Lecture 1.

Note Title 9/19/2009

1. Introduction.

The outline of the framework of QFT:

A d-dimensional local quantum field theory assigns:

- a vector space $\mathcal{H}(N)$ to each $(d-1)$ dimensional manifold
- a vector $Z(M) \in \mathcal{H}(N)$ to each d-dimensional manifold with $N = \partial M$

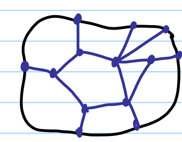
These pairs $Z(M) \in \mathcal{H}(\partial M)$ should satisfy some natural axioms

To illustrate them consider an example.

<u>QFT of 2-dimensional local vertex models in statistical mechanics</u>

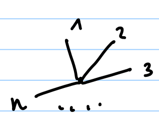
Space-time in such quantum field theory is a pair $(\Gamma \underset{\varphi}{\hookrightarrow} D)$ where

- Γ is a graph
- D - domain in $\mathbb{R}^2$
- $\varphi : \Gamma \hookrightarrow D$ is an embedding such that 1-valent vertices are all emedded into to the boundary ∂D of D.



* Assign elements $0, 1$ to each edge (this defines a state on Γ)

* Fix the weights for vertices:



$$\longrightarrow \{ w(\sigma_1, .., \sigma_n) \}_{\sigma_i = 0,1}$$

* Define the partition function with fixed boundary states as follows

$$Z_{\partial \sigma}(\Gamma \subset D) = \sum_{\tau \in \{0,1\}^{E_{inner}}} \prod_{vertices} w_{vert.}(\tau_{vert.})$$

It define the vector

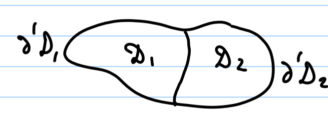
$$Z(\Gamma \subset D) \in (\mathbb{R}^2)^{\otimes E_{boundary}}$$

$E_{boundary} = (\partial \Gamma \subset \partial D)$.

Here $(\mathbb{R}^2)^{\otimes X}$ is the vector space with the basis $f : X \to 0, 1$.

The pair $Z(\Gamma \subset D) \in \mathcal{H}(\partial\Gamma \subset \partial D)$ has important <u>gluing</u> (cutting) property

* Let $D = D_1 \cup D_2$ as



$\partial D_1 = \partial' D_1 \cup \partial_{12} D_1$

$\partial D_2 = \partial' D_2 \cup \partial_{12} D_2$

$\Gamma \subset D$ is separated into $\Gamma_1 \subset D_1$ and $\Gamma_2 \subset D_2$ the intersection of Γ and $\partial_{12} D_1 = \partial_{12} D_2$ creates boundary vertices along $\partial_{12} D_1 = \partial_{12} D_2$.

*

$$Z_{(\sigma_1', \sigma_2')}(\Gamma \subset D) = \sum Z_{(\sigma_1', \sigma_{12})}(\Gamma_1 \subset D_1) Z_{(\sigma_{12}, \sigma_2')}(\Gamma_2 \subset D_2)$$

states on boundary edges of Γ_1, Γ_2 along $\partial_{12} D_1$

The gluing (cutting) property is essentially a locality because one it is imposed, it defines partition functions if vertex weights are defined.

In such vertex model the probability of finding the system in a state σ: Edges $\to$ 0,1 is

$$\underset{\sigma_{out}}{Prob}(\sigma) = \frac{\prod_{vert.} W(vert., \sigma)}{Z_{\sigma_{out}}(\Gamma \subset D)}$$

Here we assume that states on the boundary edges are fixed $\Rightarrow$ this is the conditional probability (for fixed σ_{out}).

see (1) for more details

Classical Lagrangian Mechanics (on $\mathbb{R}^n$).

Newton's law:

$$m\vec{a} = \vec{F}^{tot}$$

$\gamma(t)$ = classical trajectory = parameterized path in $\mathbb{R}^n$ $\gamma: t \mapsto \gamma(t) \in \mathbb{R}^n$, $\mathbb{R} \to \mathbb{R}^n$

$v(t) = \dot{\gamma}(t) = \frac{d\gamma}{dt}$ = the velocity = a tangent vector to $\gamma(t)$

$a(t) = \ddot{\gamma}(t) = \frac{d^2\gamma}{dt^2}$ = acceleration = tangent vector to $\dot{\gamma}(t): \mathbb{R} \to \mathbb{R}^n$

Newton's law = 2nd order ordinary differential equation (ODE)

$$m\ddot{\gamma}(t) = F^{tot}(\gamma(t))$$

$\vec{F}$ is a potential forse if

$$\vec{F} = -\vec{\nabla} U, \quad U = \text{the potential, potential energy}$$

In this case Newton's equations are

$$m\ddot{\gamma}^i(t) = -\frac{\partial U}{\partial x^i}(\gamma(t)),$$

Define the Lagrangian function on $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$

(the isomorphism is given by the basis $\frac{\partial}{\partial x^i}$ in $T_x\mathbb{R}^n$)

$$\mathcal{L}(\xi, x) = \frac{m}{2}(\xi, \xi) - U(x)$$

$$(*) \quad S[\gamma] = \int_{t_1}^{t_2} \mathcal{L}(\dot{\gamma}(t), \gamma(t))\, dt$$

The variation of S:

- informally δS = "linear part (in $\delta\gamma$) of $S(\gamma+\delta\gamma) - S(\gamma)$"
- more precisely: let γ_s be a one parametric family of paths ($s \in [0, ...]$) and let $\delta\gamma = \frac{d\gamma_s}{ds}\Big|_{s=0} ds$

then $\delta S = \frac{d}{ds} S(\gamma_s)\Big|_{s=0} ds$

see (2) for more details.

For the variation of (*) we have

$$\delta S[\gamma] = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \xi^i}(\dot{\gamma}, \gamma)\,\delta\dot{\gamma}^i + \frac{\partial \mathcal{L}}{\partial q^i}(\dot{\gamma}, \gamma)\,\delta\gamma^i \right) dt$$

Integrating by parts:

$$\delta S[\gamma] = \frac{\partial \mathcal{L}}{\partial \xi^i}(\dot{\gamma}(t), \gamma(t))\,\delta\gamma^i(t)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi^i}(\dot{\gamma}, \gamma) + \frac{\partial \mathcal{L}}{\partial q^i}(\dot{\gamma}, \gamma) \right) \delta\gamma^i\, dt$$

<u>Proposition</u>. Solutions to the boundary value problem

" find solutions to $\ddot{\gamma}(t) = -\frac{\partial U}{\partial q}(\gamma(t))$ with $\gamma(t_1) = q_1$, $\gamma(t_2) = q_2$ "

are extrema of the action functional.

<u>Proof</u>. The "bulk" term of the variation gives Euler-Lagrange equations

$$-\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \xi^i}(\dot{\gamma},\gamma) + \frac{\partial \mathcal{L}}{\partial q^i}(\dot{\gamma},\gamma) = 0$$

which are equivalent to Newton's equations. The boundary variation vanishes because $\delta\gamma(t_1) = \delta\gamma(t_2) = 0$ ∎

<u>Remark</u>. In case of initial value problem: "find the trajectory with $\gamma(0) = q$, $\dot{\gamma}(0) = V$" the Euler-Lagrange equations remain the Newton's equations but boundary terms do not vanish. In this sense such probem is not quite equivalent to the variational problem, only up to boundary terms.