Lectures on quantization of gauge systems

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Abstract A gauge system is a classical field theory where among fields there are connections in a principal G-bundle over the space-time manifold and the classical action is either invariant or transforms appropriately with respect to the action of the gauge group. The lectures are focused on the path integral quantization of such systems. Here two main examples of gauge systems are Yang-Mills, and Chern-Simons.

1 Introduction

Gauge field theories are examples of classical field theories with the degenerate action functional. The degeneration is due to the action of the infinite dimensional gauge group. Among most known examples are the Einstein gravity and Yang-Mills theory. The Faddeev-Popov (FP) method gives a recipe how to construct a quantization of a classical gauge field theory in terms of Feynman diagrams. Such quantizations are known as perturbative, or semiclassical quantizations. The appearance of so-called ghost fermion fields is one of the important aspects of the FP method [20].

The ghost fermions appear in the FP approach as a certain technical tool. Their natural algebraic meaning is clarified in the BRST approach. In the BRST setting fields and ghost fermions are considered together as coordinates on a super-manifold. Functions on this super-manifold are interpreted as elements of the Chevalley complex of the Lie algebra of gauge transformations. In this setting the FP action is a specific cocycle and the fact the integral with the FP action is equal

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to the original integral with the degenerate action is a version of the Lefschetz fixed point formula.

Among all gauge systems the Yang-Mills theory is most interesting for physics because of its role in the standard model in high energy physics [56]. At the moment there is a mathematically acceptable semiclassical (perturbative) definition of the Yang-Mills theory where the partition functions (amplitudes) are defined as formal power series of Feynman integrals. The ultraviolet divergencies in Feynman diagrams involving FP ghost fields can be removed by the renormalization [32], and the corresponding renormalization is asymptotically free [30]. All these properties make the Yang-Mills theory so important for the high energy physics.

A mathematically acceptable definition of the path integral in the 4-dimensional Yang-Mills theory which goes beyond the perturbation theory is still an open problem. One possible direction which may give such definition is the constructive field theory, where the path integral is treated as a limit of finite-dimensional approximations.

Nevertheless, even mathematically loosely defined path integral remains a powerful tool for ”phenomenological” mathematical and physics research in quantum field theory. It predicted many interesting conjectures many of which were proven later by rigorous methods.

The main goal of these notes is a survey of the semiclassical quantization of the Yang-Mills and of the Chern-Simons theories. These lectures can be considered as a brief introduction to the framework of quantum field theory (along the lines outlined by Atiyah and Segal for topological and conformal field theories). The emphases are given to the semiclassical quantization of classical field theories.

In the Einstein gravity the metric on a space time is a field. It is well known in dimension four that the semiclassical (perturbative) quantization of Einstein gravity fails to produce renormalizable quantum field theory. It is also known that three dimensional quantum gravity is related to the Chern-Simons theory for non-compact Lie groups SL_2. In this lectures we will not go as far as the discussion of this theory, but will focus on the quantum Chern-Simons field theory for compact Lie groups.

We start with a sketch of classical field theory, with some examples such as a non-linear sigma model, the Yang-Mills theory and the Chern-Simons theory. Then we outline the framework of quantum field theory following Atiyah and Segal description of basic structures in topological and conformal field theories. The emphasis are given to the semiclassical quantization. Then Feynman diagrams are introduced on the example of finite dimensional oscillatory integral. The Faddeev-Popov and BRST methods are also introduced first in the finite dimensional setting.

Last two sections contain the definition of the semiclassical quantization of the Yang-Mills and of the Chern-Simons theories in which the partition functions in such theories are given by formal power series where the coefficients are determined by Feynman diagrams.

These notes are based on lectures given when the author visited Aarhus University in 2006-2007 and on lectures given during the summer school at Holbæk, Denmark. The notes benefited from the author’s discussions with many people. Special thanks to J. Andersen, L. Faddeev, V. Fock, D. Freed, D. Kazhdan, A. Losev, J. Lott,

P. Mnev, and A. Schwarz. The author also is grateful to referees for many corrections and suggestions which improved the original draft, and to B. Booss-Bavnbek for help and patience, and to A. Budtz for helping with figures.

This work was supported by the Danish National Research Foundation through the Niels Bohr initiative, the author is grateful to the Aarhus University for the hospitality. It was also supported by the NSF grant DMS-0601912, and by DARPA.

2 Local Lagrangian classical field theory

2.1 Space-time categories

Here we will focus on Lagrangian quantization of Lagrangian classical field theories.

In most general terms *objects* of a *d-dimensional space-time category* are $(d-1)$-dimensional manifolds (space manifolds). In specific examples of space-time categories space manifolds are equipped with a structure (orientation, symplectic structure, metric, etc.).

A *morphism* between two space manifolds Σ_1 and Σ_2 is a d-dimensional manifolds M , possibly with a structure (orientation, symplectic, Riemannian, etc.), together with the identification of $\Sigma_1 \sqcup \overline{\Sigma_2}$ with the boundary of M. Here $\overline{\Sigma}$ is the manifold Σ with reversed orientation.

Composition of morphisms is the gluing along the common boundary. Here are examples of space-time categories.

The d-dimensional topological category. Objects are smooth, compact, oriented $d-1$ dimensional manifolds. A morphism between Σ_1 and Σ_2 is the homeomorphism class of a d-dimensional compact oriented manifold with $\partial M = \Sigma_1 \sqcup \overline{\Sigma_2}$ with respect to homeomorphisms constant at the boundary. The orientation on M should agree with the orientation of Σ_i in a natural way.

The composition consists of gluing two morphisms along the common boundary and then taking the homeomorphism class of the result with respect to homeomorphisms constant at the remaining boundary.

d-dimensional Riemannian category. Objects are $d-1$ Riemannian manifolds. Morphisms between two oriented $d-1$-dimensional Riemannian manifolds N_1 and N_2 are oriented d-dimensional Riemannian manifolds M, such that $\partial M = N_1 \sqcup \overline{N_2}$. The orientation on all three manifolds should naturally agree, and the metric on M agrees with the metric on N_1 and N_2 on a collar of the boundary. The composition is the gluing of such Riemannian cobordisms. For the details see [50].

This category is important for many reasons. One of them is that it is the underlying structure for statistical quantum field theories [33] *d*-**dimensional metrized cell complexes**. Objects are $d-1$ dimensional oriented metrized cell complexes (edges are given length, 2-cells have area, etc.). A morphism between two such complexes C_1 and C_2 is a metrized complex C together with two embeddings of metrized cell complexes $i: C_1 \hookrightarrow C$, $j: \overline{C_2} \hookrightarrow C$ where i is orientation reversing and j is orientation preserving. The composition is the gluing of such triples along the common $d-1$-dimensional subcomplex.

This category has a natural subcategory which consists of metrized cell approximations of Riemannian manifolds.

It is the underlying category for all lattice models in statistical mechanics.

Pseudo-Riemannian category The difference between this category and the Riemannian category is that morphisms are pseudo-Riemannian with the signature $d-1,1$. This is the category most interesting for physics. When $d=4$ it represents the space-time structure of our universe.

2.2 Local Lagrangian classical field theory

The basic ingredients of a d-dimensional local Lagrangian classical field theory are:

- For each space-time we assign the space of fields. Fields can be sections of a fiber bundle on a space-time, connections on a fiber bundle over a space-time, etc.
- The dynamics of the theory is determined by a local Lagrangian. It assigns to a field a volume form on M which depends locally on the field. Without giving a general definition we will give illustrating examples of local actions. Assume that fields are functions $\phi: M \to F$, and that F is a Riemannian manifold. An example of an (ultra)local Lagrangian for a field theory in a Riemannian category with such fields is

$$\mathscr{L}(\phi(x), d\phi(x)) = (\frac{1}{2}(d\phi(x), d\phi(x))_F - V(\phi(x)))dx, \tag{1}$$

  where $(.,.)_F$ is the metric on F and the scalar product on forms induced by the metric on M, and dx is the Riemannian volume form on M.
  The action functional is the integral

$$S_M[\phi] = \int_M \mathscr{L}(\phi, d\phi).$$

  Solutions to the Euler-Lagrange equations for S_M form a (typically infinite dimensional) manifold X_M.
- A boundary condition is a constraint on boundary values of fields which in "good cases" intersects with X_M over a discrete set. In other worlds there is a discrete set of solutions to the Euler-Lagrange equations with given boundary conditions.

A d-dimensional classical field theory can be regarded as the functor from the space-time category to the category of sets. It assigns to a $d-1$ dimensional space the set of possible boundary values of fields, and to a space-time the set of possible solutions to the Euler-Lagrange equations with these boundary values.

Some examples of local classical field theories are outlined in the next sections.

2.3 Classical mechanics

In classical mechanics the space time is a Riemannian 1-dimensional manifold with flat metric, that is an interval. Fields in classical Lagrangian mechanics are smooth mapping of an interval of the real line to a smooth finite dimensional manifold N called the configuration space (parametrized paths).

The action in the classical mechanics is determined by a choice of the Lagrangian function $\mathscr{L} : TN \to \mathbb{R}$ and is

$$S_{[t_2,t_1]}[\gamma] = \int_0^t \mathscr{L}(\dot{\gamma}(\tau), \gamma(\tau))d\tau,$$

where $\gamma = \{\gamma(t)\}_{t_1}^{t_2}$ is a parametrized path in N.

Euler-Lagrange equations in terms of local coordinates $q = (q^1, \ldots, q^n) \in N$ and $\xi = (\xi^1, \ldots, \xi^n) \in T_qN$ are

$$-\sum_{i=1}^n \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \xi^i}(\dot{\gamma}(t), \gamma(t)) + \frac{\partial \mathscr{L}}{\partial q^i}(\dot{\gamma}(t), \gamma(t)) = 0$$

where $\mathscr{L}(\xi, q)$ is the value of the Lagrangian at the point $(\xi, q) \in TN$.

Euler-Lagrange equations is a non-degenerate system of second order differential equations if $\frac{\partial^2 \mathscr{L}}{\partial \xi^i \partial \xi^j}(\xi, q)$ is non-degenerate for all (ξ, q). In realistic systems it is assumed to be positive.

