:*

$$\begin{aligned} (id_B \otimes f^*) \circ i_B &= (f \otimes id_{A^*}) \circ i_A \\ e_A \circ (f^* \otimes id_A) &= e_B \circ (id_{B^*} \otimes f) \end{aligned}$$

PROOF. (1) clear

(2) Equation (1.1) becomes

$$\mathbb{1} \xrightarrow{u^{-1}} \mathbb{1}\otimes\mathbb{1} \xrightarrow{i_A\otimes id} (\mathbb{1}\otimes\mathbb{1}^*)\otimes\mathbb{1} \xrightarrow{a^{-1}_{\mathbb{1},\mathbb{1}^*,\mathbb{1}}} \mathbb{1}\otimes(\mathbb{1}^*\otimes\mathbb{1}) \xrightarrow{id\otimes e_A} \mathbb{1}\otimes\mathbb{1} \to \mathbb{1}$$

which holds since $a^{-1}_{\mathbb{1},\mathbb{1}^*,\mathbb{1}} \circ (u^{-1}\otimes id)\circ u^{-1} = (u\circ(id\otimes u))^{-1}$ by definition of an identity object. The second condition for the dual holds for similar reasons.

(3) It is clear from the definition of duals that the following morphisms

$$\begin{aligned}
\mathbb{1} &\to B\otimes B^* \xrightarrow{id_B\otimes f^*} B\otimes A^* \\
\mathbb{1} &\to B\otimes B^* \to B\otimes B^*\otimes A\otimes A^* \xrightarrow{f} B\otimes B^*\otimes B\otimes A^* \to B\otimes A^* \\
\mathbb{1} &\to A\otimes A^* \xrightarrow{id\otimes id\otimes f\otimes id} B\otimes A^* \to B\otimes B^*\otimes B\otimes A^* \to B\otimes A^* \\
\mathbb{1} &\to A\otimes A^* \xrightarrow{f\otimes id} B\otimes A^* \xrightarrow{id} B\otimes A^*
\end{aligned}$$

are equal to $(id_B\otimes f^*)\circ i_B$. To simplify formulae we omitted brackets and associativity constraints. This proves the first identity. The proof of the second identity is similar.

□

Recall that the monoidal category $\mathcal{C}^{op}$ is the category with the same objects, with $Hom_{\mathcal{C}^{op}}(A,B) = Hom_{\mathcal{C}}(B,A)$, and with the tensor product $A\otimes_{\mathcal{C}^{op}} B = B\otimes_{\mathcal{C}} A$.

Proposition 1.5. (1) *For any pair of objects $A, B$ the assignment $f\mapsto f^*$ is a bijection from $Hom_{\mathcal{C}}(A,B)$ to $Hom_{\mathcal{C}}(B^*,A^*)$.*

(2) *$*:\mathcal{C}\to\mathcal{C}^{op}$ is a contravariant functor which is an anti-selfequivalence of the monoidal category $\mathcal{C}$.*

Proof. (1) To prove the first part of the proposition let us show that the morphism

$$A \to (B\otimes B^*)\otimes A \xrightarrow{f^*} (B\otimes A^*)\otimes A \to B\otimes(A^*\otimes A)\to B$$

coincides with $f$ and that it is inverse to the map $f\to f^*$.

Indeed, we have the following sequence of identical morphisms.

$$\begin{aligned}
A \to (B\otimes B^*)\otimes A \to (B\otimes(B^*\otimes(A\otimes A^*)))\otimes A \\
\to B\otimes((B^*\otimes B)\otimes(A^*\otimes A)) \to B
\end{aligned}$$

$$A\to A\otimes(A^*\otimes A) \xrightarrow{f\otimes id\otimes id} B\otimes(A^*\otimes A)\to B\otimes((B^*\otimes B)\otimes(A^*\otimes A))\to B$$

$$A\to A\otimes(A^*\otimes A) \xrightarrow{f\otimes id\otimes id} B\otimes(A^*\otimes A)\to B\to B\otimes((B^*\otimes B)\to B$$

$$A\to A\otimes(A^*\otimes A)\to A\xrightarrow{f} B\xrightarrow{id} B$$

$$A\xrightarrow{id} A\xrightarrow{f} B\xrightarrow{id} B$$

A shorter proof is to use part 3. of Lemma 1.5.

(2) In order to prove the (contravariant) functoriality of the map $*$ we have to show that $(g\circ f)^* = f^*\circ g^*$. The bijectivity proven above will imply that this functor is an anti-selfequivalence.

By the definition of the dual, the morphism $f^*\circ g^*$ is equal to the composition

$$C^* \to C^* \otimes (B \otimes B^*) \stackrel{a\circ(id\otimes g\otimes id)}{\longrightarrow} (C^* \otimes C) \otimes B^* \to B^*$$
$$\to B^* \otimes (A \otimes A^*) \stackrel{a\circ(id\otimes f\otimes id)}{\longrightarrow} (B^* \otimes B) \otimes A^* \to A^*$$

