Homework

Note Title 9/19/2009

Week 1. Describe S^2 in two coordinate charts (use spherical angles).

$$S^2 = \{ \vec{x} \in \mathbb{R}^3 \,/\, x_1^2 + x_2^2 + x_3^2 = 1 \},$$

$$x_1 = \cos\theta \cos\varphi$$
$$x_2 = \cos\theta \sin\varphi$$
$$x_3 = \sin\theta$$

Week 2. ① Find Legendre transform to

$$f(x) = |x|^\alpha, \quad \alpha > 1$$

② Prove the Theorem about the invertibility of Legendre transform for convex functions on $\mathbb{R}$

③ Prove that if $f: \mathbb{R} \to \mathbb{R}$ is twice differentiable and $f'' > 0$, then f is convex.

④ Prove theorems (H) and (H1)

⑤ Find general solutions to

$$m\ddot{q} = -U(q), \quad U(q) = \frac{\alpha}{q^2}$$

on level surfaces $E = \frac{p^2}{2m} + U(q)$ (of constant energy)

Analyse both cases, $\alpha > 0$, and $\alpha < 0$)

Week 3.

① Prove:

- **Thm 1** (Euler-Lagrange = restricted Newton)
- **Thm 2** $(\dot{\gamma}(t), \gamma(t)) \xrightarrow{\text{Legendre}}$ solutions to Hamilton's equations
- $d\omega = 0$ if and only if $\{F, G\} = \omega^{-1}(dF \wedge dG)$ satisfies the Jacobi identity

② Prove that

$$\{F, G\} = \sum_{a,b,c=1}^{3} \varepsilon_{abc}\, x^a \frac{\partial F}{\partial x^b} \frac{\partial G}{\partial x^c}$$

defines a Poisson structure on $C^\infty(\mathbb{R}^3)$

③ Prove that if $pq - qp = h$ then

$$p^n q^m = \sum_{N \geq 0} h^N \binom{n}{n-N} \binom{m}{m-N} N!\, q^{m-N} p^{n-N}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ when $0 \leq k \leq n$,

and $\binom{n}{k} = 0$, when $k < 0$.

Hint. Prove first

$$e^{ap} e^{bq} = e^{abh} e^{bq} e^{ap}$$

Week 4. Quantum Hamiltonian describing 2-spin system has the form

$$H = a + b \sum_{i=1}^{3} \sigma_i \otimes \sigma_i ,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices (2-dim representation of the Lie algebra $su(2)$).

Find the probability of the even when the system is in the pure state P_φ with $\varphi = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}$ at time zero and is in the state with $\psi = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}$.

Do the same for $\varphi = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}$ and $\psi = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}$.

Week 5.

① Prove that if $*_h$ is a family of associative multiplications on A and

$$a *_h b = ab + h\, m_1(a,b) + O(h^2), \text{ as } h \to 0$$

and $ab = ba$, then $\{a,b\} = m_1(a,b) - m_1(b,a)$ is a Poisson structure on the commutative algebra A (with the multiplication ab).

② Verify that $(C^\infty_{pol}(T^*\mathbb{R}^n), *_h)$ defined in Lecture 5 is a deformation quantization of the Poisson algebra (1).

③ (i) Find the solution to the initial problem:

$$\begin{cases} ih \dfrac{\partial \psi_t}{\partial t} = -\dfrac{h^2}{2m}\Delta \psi_t \\ \psi_0 = \varphi \end{cases}$$

where $\varphi \in L_2(\mathbb{R}^N)$. Hint: write it terms of a Fourier transform.

(ii) Find the propagation kernel in this case, i.e. the solution to the initial problem:

$$\begin{cases} ih \dfrac{\partial \mathcal{U}_t(q_1,q_2)}{\partial t} = -\dfrac{h^2}{2m}\Delta_{q_1} \mathcal{U}_t(q_1,q_2) \\ \lim\limits_{t\to+0} \mathcal{U}_t(q_1,q_2) = \delta(q_1-q_2) \end{cases}$$

(iii) Verify that (i) agrees with (ii), i.e. that

$$\psi_t(q) = \int_{\mathbb{R}^n} \mathcal{U}_t(q,q')\,\varphi(q')\,dq'$$

Week 6. 1. Prove Thm 1 (or its parts)

2. Prove Lemma 1.