Even when the Euler-Lagrange equations are satisfied, the variation of the action is still not necessary vanishes. It is given by boundary terms:

$$\delta S_{[t_2,t_1]}[\gamma] = \frac{\partial \mathscr{L}}{\partial \xi^i}(\dot{\gamma}(t), \gamma(t))\delta\gamma(t)^i|_{t_1}^{t_2} \tag{2}$$

Imposing Dirichlet boundary conditions means fixing boundary points of the path: $\gamma(t_1) = q_1 \in N$, and $\gamma(t_2) = q_2 \in N$. With these conditions the variation of γ at the boundary of the interval is zero and the boundary terms in the variation of the action vanish.

A concrete example of a classical Lagrangian mechanics is the motion of a point-wise particle on a Riemannian manifold in the potential force field. In this case

$$\mathscr{L}(\xi, q) = \frac{m}{2}(\xi, \xi) + V(q), \tag{3}$$

where $(.,.)$ is the metric on N and $V(q)$ is the potential.

2.4 First order classical mechanics

The non-degeneracy condition of $\frac{\partial^2 \mathscr{L}}{\partial \xi^i \partial \xi^j}(\xi, q)$ is violated in an important class of first order Lagrangians.

Let α be a 1-form on N and b be a function on N. Define the action

$$S_{[t_2,t_1]}[\gamma] = \int_{t_1}^{t_2} (< \alpha(\gamma(t)), \dot{\gamma}(t) > + b(\gamma(t)))dt,$$

where γ is a parametrized path.

Euler-Lagrange equations for this action are:

$$\omega(\dot{\gamma}(t)) + db(\gamma(t)) = 0,$$

where $\omega = d\alpha$. Naturally, the first order Lagrangian system is called non-degenerate if the form ω is non-degenerate. It is clear that non-degenerate first order Lagrangian system defines a symplectic structure on a manifold N. The Euler-Lagrange equations for such system are equations for flow lines of the Hamiltonian on the symplectic manifold (N, ω) generated by the Hamiltonian $H = -b$. It is also clear that the action of a non-degenerate first order system is exactly the Hamilton-Jacobi action for this Hamiltonian system.

Assuming that γ satisfies the Euler-Lagrange equations the variation of the action does not yet vanishes. It is given by the boundary terms (2):

$$\delta S_{[t_2,t_1]}[\gamma] = < \alpha(\gamma(t)), \delta\gamma(t) > |_{t_1}^{t_2}$$

If $\gamma(t_1)$ and $\gamma(t_2)$ are constrained to Lagrangian submanifolds in N these terms vanish.

Thus, constraining boundary points of γ to a Lagrangian submanifold is a natural boundary condition for non-degenerate first order Lagrangian systems. As we will see, this is a part of the more general concept where Lagrangian submanifolds are natural boundary conditions for Hamiltonian systems.

2.5 Scalar field

The space-time in such theory is a Riemannian category. Fields are smooth mappings from a space time to $\mathbb{R}$ (sections of the trivial fiber bundle $M \times \mathbb{R}$). The action function is

$$S_M[\phi] = \int_M \left(\frac{1}{2}(d\phi(x), d\phi(x)) - V(\phi(x)) \right) dx,$$

where the first term is determined by the metric on M and dx is the Riemannian volume form. The Euler-Lagrange equations are:

$$\Delta\phi + V'(\phi) = 0. \tag{4}$$

The Dirichlet boundary conditions fix the value of the field at the boundary $\phi|_{\partial M} = \eta$ for some $\eta : \partial M \to \mathbb{R}$. The normal derivative of the field at the boundary varies for this boundary conditions.

2.6 *Pure Euclidean d-dimensional Yang-Mills*

2.6.1 Fields, the classical action, and the gauge invariance

The space-time is a Riemannian d-dimensional manifold. Fields are connections on a principle G-bundle P over M, where G is a compact Lie group (see for example [23]) for basic definitions). Usually it is a simple (or Abelian) Lie group.

The action functional is given by the integral

$$S_M[A] = \int_M \frac{1}{2} tr < F(A), F(A) > dx,$$

where $< .,. >$ is the scalar product of two-forms on M induced by the metric, $tr(AB)$ is the Killing form on the Lie algebra $\mathfrak{g} = Lie(G)$, $F(A)$ is the curvature of A, and dx is the volume form.

Euler-Lagrange equations for the Yang-Mills action are:

$$d_A^* F(A) = 0.$$

The Yang-Mills action is invariant with respect to gauge transformations. Recall that gauge transformations are bundle automorphisms (see for example [23]). Locally, a gauge transformation acts on a connection as

$$A \to A^g = g^{-1}Ag + g^{-1}dg$$

Here we assume that G is a matrix group, and $g^{-1}dg$ is the Maurer-Cartan form on G. Now let us describe Dirichlet boundary conditions for the Yang-Mills theory. Fix a connection A^b on $P|_{\partial M}$. The Dirichlet boundary conditions on the connection A for the Yang-Mills theory require that A^b is the pull-back of A to the boundary induced by the embedding $i : \partial M \to M$, i.e. $i^*(A) = A^b$.

The Yang-Mills action is invariant with respect to bundle automorphisms (gauge transformations). Generically, gauge class of Dirichlet boundary conditions defines finitely many gauge classes of solutions to the Yang-Mills equations.

2.7 Yang-Mills field theory with matter

Let V be a finite dimensional representation of the Lie group G, and $V_P = P\times_G V$ be the vector bundle over M associated to a principal G-bundle P. Assume that V has an invariant scalar product $(.,.)$.

The classical Yang-Mills theory with matter fields, which are sections of V_P, has the action functional

$$S[\Phi,A]=\int_M \left(\frac{1}{2}tr<F(A),F(A)>+\frac{1}{2}(<d_A\Phi,d_A\Phi>)+U(\Phi)\right)dx,$$

where U is a G-invariant function on V and $<.,.>$ is the scalar product on forms defined by the metric on M. The function U describes the self-interaction of the scalar field Φ.

Euler-Lagrange equations in this theory are

$$*d_AF(A)+j_A=0,\quad d_A^*d_A\Phi-U'(\Phi)=0,$$

where $j_A\in\Omega^1(M,\mathfrak{g})$ is the one form defined as $tr<\omega,j_A>=<\omega\Phi,d_A\Phi>$.

Dirichlet boundary conditions in this theory are determined by Dirichlet boundary conditions for the connection A and for the scalar field Φ.

2.8 3-dimensional Chern-Simons theory

In this case the space-time category is the category of 3-dimensional topological cobordisms. Fix a smooth 3-dimensional manifold M. The space of fields of the Chern-Simons theory is the space of connections on a principal G-bundle P over M (just as in the Yang-Mills theory). The choice of a simple compact Lie group G is part of the data.

The Chern-Simons form is the 3-form on P:

$$\alpha(A)=tr\left(A\wedge dA-\frac{2}{3}A\wedge[A\wedge A]\right).$$

Any principal G-bundle over a manifold of dimension 3 or smaller is trivializable, i.e. admits a section $p: M\to P$. The Chern-Simons action is

$$CS_M(A,p)=\int_M p^*\alpha(A),$$

where $p^*\alpha(A)$ is the pull-back of the 3-form $\alpha(A)$ on P to a 3-form on M.

This action is of the first order (in derivatives of A). It is very different from the Yang-Mills theory where the action is of the second order.

The variation of the Chern-Simons action is

$$\delta CS_M(A,p) = \int_M tr\,(F(A)\wedge \delta A) + \int_{\partial M} tr\,(p^*(A_\tau)\wedge \delta A_\tau)\,,$$

where $A_\tau, \delta A_\tau$ are pull-backs to the boundary of A and δA.

Euler-Lagrange equations for this Lagrangian are

$$F(A) = 0.$$

They guarantee that the first term (the bulk) in the variation vanishes. Solutions to Euler-Lagrange equations are flat connections in P over M. On the space of solutions to the Euler-Lagrange equations

$$\delta CS_M(A,p) = (\Theta, \delta A_\tau),$$

where Θ is the one form on the space $C_{\partial M}$ of connections on $P|_{\partial M} \to \partial M$. Let D be the differential acting on forms on the space $C_{\partial M}$. The form $\omega = D\Theta$ is non-degenerate and defines a symplectic structure on $C_{\partial M}$.

$$\omega(\delta A, \delta B) = \int_{\partial M} tr \delta A \wedge \delta B \tag{5}$$

A natural boundary condition for the Chern-Simons Lagrangian is the isotropic subspace in $(C_{\partial M}, \omega)$ of flat connections on ∂M which continue to flat connections on M. The form Θ restricted to such subspace L vanishes on tangent vectors to L.

The Chern-Simons action is gauge invariant (for details see [23]). The action of the gauge group is Hamiltonian on $(C_{\partial M}, \omega)$. The result of the Hamiltonian reduction of this symplectic space with respect to the action of the gauge group is the finite dimensional moduli space $F(\partial M)$ of gauge flat connections together with reduced symplectic structure.

Gauge orbits through flat connections from $C_{\partial M}$ which continue to flat connections on P over M form a Lagrangian submanifold $L_M \subset F(\partial M)$. The corresponding first order Hamiltonian system describes the reduced Chern-Simons theory as a classical Hamiltonian field theory. For details see for example [23], [7], and references therein.

3 Hamiltonian local classical field theory

3.1 The framework

An n-dimensional Hamiltonian field theory in a category of space-times is an assignment of the following data to manifolds which are objects and morphisms of this category:

- A symplectic manifold $S(M_{n-1})$ to an $(n-1)$-dimensional manifold M_{n-1}.
- A Lagrangian submanifold $L(M_n) \subset S(\partial M_n)$ to each n-dimensional manifold M_n.