This composition coincides with the maps

$$C^* \to C^* \otimes (B \otimes (B^* \otimes (A \otimes A^*)))$$
$$\stackrel{a\circ(id\otimes g\otimes id\otimes f\otimes id)}{\longrightarrow} ((C^* \otimes C) \otimes (B^* \otimes B)) \otimes A^* \to A^*$$

$$C^* \to C^* \otimes (A \otimes A^*) \stackrel{id\otimes f\otimes id}{\longrightarrow} C^* \otimes (B \otimes A^*) \to C^* \otimes (((B \otimes B^*) \otimes B) \otimes A^*)$$
$$\to C^* \otimes ((B \otimes (B^* \otimes B)) \otimes A^*) \to C^* \otimes (B \otimes A^*) \stackrel{id\otimes g\otimes id}{\longrightarrow} C^* \otimes (C \otimes A^*) \to A^*$$

$$C^* \to C^* \otimes (A \otimes A^*) \stackrel{id\otimes f\otimes id}{\longrightarrow} C^* \otimes (B \otimes A^*) \stackrel{id\otimes g\otimes id}{\longrightarrow} C^* \otimes (C \otimes A^*) \to A^*$$

But this morphism is, by definition, exactly $(g \circ f)^*$.

□

If $\mathcal{C}$ is a rigid category the map $\sigma : [\mathcal{C}] \to [\mathcal{C}]$ which maps $[V]$ to $[V^*]$ is an automorphism of the Grothendieck ring of this category. Notice that, in general, $\sigma$ is not an involution.

### 1.3. Quasi-Hopf algebras.

DEFINITION 1.3. *A quasi-Hopf algebra is a quasibialgebra together with an antiautomorphism $S : A \to A$ and an invertible elements $\alpha, \beta \in A$, satisfying the following relations:*

$$\begin{aligned}
\sum_a S(a^{(1)})\alpha a^{(2)} &= \epsilon(a)\alpha \\
\sum_a a^{(1)}\beta S(a^{(2)}) &= \epsilon(a)\beta \\
\sum_i X_i \beta S(Y_i)\alpha Z_i &= 1 \\
\sum_i S(\overline{X_i})\alpha \overline{Y_i}\beta S(\overline{Z_i}) &= 1
\end{aligned}$$

*Here we used notations $\sum_a a^{(1)} \otimes a^{(2)}$ for $\Delta(a)$, $\sum_i X_i \otimes Y_i \otimes Z_i$ for $\Phi$, and $\sum_i \overline{X_i} \otimes \overline{Y_i} \otimes \overline{Z_i}$ for $\Phi^{-1}$.*

*A Hopf algebra is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$.*

The characteristic property of quasi-Hopf algebras is that its category of finite dimensional representations is an abelian rigid monoidal category with the dual representations defined as $\pi_{V^*}(a) = \pi_V^*(S(a))$. The evaluation map $e_V : V^* \otimes V \to 1$ (here 1 is the one dimensional module with the module map $\epsilon : A \to \kappa$) acts as $e_V(l \otimes x) = l(\alpha x)$. The injection morphism $i_V : 1 \to V \otimes V^*$ is a linear map acting as $i_V(1) = \sum_j e^j \otimes \beta e_j$ where $1 \in \kappa$ is the unit element is the base ring.

The isomorphism between a module and its double dual is determined by the following property of the antipode $S^2(a) = uau^{-1}$ where $u = S(\beta)\alpha$.

## 2. Balanced categories

In a rigid category any object $A$ induces a sequence of multiple duals $\dots, A, A^*, A^{**}, \dots$.

Because $(A \otimes B)^* \simeq B^* \otimes A^*$ the double dual mapping $** : \mathcal{C} \to \mathcal{C}$ is a monoidal functor.

Definition 2.1. B*alancing is an isomorphism of monoidal functors* $\beta : id \simeq **$, *i.e. it is a collection of functorial isomorphisms* $\beta_A : A \to A^{**}$ *such that the diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\beta_A} & A^{**} \\ f \downarrow & & \downarrow f^{**} \\ B & \overset{\sim}{\underset{\beta_B}{\longleftarrow}} & B^{**} \end{array} \tag{4}$$

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\beta_{A \otimes B}} & (A \otimes B)^{**} \\ id \downarrow & & \downarrow ** \\ A \otimes B & \overset{\sim}{\underset{\beta_A \otimes \beta_B}{\longleftarrow}} & A^{**} \otimes B^{**} \end{array} \tag{5}$$

*commute.*

In such categories one can define the natural notion the $End(\mathbb{1})$-valued trace of an endomorphism. For $f : A \to A$ define

$$tr_A(f) : \mathbb{1} \xrightarrow{i_A} A \otimes A^* \xrightarrow{f \otimes id} A \otimes A^* \xrightarrow{\beta_A \otimes id} A^{**} \otimes A^* \xrightarrow{e_{A^*}} \mathbb{1}$$

$$dim(A) = tr_A(id)$$

Assume $Hom(\mathbb{1}, \mathbb{1}) \simeq k$.

Proposition 2.1. (1) *If* $f_1 : A_1 \to A_1$ *and* $f_2 : A_2 \to A_2$, *then*

$$tr_{A_1 \oplus A_2}(f_1 \oplus f_2) = tr_{A_1}(f_1) + tr_{A_2}(f_2)$$

(2) *If* $f_1 : A_1 \to A_1$ *and* $f_2 : A_2 \to A_2$, *then*

$$tr_{A_1 \otimes A_2}(f_1 \otimes f_2) = tr_{A_1}(f_1) tr_{A_2}(f_2)$$

(3) *If* $\mathcal{C}$ *is semisimple, then for* $f : A \to B$ *and* $g : B \to A$

$$tr_A(gf) = tr_B(fg)$$

We leave the proof of this proposition as an exercise. It can be found in [**?**].