Homework 3.

Note Title 2/22/2008

① A quasi-Hopf algebra is a collection $(H, \Delta, \varepsilon, S, \Phi, \alpha, \beta)$ where

- H is an algebra
- $\Delta : H \to H \otimes H$ is an algebra homomorphism
- $\Phi \in H^{\otimes 3}$. This element makes the comultiplication Δ coassociative:

$$\big((id \otimes \Delta) \circ \Delta\big)(a) = \Phi \cdot \big((\Delta \otimes id) \circ \Delta\big)(a) \cdot \Phi^{-1}$$

- $\varepsilon : H \to k$ is the counit it is an algebra homomorphism
- $S : H \to H$ is the antipode.

It is an algebra and coalgebra anti-automorphism:

$$S(ab) = S(b)S(a), \quad (S \otimes S) \circ \Delta(a) = \Delta^{op}(S(a))$$

- Invertible $\lambda, \rho \in H$.

These data satisfy certain identities, so that the category of finite-dimensional modules $\underline{H\text{-mod}}$ is a rigid monoidal category with

- $(\pi_V, V) \otimes (\pi_U, U) = ((\pi_V \otimes \pi_U) \circ \Delta, V \otimes U)$
- $a_{U,V,W} : ((\pi_U, U) \otimes (\pi_V \otimes V)) \otimes (\pi_W, W) \xrightarrow{\sim}$

$$\xrightarrow{\sim} (\pi_U, U) \otimes ((\pi_V, V) \otimes (\pi_W, W))$$

given by

$$a_{U,V,W} = (\pi_U \otimes \pi_V \otimes \pi_W)(\Phi)$$

- $\mathbb{1} = (\varepsilon, k)$
- $\ell_V : V \otimes \mathbb{1} \simeq V, \quad \ell_V(x \otimes a) = \pi_V(\lambda) a x$

$r_V : \mathbb{1} \otimes V \simeq V, \quad r_V(a \otimes x) = \pi_V(\rho) a x$

- $(\pi_V, V)^* = (\pi_V^* \circ S, V^*)$

Derive the identities for $\Delta, \Phi, \lambda, \mu, \varepsilon, S$ which will guaratee that H-mod is a rigid monoidal category (and complete the definition of a quasi-Hopf algebra).

② Prove that in a quasitriangular Hopf algebra $(H, R = \sum_i \alpha_i \otimes \beta^i \in H \otimes H)$ the element $u = \sum_i S(\beta^i)\alpha_i$ satisfies the identities

(i) $\varepsilon(u) = 1$

(ii) $\Delta u = (u \otimes u) \cdot (R\,\sigma(R))^{-1}$

$$\sigma(R) = \sum_i \beta^i \otimes \alpha^i$$

③ For the quasitriangluluar Hopf algebra $(U_h(sl_2), R)$, generated by E, F, H with defining relations

$$[H,E] = 2E, \quad [H,F] = -2F$$

$$[E,F] = \frac{sh(\frac{hH}{2})}{sh(\frac{h}{2})},$$

comultiplication $\Delta H = H\otimes 1 + 1\otimes H$

$$\Delta E = E\otimes e^{\frac{hH}{2}} + 1\otimes E,$$

$$\Delta F = F\otimes 1 + e^{-\frac{hH}{2}}\otimes F$$

(i) find the action of S^2 on generators

(ii) Compare it with the formula

$$S^2(a) = u\, a\, u^{-1}$$

and observe that $\tau = u e^{-\frac{hH}{2}}$ is a ribbon element, i.e.

- τ is central
- $\varepsilon(\tau) = 1$
- $S(\tau) =$
- $\Delta\tau = (\tau\otimes\tau)(R\,\sigma(R))^{-1}$

(iii) Find the action of τ on irreducible representations V_λ.

Here $\lambda \in \mathbb{Z}_{\geq 0}$, V_λ has a basis $e_\lambda, e_{\lambda-2}, \dots, e_{-\lambda}$, generators act as

$$H e_{\lambda-2i} = (\lambda-2i)\, e_{\lambda-2i}$$

$$E e_{\lambda-2i} = e_{\lambda-2i+2}$$

$$F e_{\lambda-2i} = a_i\, e_{\lambda-2i-2}$$

(iv) Find a_i.