These data should satisfy the following axioms:

- $S(\emptyset) = \{0\}$
- $S(M_1 \sqcup M_2) = S(M_1) \times S(M_2)$
- $L(M_1 \sqcup M_2) = L(M_1) \times L(M_2)$ with $L(M_i) \subset S(\partial M_i)$.
- $(S(\overline{M}), \omega) = (S(M), -\omega)$
- An orientation preserving diffeomorphism $f : M_1 \to M_2$ of $(n-1)$-dimensional manifolds lifts to a symplectomorphism $s(f) : S(M_1) \to S(M_2)$.
- Assume that $\partial M = (\partial M)_1 \sqcup (\partial M)_2 \sqcup \overline{(\partial M)'}$ and that there is an orientation reversing diffeomorphism $f : (\partial M)_1 \to \overline{(\partial M)_2}$. Denote by M_f the result of gluing M along $(\partial M)_1 \simeq \overline{(\partial M)_2}$ via f:

$$M_f = M / < (\partial M)_1 \simeq \overline{(\partial M)_2} > .$$

The Lagrangian submanifold corresponding to the result of the gluing should be

$$L(M_f) = \{x \in S((\partial M)')| \text{ such that there exists } y \in S(\partial M)_1 \text{ with } (y, s(f)(y), x) \in L(M)\}, \quad (6)$$

Notice that $\partial M_f = (\partial M)'$ by definition. This axiom is known as the gluing axiom. In classical mechanics the gluing axiom is the composition of the evolution at consecutive intervals of time.

A *boundary condition* in the Hamiltonian formulation is Lagrangian submanifold in the symplectic manifold assigned to the boundary of the manifold, $L^b(\partial M) \subset S(\partial M)$. It factorizes into the product of Lagrangian submanifolds corresponding to connected components of the boundary:

$$L^b((\partial M)_1 \sqcup (\partial M)_2) = L^b((\partial M)_1) \times L^b((\partial M)_2)$$

Classical solutions with given boundary conditions are intersection points $L^b(\partial M) \cap L(M)$.

In order to glue classical solutions along the common boundary (composition of classical trajectories in classical mechanics) let us assume that boundary Lagrangian submanifolds are fibers of Lagrangian fiber bundles. That is, we assume that for each connected component of the boundary $(\partial M)_i$ that $S((\partial M)_i)$ is given together with the Lagrangian fiber bundle $\pi_i : S((\partial M)_i) \to B((\partial M)_i)$.

3.2 Hamiltonian formulation of local Lagrangian field theory

Here again, instead of giving general definitions we will give few illustrating examples.

3.2.1 Classical Hamiltonian mechanics

1. Let $H \in C^\infty(M)$ be the Hamiltonian function generating Hamiltonian dynamics on a symplectic manifold M[1]. Here is how such system can be reformulated in the framework of a Hamiltonian field theory.

Objects of the corresponding space time category are points, morphisms are intervals $\mathbb{R}$ with the flat metric. The boundary of each interval consists of two points (with different orientation). The symplectic manifold assigned to the boundary of the space time is

$$M(I) = \overline{M} \times M,$$

where M is the phase space of the Hamiltonian system and $\overline{M}$ is the phase space with the opposite sign of the symplectic form.

The Lagrangian subspace $L(I)$ in $M(I)$ is the set of pairs of points (x,y) where x is the initial point of a classical trajectory generated by H and y is a target point of this trajectory.

A pair of Lagrangian fiber bundles $\pi_1 : M \to B_1$ and $\pi_2 : M \to B_2$ defines a complete family of boundary conditions is corresponding to an interval I with fiber bundles corresponding to two components of the boundary of I.

Classical trajectories with such boundary conditions are intersection points of $\pi^{-1}(b_1) \times \pi^{-1}(b_2)$ with $L(I)$.

2. The Lagrangian mechanics on N (see section 2.3) is equivalent (for non-degenerate Lagrangians) to the Hamiltonian mechanics on $M = T^*N$ with the canonical symplectic form. The Hamiltonian functions given by the Legendre transform of the Lagrangian:

$$H(p,q) = \max_{\xi \in T_q N} (p(\xi) - L(\xi, q))$$

The boundary conditions $q(t_1) = q_1, q(t_2) = q_2$ correspond to Lagrangian fiber bundles $T^*N \to N$ for each component of the boundary of the interval.

The Hamiltonian of a point-wise particle on a Riemannian manifold is :

$$H(p,q) = \frac{m}{2}(p,p) + V(q)$$

where (p,p) is uniquely determined by the metric on N.

3. A non-degenerate first order Lagrangian defines the symplectic structure on the configuration space M given by $\omega = d\alpha$. Solutions to the Euler-Lagrange in such system are flow lines of the Hamiltonian vector field generated by the function $b(q)$, see section 2.4. So, first order non-degenerate Lagrangian systems are simply Hamiltonian systems on exact symplectic manifolds (i.e. on symplectic manifolds where the form ω is exact).

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[1] Recall that Hamiltonian mechanics is a dynamical system on a symplectic manifold (M,ω) with trajectories being flow lines of the Hamiltonian vector field v_H generated by the function $H \in C^\infty(M)$, $v_H = \omega^{-1}(dH)$. Here $\omega^{-1} : T^*M \to TM$ is the isomorphism induced by the symplectic structure on M.

3.2.2 Bose field theory

In this case the symplectic manifold $S(N)$ assigned to a $(d-1)$-dimensional manifold N is an infinite dimensional linear symplectic manifold which is the cotangent bundle to the space of real-valued smooth functions on N.

Since $C^\infty(N)$ is a linear space its tangent space at any point (can be thought as the space of infinitesimal variations of functions on N) can be naturally identified with the $C^\infty(N)$ itself.

$$\omega((\delta\eta_1,\delta f_1),(\delta\eta_2,\delta f_2)) = \int_N (\delta\eta_1\delta f_2 - \delta\eta_2\delta f_1)dx,$$

where $\eta_i \in T_{f_i}C^\infty(N)$ and $(\delta\eta_i,\delta f_i)$ are tangent vectors from $T_{(\eta_i,f_i)}(T^*C^\infty(N))$. In this formula we identified the tangent space to $T^*C^\infty(N)$ with $C^\infty(N)\oplus C^\infty(N)$.

The Lagrangian fibration corresponding to the Dirichlet boundary conditions is the standard projection $\pi : T^*C^\infty(N) \to C^\infty(N)$.

The Lagrangian submanifold $L(M) \subset S(\partial M)$ is the space of solutions to the Euler-Lagrange equations. Solutions to the Euler-Lagrange equations with given Dirichlet boundary condition $\phi|_{\partial M} = \eta$ are intersection points of $L(M)$ with the Lagrangian fiber $\pi^{-1}(\eta)$.

3.2.3 Yang-Mills theory

Here we will discuss only the Yang-Mills theory where fields are connections in a trivial principal G-bundle. The symplectic manifold $S(N)$ assigned to the $(d-1)$-dimensional manifold N in such field theory is the cotangent bundle to the space of connections in trivial principal G-bundle over N with the natural symplectic structure.

As in the case of a scalar Bose field the symplectic manifold is the cotangent bundle to the space of all possible Dirichlet boundary conditions. Since the space $\mathscr{A}(N)$ of all smooth connections on N is linear, its tangent bundle can naturally be identified with $\mathscr{A}(N)\oplus\mathscr{A}(N)$. The symplectic form is

$$\omega((\delta\eta_1,\delta A_1),(\delta\eta_2,\delta A_2)) = \int_N (tr < \delta\eta_1,\delta A_2 > -tr < \delta\eta_2,\delta A_1 >)dx$$

Here tangent vectors $(\delta\eta_i,\delta A_i) \in T_{(\eta_i,A_i)}(T^*\mathscr{A}(N)$ are $\mathfrak{g}$-valued 1-forms on N. The scalar product $< .,. >$ is the scalar product on 1-forms induced by the metric on N.

3.2.4 Chern-Simons

The main difference between the Yang-Mills theory and the Chern-Simons field theory is that the YM theory is the second order theory and the CS is the first order theory. Solutions to the Euler-Lagrange equations are flat connection on M and their pull-backs to the boundary are flat connections on the boundary ∂M.

Without taking into account the gauge invariance of the Chern-Simons theory, this suggests the its Hamiltonian formulation with the symplectic manifold attached to the boundary of M being the space of connections on the principal G-bundle $P|_{\partial M}$ with symplectic structure (5). The submanifold $L(M)$ of solutions to the Euler-Lagrange equations (flat connections on P) is an isotropic subspace of flat connections on $P_{\partial M}$ which continue to flat connections on P over M. The submanifold $L(M)$ is not Lagrangian, and this due to the gauge invariance.

The correct Hamiltonian formulation of the Chern-Simons theory is the Hamiltonian reduction of the construction above with respect to the action of the gauge group. The symplectic manifold assigned to the boundary is the moduli space of flat connections on $P_{\partial M}$ and the Lagrangian submanifold $L(M)$ is the space of gauge classes of flat connections on $P_{\partial M}$ which continue to flat connections on P.

4 Quantum field theory framework

4.1 General framework of quantum field theory

We will follow the framework of local quantum field theory which was outlined by Atiyah and Segal for topological and conformal field theories. In a nut-shell it is a functor from a category of cobordisms to the category of vector spaces (or, more generally to some ”known” category).

All known local quantum field theories can be formulated in this way at some very basic level. It does not mean that this is a final destination of our understanding of quantum dynamics at the microscopical scale. But at the moment this general setting includes the standard model, which agrees with most of the experimental data in high energy physics. In this sense this is the accepted framework at the moment, just as at different points of history classical mechanics, classical electromagnetism, and quantum mechanics were playing such a role [2].

A quantum field theory in a given space-time category can be defined as a functor from this category to the category of vector spaces (or to another ‘standard’, ‘known’ category).

It assigns a vector space to the boundary and a vector in this vector space to the manifold:

$$N \mapsto H(N), \;\; M \mapsto Z_M \in H(\partial M).$$

The vector space assigned to the boundary is the space of pure of the system on M. It may depend on the extra structure at the boundary (it can be a vector bundle

[2] The string theory goes beyond such framework and beyond scales of present experiments. It is a necessary step further, and it already produced a number of outstanding mathematical ideas and results. One of the differences between the string theory and the quantum field theory is that the concept of non-perturbative string theory is still developing .

over the moduli space of such structures). The vector $Z(M)$ is called the partition function, or the amplitude.

These data should satisfy natural axioms, such as

$$H(\emptyset) = \mathbb{C}.$$

$$H(N_1 \sqcup N_2) = H(N_1) \otimes H(N_2), \tag{7}$$

$$Z_{M_1 \sqcup M_2} = Z_{M_1} \otimes Z_{M_2} \in H(\partial M_1) \otimes H(\partial M_2). \tag{8}$$

An isomorphism $f : N_1 \to N_2$ lifts to a linear isomorphism

$$\sigma(f) : H(N_1) \to H(N_2).$$

The pairing

$$< .,. >_N : H(\overline{N}) \otimes H(N) \to \mathbb{C}$$

is defined for each N. This pairing should agree with partition functions in the following sense. Let $\partial M = N \sqcup \overline{N} \sqcup N'$, then

$$(\langle .,. \rangle \otimes id) Z_M = Z_{M_N} \in H(N') \tag{9}$$

where M_N is the result of gluing of M along N. The operation is known as the gluing axiom. We outlined its structure. The precise definition involve more details (see [4], [48]). The gluing axiom in particular implies the functoriality of Z:

$$Z_{M_1 \circ M_2} = Z_{M_1} * Z_{M_2}.$$

Originally this framework was formulated by Atiyah and Segal for topological and conformal field theories but it is natural to extend it to more general and more realistic quantum field theories, including the standard model.

This framework is very natural in models of statistical mechanics on cell complexes with open boundary conditions, also known as lattice models.

The main physical concept behind this framework is the locality of the interaction. Indeed, we can cut our space-time manifold in small pieces and the resulting partition function Z_M in such framework is expected to be the composition of partition functions of small pieces. Thus, the theory is determined by its structure on ‘small’ space-time manifolds, or at ‘short distances’. This is the concept of locality.

4.2 Constructions of quantum field theory

4.2.1 Quantum mechanics

Quantum mechanics fits into the framework of quantum field theory as a one-dimensional example. One-dimensional space time category is the same as in classical Lagrangian mechanics.

In quantum mechanics of a point particle on a Riemannian manifold N the vector space assigned to a point is $L_2(N)$ with the usual scalar product. The quantized Hamiltonian is the second order differential operator acting in $L_2(N)$

$$\hat{H} = -\frac{mh^2}{2}\Delta + V(q),$$

where Δ is the Laplace operator on N, $V(q)$ is the potential, and h is the Plank constant.

The operator

$$U_{t_2-t_1} = e^{\frac{i}{h}\hat{H}(t_2-t_1)} \tag{10}$$

is known as the propagator, or evolution operator in quantum mechanics. It is a unitary operator in $L_2(N)$ (assume N is compact and $V(q)$ is sufficiently good). It can be written as an integral operator:

$$U_{t_2-t_1}(f)(q) = \int_N U_{t_2-t_1}(q,q')f(q')dq', \tag{11}$$

where dq' is the volume measure on N induced by the metric.

The kernel $U_t(q,q')$ is a solution to the Schrödinger equation

$$(ih\frac{\partial}{\partial t} - \frac{h^2}{2m}\Delta + V(q))U_t(q,q') = 0 \tag{12}$$

for $t > 0$ with the initial condition

$$\lim_{t\to+0} U_t(q,q') = \delta(q,q').$$

Quantum mechanics of a point particle on a Riemannian manifold N viewed as a 1-dimensional quantum field theory assigns the vector space $L_2(N)$ to a point, and the vector $Z(I)(q_1,q_2) = U_{t_2-t_1}(q_2,q_1) \in H(\partial I) = \overline{L_2(N) \otimes L_2(N)}$ to the interval $[t_2,t_1]$. Here $\overline{L_2(N) \otimes L_2(N)}$ is a certain completion of the tensor product which can be identified with a space of operators in $L_2(N)$, for details see any mathematically minded textbook on quantum mechanics, for example [51]. For a variety of reasons it is better to think about the space attached to a point not as $L_2(N)$ but as the space of 1/2-densities on N. Given two 1/2-densities a and b, their scalar product is

$$(a,b) = \int_N \bar{a}b$$

where $\bar{a}b$ is now a density and can be integrated over N (for details see for example [10]). In terms of 1/2-densities the kernel of the evolution operator is a 1/2-density on $N \times N$ and

$$U_t(a)(q) = \int_N U_t(q,q')a(q'),$$

where $U_t(q,q')a(q')$ is a density in q' and can be integrated over N.

4.2.2 Statistical mechanics

Lattice models in statistical mechanics also fit naturally in the framework of quantum field theory. The space time category corresponding to these models is a combinatorial space category of cell complexes.

A simple combinatorial example of combinatorial quantum field theory with the dimer partition function can be found in [18].

The combinatorial construction of the TQFT (Topological Quantum Field Theory) based on representation theory of quantized universal enveloping algebras at roots of unity is given in [45] or, more generally, on any modular category.

Another combinatorial construction of TQFT, based on triangulations is given in [53]. This TQFT is the double of the construction from [45], for details see for example [54].

4.2.3 Path integral and the semiclassical quantization

If we were able to integrate over the space of fields in a Lagrangian classical field theory (as in lattice models in statistical mechanics) we could construct a quantization of a d-dimensional classical Lagrangian system as follows:

- Assign the space of functionals on boundary values of fields to a $(d-1)$-manifolds. here we assume that a choice of boundary conditions was made.
- To a d-dimensional manifold we assign the functional on boundary fields given by the integral
$$Z_M(b) = \int_{\phi|_{\partial M}=b} \exp(\frac{iS[\phi]}{h})D\phi.$$
If one treat the integral as a formal symbol which satisfies the Fubini's theorem (the iterated integral is equal to the double integral), such assignment satisfies all properties of QFT. The problem is that the integral is, usually not defined, unless the the space of fields is finite or finite-dimensional (as in statistical mechanics of cell complexes) this really defines a QFT. Thus, one should either make sense of the integral, and check if the definition satisfies the Fubini's theorem, or define the QFT by some other means.

There are two approaches on how to make sense of path integrals. The approach of constructive field theory, is based on the approximating the path integral by a finite dimensional integrals and then proving that when the mesh of the approximation goes to zero, the finite QFT has a limit. For details on this approach see for example in [29].

Another approach is known as perturbation theory, or semiclassical limit. The main idea is to define the path integral in the way its asymptotic expansion as $h \to 0$ would look like if the integral were defined. The coefficients of this asymptotic expansion are given by Feynman diagrams. Under the right assumptions the first few coefficients would approximate the desired quantity sufficiently well. The numbers derived from this approach are the base for the comparison of quantum field theoretical models of particles with the experiment.

In the next sections we will outline this approach on several examples.

When M is a cylinder $M = [t_1, t_2] \times N$, the partition function Z_M is an element of $H(N) \times H(N)^*$ [3] and therefore can be regarded as operators in $H(N)$. Classical observables become operators acting in $H(N)$. Thus, a quantization of classical field theories for space-time cylinders can be regarded as passing from classical commutative observables to quantum non-commutative observables. The partition function for the torus has a natural interpretation as a trace of the partition function for the cylinder (see for example [29] for more details).

5 Feynman diagrams

5.1 Formal asymptotic of oscillatory integrals

Let $\mathscr{M}$ be a compact smooth manifold with a volume form on it. In this section we will recall the diagrammatic formula for the asymptotic expansion of the integral

$$I_h(f) = \int_{\mathscr{M}} \exp\left(i\frac{f(x)}{h}\right) dx \tag{13}$$

where f is a smooth function on $\mathscr{M}$ with finitely many isolated critical points.

Lemma 1. *We have the following identity*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} e^{i(x,Bx)/2 - \varepsilon(x,x)} x_{i_1} \cdots x_{i_n} d^N x$$
$$= (2\pi)^{\frac{N}{2}} i^{\frac{n}{2}} \frac{1}{\sqrt{|det(B)|}} e^{\frac{i\pi}{4} sign(B)} \sum_m B^{-1}_{i_{m_1} i_{m_2}} B^{-1}_{i_{m_1} i_{m_2}} \dots B^{-1}_{i_{m_{n-1}} i_{m_n}} \tag{14}$$

[3] In this rather general discussion of the basic structures of a local quantum field theory we are deliberately somewhat vague about such details as the completion of the tensor product and similar topological questions. Such questions are better answered on a case-by-case basis.

Here the sum is taken over perfect matchings m on the set $\{1,2,\dots,n\}$*, sign*(B) *is the signature of the real symmetric matrix B (the number of positive eigenvalues minus the number of negative eigenvalues. If n is odd, this integral is zero.*

Proof. First notice that:

$$\lim_{\varepsilon\to 0}\int e^{\frac{i}{2}(x,Bx)-\varepsilon(x,x)}x_{i_1}\cdots x_{i_n}d^Nx=\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}}\int e^{\frac{i}{2}(x,Bx)+(y,x)}d^Nx\Big|_{y=0}$$

After change of variables $z=x-iB^{-1}y$ in the Gaussian integral

$$\int_{\mathbb{R}^N}e^{\frac{i}{2}(x,Bx)}d^Nx=(2\pi)^{\frac{N}{2}}\frac{1}{\sqrt{|det(B)|}}e^{\frac{i\pi}{4}sign(B)}$$

we have:

$$\int_{\mathbb{R}^N}e^{\frac{i}{2}(x,Bx)+(y,x)}d^Nx=(2\pi)^{\frac{N}{2}}\frac{1}{\sqrt{|det(B)|}}e^{\frac{i\pi}{4}sign(B)}e^{\frac{i}{2}(B^{-1}y,y)}$$

Expanding the right side in powers of y we obtain the contribution of monomials of degree $2k$.

$$\frac{i^k}{2^k!}\sum_{(i)(j)}(B^{-1})_{i_1j_1}\dots(B^{-1})_{i_{2k}j_{2k}}y_{i_1}\dots y_{i_k}y_{j_1}\dots y_{j_k}=\frac{i^k}{2^k!}\sum_{i_1\le\cdots\le i_{2k}}\frac{y_{i_1}\dots y_{i_{2k}}}{m_1(i)!\dots m_{2k}(i)!}$$
$$\sum_{\sigma\in S_{2k}}(B^{-1})_{\sigma(i_1)\sigma(i_2)}(B^{-1})_{\sigma(i_3)\sigma(i_4)}\cdots(B^{-1})_{\sigma(i_{2k-1})\sigma(i_{2k})}$$

Here $m_1(i)$ is the number of smallest entries in the sequence $i_1,\dots,i_{2k}$, $m_2(i)$ is the number of smallest entries after the elimination of i_1 etc.

Taking derivatives with respect to y and taking into account that

$$\frac{1}{2^k!}\sum_{\sigma\in S_{2k}}(B^{-1})_{\sigma(i_1)\sigma(i_2)}(B^{-1})_{\sigma(i_3)\sigma(i_4)}\cdots(B^{-1})_{\sigma(i_{2k-1})\sigma(i_{2k})}=$$
$$\sum_m(B^{-1})_{i_{m_1}i_{m_2}}(B^{-1})_{i_{m_1}i_{m_2}}\dots(B^{-1})_{i_{m_{2k-1}}i_{m_{2k}}}$$

where the sum is taken over perfect matchings on the set $\{1,2,,\dots,2k\}$ we obtain the desired formula.

For example when $n=4$, then this integral is equal to

$$(B^{-1})_{12}(B^{-1})_{34}+(B^{-1})_{13}(B^{-1})_{24}+(B^{-1})_{14}(B^{-1})_{23}.$$

These three terms correspond to perfect matching shown on Fig. 1.

Theorem 1. *We have the following identity of power series*

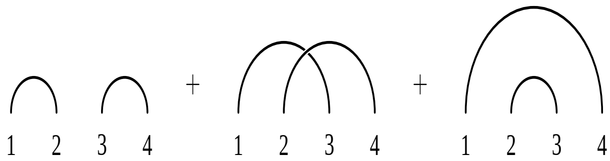


Fig. 1 Perfect matching for $n=4$

$$\int_{\mathbb{R}^N} \exp(i(x,Bx)/2 - \sum_{n\geq 3} \frac{i}{n!} V^{(n)}(x) h^{n/2-1})\, d^N x =$$
$$(2\pi)^{\frac{N}{2}} \frac{1}{\sqrt{|\det(B)|}} e^{\frac{i\pi}{4} sign(B)} \sum_{\Gamma} \frac{(ih)^{-\chi(\Gamma)} F(\Gamma)}{|Aut(\Gamma)|}$$

where sum is taken over graphs with vertexes of valency ≥ 3, $F(\Gamma)$ is the state sum corresponding to Γ described below, $|Aut(\Gamma)|$ is the number of elements in the automorphism group of Γ, $\chi(\Gamma)$ is the Euler characteristic of the graph $\chi(\Gamma) = |V| - |E|$, where $|E|$ is the number of edges of Γ, and $|V|$ is the number of vertices of Γ. Each term in the formal power series on the left side is the limit of a convergent integral (as in the previous lemma).

Proof. Expanding the integral in formal power series in h we have:

$$\int_{\mathbb{R}^N} e^{\frac{i}{2}(x,Bx) + \sum_{n\geq 3} \frac{i}{n!} V^{(n)}(x) h^{n/2-1}}\, dx = \sum_{n_3\geq 0, n_4 \geq 0 \cdots} \frac{h^{(3n_3+4n_4+\cdots)/2 - n_3 - n_4 - \cdots} i^{n_3+n_4+\ldots}}{n_3!(3!)^{n_3} n_4!(4!)^{n_4}\cdots}$$
$$\int_{\mathbb{R}^N} e^{i(x,Bx)/2} (V^{(3)}(x))^{n_3} (V^{(4)}(x))^{n_4} \cdots d^N x \quad (15)$$

Here
$$V^{(n)}(x) = \sum_{i_1,\ldots,i_n} V^{(n)}_{i_1,\ldots,i_n} x^{i_1} \ldots x^{i_n}$$

For a graph Γ define the state sum $F(\Gamma)$ as follows.

- Enumerate vertices, for each vertex enumerate edges adjacent to it. This defines a total ordering on endpoints of edges (the ordering from left to right on Fig. 2).
- The graph Γ defines a perfect matching between edges adjacent to vertices as it is shown on Fig. 2. Denote by Γ_m the graph corresponding to the perfect matching m.
- Assign indices $i_1, i_2, \ldots$ to endpoints of edges, $i_\alpha = 1, 2, \ldots, N$.
- Define $F(\Gamma)$ as

$$F(\Gamma) = \sum_{\{i\}} \prod_{e\in E(\Gamma_m)} (B^{-1})_{e_l, e_r} V^{(n_1)}_{i_1,\ldots,i_{n_1}} V^{(n_1)}_{i_{n_1+1},\ldots,i_{n_1+n_2}} V^{(n_1+n_2+1)}_{i_1,\ldots,i_{n_1+n_2+n_3}} \ldots$$

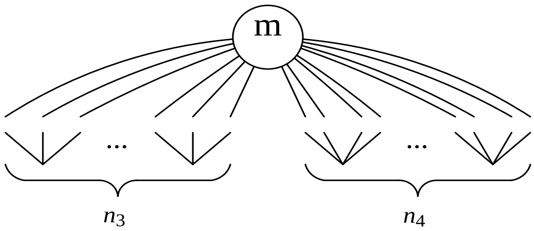


Fig. 2 Perfect matchings and Feynman diagrams.

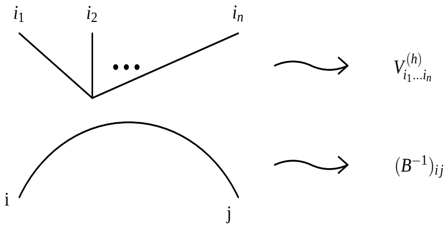


Fig. 3 Weights of vertices and edges in Feynman diagrams.

where e_l is the index corresponding to the left end of the edge e, e_r correspond to the right side. The state sum $F(\Gamma)$ is the sum over $\{i\}$ of the product of weights assigned to vertices and edges according to the rules from Fig. 3. [4]

Lemma 1 gives the following expression for (15)

$$(2\pi)^{\frac{N}{2}}\frac{1}{\sqrt{|\det(B)|}}e^{\frac{i\pi}{4}sign(B)}i^{|V|}$$
$$\sum_{n_3\geq 0,n_4\geq 0\cdots}\frac{(ih)^{|E|-|V|}}{n_3!(3!)^{n_3}n_4!(4!)^{n_4}\cdots}\sum_m F(\Gamma_m) \quad (16)$$

Here the sum is taken over perfect matchings, and Γ_m is the graph corresponding to the matching m, see Fig. 2, $|E|$ is the number of edges and $|V|$ is the number of vertices of the graph Γ_m.

Some perfect matchings produce the same graphs. Denote $N(\Gamma)$ the number of perfect matchings corresponding to Γ. In the formula (16) the contribution from the diagram Γ will have the combinatorial factor

[4] Equivalently $F(\Gamma)$ can be defined as follows. Assign elements $1,\ldots,N$ to end points of edges of Γ. This defines assignment of indices to endpoints of stars of vertices. The state sum is defined as

$$F(\Gamma)=\sum_{\{i\}}\prod_{e\in E(\Gamma)}(B^{-1})_{i_e,j_e}\prod_{v\in V(\Gamma)}(\text{ weight of } v)_i$$

Here weights of vertices are defined on Fig. 3, the indices i_e, j_e correspond to two different endpoints of e (since B is symmetric, it does not matter this pair is defined up to a permutation).

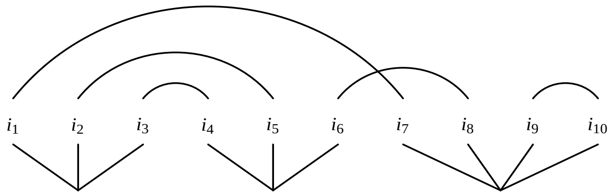


Fig. 4 An example of a perfect matching with a state $\{i\}$.

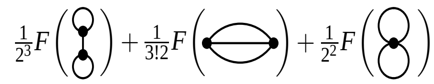


Fig. 5 Contributions from Feynman diagrams of order one.

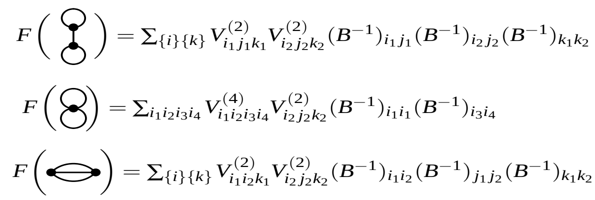


Fig. 6 Weights of Feynman diagrams of order one.

$$\frac{N(\Gamma)}{n_3!(3!)^{n_3} n_4!(4!)^{n_4} \cdots} = \frac{1}{|\mathrm{Aut}(\Gamma)|}$$

This finishes the proof.

There is a simple rule how to check powers of $i = \sqrt{-1}$. These factors disappear if we replace $B \to iB$ and $V^{(n)} \to iV^{(n)}$.

A Feynmann diagram *has order n* if it appears as a coefficient in h^n, i.e. when $n = |E| - |V|$ (or $n = -\chi(\Gamma)$) in the expansion above. As an example, order one Feynman diagrams are given on Fig. 5 and Fig. 6.

Now let us focus on the asymptotic expansion of the integral (13). The standard asymptotic analysis applied to this integral shows that the leading contributions to the asymptotics of the integral as $h \to 0$ come from the infinitesimal (of the diameter of order $h^{-1/2}$) neighborhoods of critical points of $f(x)$. The contribution to the integral (13) from the critical point a localizes to the integral (15) with $(B_a)_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}(a)$ and $(V_a^{(n)})_{i_1,\dots,i_n} = -\frac{\partial^n f}{\partial x^{i_1} \dots x^{i_n}}(a)$.

Choose local coordinates such that $dx = dx_1 \dots dx_N$. Denote by $F_a(\Gamma)$ the state sum on the graph Γ with such matrices B and $V(n)$. The asymptotic expansion of the integral (13) has the following form:

$$\int_{\mathscr{M}} \exp\left(i\frac{f(x)}{h}\right) dx \simeq \sum_a (2\pi h)^{\frac{N}{2}} \frac{1}{\sqrt{|\det(B_a)|}} e^{\frac{if(a)}{h} + \frac{i\pi}{4} sign(B_a)} \sum_\Gamma \frac{(ih)^{-\chi(\Gamma)+1} F_a(\Gamma)}{|\mathrm{Aut}(\Gamma)|} \tag{17}$$

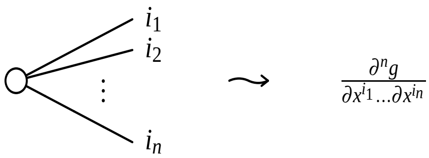


Fig. 7 Extra vertices in Feynman diagrams for integrals 18.

Here $\simeq$ is the asymptotical equivalence when $h \to 0$. Similar argument applied to the integral

$$\int_{\mathcal{M}} \exp(i \frac{f(x)}{h}) g(x) dx \tag{18}$$

give the asymptotic expansion as $h \to 0$ which looks exactly as (17) with the only difference that in each Feynman diagrams there will be exactly one of the vertices given on Fig 7. The order of the diagram is still $|E| - |V|$ where V is the number vertices given by derivatives of f, i.e. $-\chi(\Gamma) + 1$.

5.2 Integrals over Grassmann algebras

The Grassmann algebra G_n is the exterior algebra of $\mathbb{C}^n$, $G_n = \wedge^{\cdot} \mathbb{C}^n$ with the multiplication $(a,b) \to a \wedge b$. As an algebra defined in terms of generators and relations G_n is generated by $c^1, \ldots, c^n$ with defining relations $c^i c^j + c^j c^i = 0$. The Grassmann algebra G_n can also be regarded as the space of polynomial functions on the odd vector space $\mathbb{C}^{0|n}$.

Left derivatives with respect to c^i, are defined as

$$\partial_{c^i} c^{i_1} \cdots c^{i_n} = \begin{cases} 0 & i \notin \{i_1, \ldots, i_n\} \\ (-1)^k c^{i_1} \cdots \hat{c}^{i_k} \cdots c^{i_n} & i = i_k \end{cases}$$

The right derivatives are defined similarly with the sign $(-1)^{n-k}$ instead.

Recall that an *orientation* of $\mathbb{C}^n$, a basis in $\bigwedge^n \mathbb{C}^n$. Choose $c^1 \wedge \cdots \wedge c^n$ as such orientation. Any element $P \in G_n$, can be written as $p^{top} c^1 \wedge \cdots \wedge c^n +$ lower terms. The integral of P over the odd vector space $\mathbb{C}^{0|n}$ with the orientation $c_1 \wedge \cdots \wedge c_n$ is

$$\int_{\mathbb{C}^{0|n}} P \, dc := p^{top}.$$

Lemma 2. *Let $(c, Bc) = \sum_{ij=1}^n c^i B_{ij} c^j$ where B is skew-symmetric $B_{ij} = -B_{ji}$. If n is even, then*

$$\int_{\mathbb{C}^{0|n}} \exp\Big(\frac{1}{2}(c, Bc)\Big) \, dc = Pf(B), \tag{19}$$

where Pf is the Pfaffian of the matrix B. If n is odd the integral is zero.

Proof. Recall that

$$\mathrm{Pf}(B) = \sum_m (-1)^m B_{i_1 j_1} B_{i_2 j_2} \dots B_{i_{n/2} j_{n/2}}.$$

where the sum is taken over perfect matchings m. A perfect matching m is the equivalence class of a collection of pairs $((i_1, j_1), \dots, (i_{n/2}, j_{n/2}))$ obtained by a permutation σ of $(1, 2, \dots, n)$ with respect to permutations of pairs $((i_a, j_a)$ with $(i_b, j_b))$ and permutations in a pair $((i_a, j_a)$ to $(j_a, i_a))$. The sign $(-1)^m$ is the sign of the permutation σ, which is constant on the equivalence class m.

Now let us prove the formula (19). It is clear that only monomials of degree n in c will give a non-zero contribution to the integral and that they all come from the term

$$(c, Bc)^{n/2} = \sum_{i_1, \dots, i_{n/2}, j_1, \dots, j_{n/2} = 1}^{n} B_{i_1 j_1} \cdots B_{i_{n/2} j_{n/2}} c^{i^1} c^{j_1} \cdots c^{i_{n/2}} c^{j_{n/2}}.$$

Reordering factors we get

$$c^{i_1} c^{j_1} \cdots c^{i_{n/2}} c^{j_{n/2}} = (-1)^{\sigma(i|j)} c^1 \cdots c^n,$$

where $\sigma(i|j)$ is the permutation which brings $i_1, j_1, \dots, i_{n/2}, j_{n/2}$ to $1, 2, \dots, n$. Thus for the Gaussian Grassmann integral we have:

$$\int_{\mathbb{C}^{0|n}} \exp\Big(\frac{1}{2}(c, Bc)\Big)\, dc = \frac{(1/2)^{n/2}}{(n/2)!} \sum_{\sigma(i|j)} B_{i_1 j_1} \cdots B_{i_{n/2} j_{n/2}} (-1)^{\sigma(i|j)}$$

Note that the sign doesn't change when i_a is switched with j_a because the signs come in pairs. Also, the sign doesn't change when pair (i_a, j_a) and (i_b, j_b) are permuted. But such equivalence classes of permutations are exactly perfect matchings and therefore the formula becomes

$$\sum_{\substack{\sigma(i|j) \\ i_a < j_a \\ i_{a_1} < \cdots < i_{a_n}}} (-1)^{\sigma(i|j)} B_{i_1 j_1} \cdots B_{i_{n/2} j_{n/2}} = \mathrm{Pf}(B)$$

which is the Pfaffian of B.

This lemma is equivalent to the identity

$$\Big(\sum_{i<j} x^i \wedge x^j B_{ij}\Big)^{\wedge \frac{n}{2}} = \mathrm{Pf}(B) x^1 \wedge \cdots \wedge x^n$$

in the exterior algebra $\bigwedge^{\cdot} \mathbb{C}^n$.

Two important identities for Pfaffians:

$$\det B = \mathrm{Pf}(B)^2, \qquad \mathrm{Pf}\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} = \det A$$

The following formula is a Grassmann analog of the formula from Lemma 1 for integrating monomials with respect to the Gaussian measure

$$\int_{\mathbb{C}^{0|n}} \exp\Big(\frac{1}{2}(c,Bc)\Big)c^{i_1}\dots c^{i_k}\,dc = \mathrm{Pf}(B)(-1)^{\frac{k}{2}}\sum_m (-1)^m (B^{-1})^{i_{m_1}i_{m_2}}\dots(B^{-1})^{i_{m_{k-1}},i_{m_k}}. \tag{20}$$

Here the sum is taken over perfect matchings m of $1,\dots,k$, and B is assumed to be non-degenerate. The proof of this formula is parallel to the one for Gaussian oscillating integrals. The only difference is the factor $(-1)^m$ which appear when left derivatives are applied to the exponent.

Let $P(c)$ be an even element of G_n with monomials of degree at least 4, $P(c) = \sum_{k\geq 4}\frac{1}{k!}P^{(k)}(c)$ where $P^{(k)}(c) = \sum_{i_1,\dots,i_k=1}^n P^{(k)}_{i_1,\dots,i_k}c^{i_1}\dots c^{i_k}$.

Theorem 2. *The following identity holds:*

$$\int_{\mathbb{C}^{0|n}} \exp\Big(-\frac{1}{2}(c,Bc)+P(c)\Big)\,dc = Pf(-B)\sum_\Gamma (-1)^{c(D(\Gamma))}\frac{F(D(\Gamma))}{Aut(\Gamma)} \tag{21}$$

where the summation is taken over finite graphs, $D(\Gamma)$ is a mapping of Γ to $\mathbb{R}^2$, with the only singular points being crossings of edges ($D(\Gamma)$ is a diagram of the graph Γ), and $c(D(\Gamma)$ is the number of crossings of edges in the diagram $D(\Gamma)$. The number $F(D(\Gamma))$ is computed by the same rules as in the previous section. The product $(-1)^{c(D(\Gamma))}F(D(\Gamma))$ does not depend on the projection.

Proof. Expand the integral in $P(c)$:

$$\int_{\mathbb{C}^{0|n}} \exp\Big(\frac{1}{2}(c,Bc)+P(c)\Big)\,dc = \sum_{n_4,n_6,\dots\geq 0}\frac{1}{n_4!(4!)^{n_4}n_6!(6!)^{n_6}\dots} \sum_{i_1,i_2,i_3,\dots} P^{(4)}_{i_1,i_2,i_3,i_4}\dots P^{(6)}_{i_{4n_4+1},i_{4n_4+2},i_{4n_4+3},i_{4n_4+4},i_{4n_4+5},i_{4n_4+6}}\dots \int_{\mathbb{C}^{0|n}}\exp\Big(\frac{1}{2}(c,Bc)\Big)c^{i_1}c^{i_2}c^{i_3}\dots dc \tag{22}$$

Using the identity (20) we arrive to the formula

$$\mathrm{Pf}(B)\sum_{n_4,n_6,\dots\geq 0}\frac{1}{n_4!(4!)^{n_4}n_6!(6!)^{n_6}\dots} \sum_{i_1,i_2,i_3,\dots} P^{(4)}_{i_1,i_2,i_3,i_4}\dots P^{(6)}_{i_{4n_4+1},i_{4n_4+2},i_{4n_4+3},i_{4n_4+4},i_{4n_4+5},i_{4n_4+6}}\dots \sum_m (-1)^m (B^{-1})_{i_{m_1}i_{m_2}}(B^{-1})_{i_{m_3},i_{m_4}}\dots, \tag{23}$$

where m is a perfect matching on $1,2,\dots,k$, $k=\sum_{i\geq 3} in_i$. The summation over $\{i\}$ gives the number $F(D_m)$ where D_m is the diagram from Fig. 2. Some of the diagrams D_m represent projections of the same graph. It is easy to show that the combination

$(-1)^m F(D_m)$ depends only on the graph but not on its diagram and is equal to $(-1)^{c(D(\Gamma))}F(D(\Gamma))$ for any diagram $D(\Gamma)$ of Γ. Thus, if we will change the summation from n_i and m to the summation over graphs the factorials together with the number of perfect matchings corresponding to the same graph produce the combinatorial factor $1/|\mathrm{Aut}(\Gamma)|$.

Having in mind applications to oscillatory integral, it is convenient to have the formula (24) in the form

$$\int_{\mathbb{C}^{0|n}} \exp\Big(\frac{i}{2h}(c,Bc) - \frac{i}{h}P(c)\Big)\, dc = h^{-\frac{n}{2}}\mathrm{Pf}(iB)\sum_{\Gamma}(ih)^{-\chi(\Gamma)}(-1)^{c(D(\Gamma))}\frac{F(D(\Gamma))}{\mathrm{Aut}(\Gamma)} \tag{24}$$

5.3 Formal asymptotics of oscillatory integrals over supermanifolds

There is a number of equivalent definitions of super-manifolds. For our goals a super-manifold $M_{(n|m)}$ is a trivial vector bundle over a smooth n-dimensional manifold M (even part) with the fiber which is the exterior algebra of an m-dimensional vector space V (odd part). The algebra of functions on such super-manifold is the algebra of sections of this vector bundle with the point-wise exterior multiplication on fibers, i.e. if $f,g : M \to M \times \wedge V$ are two sections $x \mapsto (x, f(x))$ and $x \mapsto (x, g(x))$, their product is the section

$$x \mapsto (x, f(x) \wedge g(x)).$$

In other words, it is the tensor product of the Grassmann algebra of the fibers with the algebra of smooth functions on M, i.e.

$$C^\infty(M_{(n|m)}) = C^\infty(M) \otimes_{\mathbb{R}} < c^1, \ldots, c^m | c^\alpha c^\beta = -c^\beta c^\alpha > .$$

Elements of the algebra are polynomials in anti-commuting variables $c^1, \cdots, c^m$ with coefficients in smooth functions on M:

$$f(x,c) = f_0(x) + \sum_{k=1}^{m} \sum_{\alpha_1 < \cdots < \alpha_k} f_{\alpha_1,\ldots,\alpha_k}(x) c^{\alpha_1} \ldots c^{\alpha_k}. \tag{25}$$

Let dx be a volume form for the manifold M. Choose the orientation $c_1 \ldots c_m$ on the fibers. By definition, the integral of the function $f(x,c)$ with respect to the volume form dx and the orientation $c_1 \ldots c_m$ is

$$\int_{M_{(n|m)}} f dx dc = \int_M f_{1,\ldots,m}(x) dx$$

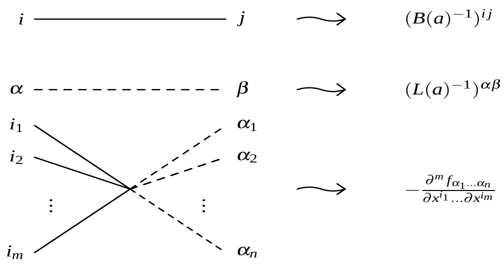


Fig. 8 Weights for Feynman diagrams in (27).

Figure missing

Fig. 9 An example of the Feynman diagram for super-integrals.

An even function on such super-manifold has only terms of even degree in (25). *Critical points of an even function* f on $M_{(n|m)}$ are, by definition, critical points of f_0 on M.

Let f be an even function $M_{(n|m)}$. Consider the following integral

$$\int_{M_{(n|m)}} \exp(\frac{if(x,c)}{h}) g(x,c) dx dc \tag{26}$$

Here we assume that M is compact, and that all functions are smooth.

Combining asymptotic analysis and the asymptotic expansion for oscillating integrals with the formulae for Grassmann integrals obtained in the previous section we arrive to the following asymptotic expansion for the integral (26)

$$\int_{M_{(n|m)}} \exp(\frac{if(x,c)}{h}) g(x,c) dx dc \simeq h^{\frac{n-m}{2}} (2\pi)^{\frac{n}{2}}$$
$$\sum_a \frac{1}{\sqrt{|\det(B(a))|}} \mathrm{Pf}(iL(a)) e^{\frac{i}{h}f(a)+\frac{i\pi}{4}sign(B(a))}$$
$$\left(1 + \sum_{\Gamma \neq \emptyset} \frac{(ih)^{-\chi(\Gamma)}(-1)^{c(D(\Gamma))} F_a(D(\Gamma))}{|\mathrm{Aut}(\Gamma)|}\right) \tag{27}$$

Here $B(a)_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}(a)$ and $L(a)_{\alpha\beta} = f_{\alpha\beta}(a)$, the summation is over finite graphs with two types of edges: fermionic edges (dashed), and bosonic edges (solid), $c(D(\Gamma))$ is the number of crossings of fermionic edges in the diagram. Weights of edges (propagators) and of vertices are given on Fig. 8. An example is given on Fig. 9.

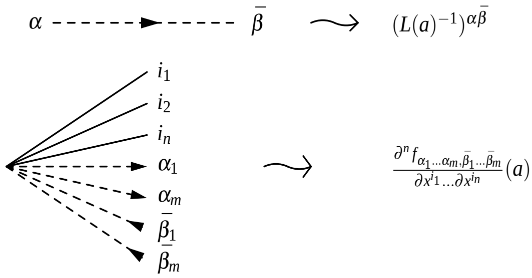


Fig. 10 Weights for Feynman diagrams in 28.

5.4 Charged fermions

Assume that $m = 2k$. Denote $c^i = c^i, \bar{c}^i = c^{k+i}$ for $i = 1, \ldots, k$. Assume that the function f in (26) has the form

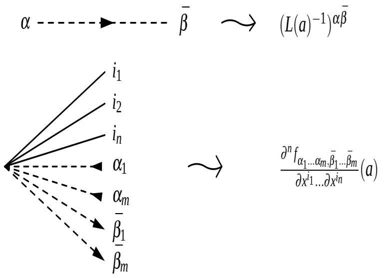
$$f(x,c,\bar{c}) = f_0(x) + \sum_{\alpha,\beta=1}^{k} f_{\alpha\bar{\beta}}(x) c^\alpha \bar{c}^{\bar{\beta}} + \ldots$$

where $\ldots$ denote terms of higher order in $c, \bar{c}$.

In this case the asymptotic expansion of the integral (26) is given by Feynman diagrams with oriented fermionic edges:

$$\int_{M_{(n|2k)}} \exp(\frac{if(x,c)}{h}) g(x,c) dxdc = h^{\frac{n-2k}{2}} (2\pi)^{\frac{n}{2}}$$
$$\sum_a \frac{1}{\sqrt{|\det(B(a))|}} \det(L(a)) \exp(\frac{i}{h} f(a) + \frac{i\pi}{4} sign(B(a)))$$
$$\left(1 + \sum_{\Gamma \neq \emptyset} \frac{(ih)^{-\chi(\Gamma)} (-1)^{c(D(\Gamma))} F_a(D(\Gamma))}{|\mathrm{Aut}(\Gamma)|}\right), \quad (28)$$

where all ingredients are the same as in (27) except the summation is taken over the graphs with oriented fermionic edges, and with weights from Fig. 10. An example is given in Fig. 5.4.



6 Finite-dimensional Faddeev-Popov quantization and the BRST differential

In this section we will study the integral (13) when a Lie group G acts on X faithfully (with no stabilizers) and the function f is invariant with respect to this action.

6.1 Faddeev-Popov trick

Let X be a manifold with the action of a Lie group G. We assume here that the action is free, i.e. that the stabilizer of every point in X is trivial. Assume also that X/G is a manifold. (Note that what is really important is the assumption that X/G is smooth near orbits where f is critical.)In this case

$$\dim(X/G) = \dim(X) - \dim(G).$$

Assume that the manifold X has a G-invariant volume form, and that X is compact. It is clear that such restrictions are too strong, but we will see in the next section how they can be relaxed to reasonable assumptions.

Let $f(x)$ be a G-invariant real analytic function. The goal of this section is to prepare the set-up for the description of the asymptotic expansion of the integral

$$I_h = \int_X \exp\left(i\frac{f(x)}{h}\right) dx \tag{29}$$

as the sum of Feynman diagrams, just as it was done in section for functions on X with simple critical points.

Since the function f is G-invariant, its critical points are not simple, except when a critical point is a fixed point of the G-action, but since we assume faithfulness, there are no such points.

Instead of assuming the simplicity of critical points of f we assume that critical variety $C_f = \{x \in X | df(x) = 0\}$ of f is the disjoint union of finitely many G-orbits.

We want to change the integration over X to the integration over the orbits of the G-action. In practice, it is convenient to describe the space of orbits in terms of a cross-section.

Let us assume that the surface

$$S_\phi = \{x \in X | \phi^a(x) = 0, a = 1, \dots, n\}$$

where $\phi^a(x), a = 1, \dots, n$ with $n = \dim(G)$ are independent functions, is a cross-section, i.e. intersects every G-orbit exactly once.

Let $x^i, i = 1, \dots, d$ be local coordinates on $U \subset X$, $e_a, a = 1, \dots, n$ be a linear basis in the Lie algebra $\mathfrak{g} = Lie(G)$. Denote by $D_a^i(x)$ the matrix describing the action of e_a as a vector field on X in terms of local coordinates x^i:

$$(e_a f)(x) = \sum_{i=1}^{d} D_a^i(x) \frac{\partial f}{\partial x^i}(x)$$

and by $L_c^b(x)$ the matrix:

$$L_a^b(x) = \sum_{i=1}^{d} D_a^i(x) \frac{\partial \phi^b}{\partial x^i}(x) = e_a \phi^b(x).$$

Since we assume that S_ϕ is a cross-section, $\det(L) \neq 0$ on this surface. Later we will relax this condition requiring only that the determinant is not vanishing in a vicinity of critical points of f.

A coordinate free way to formulate this non-degeneracy condition can be phrased as follows. For $x \in S_\phi \subset X$ let $L_x \subset T_x X$ be the subspace of the tangent space spanned by vector field describing the action of the Lie algebra $\mathfrak{g}$ on X, and $T_x S_\phi \subset T_x X$ be the tangent space to S_ϕ at this point. The non-degeneracy of L is equivalent to linear independence of L_x and $T_x S_\phi$ in $T_x X$.

Theorem 3. *(Faddeev-Popov)*[5] *The integral in question is given by*

$$\int_X \exp\left(i\frac{f(x)}{h}\right) dx = h^n |G| \int_L \exp\left(i\frac{f_{FP}(x,c,\bar{c},\lambda)}{h}\right) dx\, d\bar{c}\, dc\, d\lambda, \qquad (30)$$

where the supermanifold L is $X \times \mathfrak{g}_{odd} \times \mathfrak{g}_{odd} \times \mathfrak{g}^*$, $|G|$ is the volume of G with respect to a left invariant measure dg, and

$$f_{FP}(x,c,\bar{c},\lambda) = f(x) - ih \sum_{a,b=1}^{n} \bar{c}^a L_a^b(x) c_b + \sum_{a=1}^{n} \lambda_a \phi^b(x). \qquad (31)$$

Proof. From the G-invariance of f:

[5] Faddev and Popov derived the formula (30) in the setting of the Yang-Mills theory where the symmetry group is infinite-dimensional and only the integration over gauge classes may have a meaning. see [20]

$$\int_X \exp\left(i\frac{f(x)}{h}\right)dx = \int_{X\times G} \exp(i\frac{f(x)}{h})\Delta(x)\delta(\phi(x))dxdg \qquad (32)$$

where $\Delta(x)$ is determined by the identity

$$\Delta(x)\int_G \delta(\phi(gx))dg = 1. \qquad (33)$$

The G-orbit through x intersects the cross-section S_ϕ only once (since it is a cross-section). Denote this point g_0x (such g_0 depends on x, it exists because S_ϕ is a cross-section, and, in particular, intersect all orbits). Then, we have

$$\phi^a(g_0x) = 0.$$

In a vicinity of this point

$$\phi^a(\exp(\sum_b t^b e_b)g_0x) = \sum_{b,i} t^b D^i_b(g_0x)\frac{\partial\phi^a(g_0x)}{\partial x^i} + O(t^2) = \sum_b t^b L^a_b(g_0x) + O(t^2).$$

Thus, the identity (33) is equivalent to

$$\Delta(x)\int_{\mathbb{R}^n} \delta(L(g_0x)t)dt = 1,$$

i.e.

$$\Delta(x) = \det(L(g_0x)).$$

Here we identified $T_{g_0}G \simeq \mathbb{R}^n$. Tanking this into account we arrive to the formula

$$\int_X \exp\left(i\frac{f(x)}{h}\right)dx = |G|\int_X \exp\left(i\frac{f(x)}{h}\right)\det(L(g_0x))\delta(\phi(g_0x))dx.$$

Changing the variable x to $g_0^{-1}x$ and using G-invariance of $f(x)$ and of the measure dx we obtain

$$\int_X \exp\left(i\frac{f(x)}{h}\right)dx = |G|\int_X \exp\left(i\frac{f(x)}{h}\right)\det(L(x))\delta(\phi(x))dx.$$

Expressing $\det(L(x))$ as a fermionic integral and taking into account

$$\delta(\phi) = \int_{\mathbb{R}^n} \exp(i(\phi,\lambda))d\lambda$$

we arrive to the formula (30).

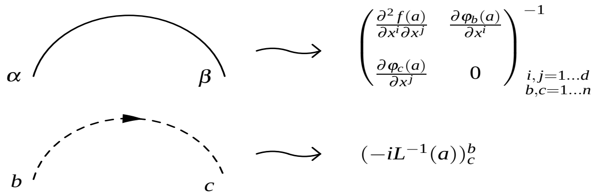


Fig. 11 Weights of edges for Feynman diagrams in (34).

6.2 Feynman diagrams with ghost fermions

Now let us use the formula (30) to derive the Feynman diagram expansion of the integral (29).

Critical points of the function f_{FP} on the supermanifold L are, by definition, critical points of

$$\tilde{f}(x,\lambda) = f(x) + \sum_a \lambda_a \phi^a(x).$$

This is simply the Lagrange multiplier method and by the assumption which we made above critical points of this function on $X \times \mathfrak{g}^*$ are simple. In particular the matrix of second derivatives is non-degenerate near each critical point of this function.

Thus, we can describe the asymptotic expansion of the integral (29) by Feynman diagrams. Applying the formula (28) to the integral (30) we obtain the following asymptotic expansion:

$$\int_L \exp\left(\frac{if_{FP}(x,\bar{c},c,\lambda)}{h}\right) g(x,c,\bar{c})dxdcd\bar{c}d\lambda \simeq |G| h^{\frac{d-n}{2}} (2\pi)^{\frac{d+n}{2}}$$
$$\sum_a \frac{1}{\sqrt{|\det(B(a))|}} \det(-iL(a)) \exp\left(\frac{i}{h} f(a) + \frac{i\pi}{4} sign(B(a))\right)$$
$$\left(1 + \sum_{\Gamma \neq \emptyset} \frac{(ih)^{-\chi(\Gamma)}(-1)^{c(D(\Gamma))} F_a(D(\Gamma))}{|\mathrm{Aut}(\Gamma)|}\right), \quad (34)$$

Here the first summation is over the set of critical points of $\tilde{f}$. Feynmann diagrams in this formula have bosonic edges and fermionic oriented edges, $c(D(\Gamma))$ is the number of crossings of fermionic edges. The structure of Feynman diagrams is the same as in (28). The propagators corresponding to Bose and Fermi edges are Fig. 11 . The weights of vertices are shown on Fig. 12[6].

The asymptotic expansion (34) depends only on how the the cross section S_ϕ intersects G-orbits in the infinitesimal neighborhood of critical points of f. In other words, the expansion is defined as long as linear operators $B(a)$ and $L(a)$ are invert-

[6] Each fermionic propagator contributes to the weight of the diagram an extra factor h^{-1}. Each vertex with two adjacent fermionic (dashed) edges contributes the factor of h. Because fermionic lines form loops, these factors cancell each other.

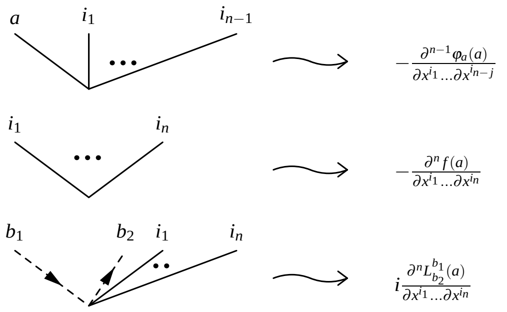


Fig. 12 Weights of vertices for Feynman diagrams in (34).

ible at all critical points of the function $\tilde{f}(x,\lambda)$. This is equivalent to the condition $T_aS_\phi \cap \mathfrak{g}_a = \{0\}$ where $T_aS_\phi \subset T_aX$ is the tangent space to S_ϕ at a, and $\mathfrak{g}_a$ is the subspace in T_aX spanned by vector fields describing the infinitesimal action of the Lie algebra of G.

The main moral of this observation is that in order to have the asymptotic expansion of the integral in terms of Feynman diagrams we just have to choose a constraint which is a cross-section through the orbits in an infinitesimal neighborhood of critical orbits.

6.3 Gauge independence

The asymptotic expansion of the integral (29) does not depend on the choice of the constraint ϕ (as long as it is a cross-section through the G-orbits of tangent spaces at critical points).

However, it is not obvious from the Feynman diagram formula for the asymptotic expansion. Let us check that the semiclassical term of the expansion does not depend on ϕ. Till the end of this section we work in a vicinity of a critical point of f_{FP}. The semiclassical term is

$$\det(B)^{-\frac{1}{2}}\det(L)$$

where

$$B = \begin{pmatrix} \frac{\partial^2 f}{\partial x^i \partial x^j} & \frac{\partial \phi^b}{\partial x^i} \\ \frac{\partial \phi^a}{\partial x^j} & 0 \end{pmatrix}, \tag{35}$$

and

$$L^b_c = \sum_i l^i_c \frac{\partial \phi^b}{\partial x^i} \tag{36}$$

Let us make an infinitesimal variation of the constraint $\phi^a(x) \to \phi^a(x) + \varepsilon^a(x)$. The product of the determinants will change as

$$\det(B)^{-\frac{1}{2}}\det(L) \to \det(B)^{-\frac{1}{2}}\det(L)\left(1-\frac{1}{2}tr(B^{-1}\delta B)+tr(L^{-1}\delta L)+\dots\right)$$

where $\dots$ are higher order terms. We have to prove that the first order terms vanish. The matrix B has the block form, so is the matrix B^{-1}. Both of these matrices are symmetric, therefore

$$-\frac{1}{2}tr(B^{-1}\delta B) = -tr((B^{-1})_{12}\delta B_{21}) = -\sum_{i,c} b^i_c \frac{\partial \varepsilon^c}{\partial x^i},$$

where b^i_a are matrix elements of the block $(B^{-1})_{12}$. They satisfy the identity $\sum_i \frac{\partial \phi^b}{\partial x^i} b^i_c = \delta^b_c$.

The second term can be written as

$$tr(L^{-1}\delta L) = \sum_{b,c,i}(L^{-1})^c_b l^i_c \frac{\partial \varepsilon^b}{\partial x^i}.$$

The identity $\sum_i (L^{-1}l)^i_b \frac{\partial \phi^c}{\partial x^i} = \delta^c_b$. From here and the identity for b we conclude that

$$-\frac{1}{2}tr(B^{-1}\delta B) + tr(L^{-1}\delta L) = 0,$$

which proves that the semiclassical factor does not depend on the choice of gauge condition.

We will leave the exercise of verifying this fact in all orders ≥ 1 to the reader.

6.4 Feynman diagrams for linear constraints

Because the asymptotic expansion depends only on the formal neighborhood of critical points of $f(x)$ on the surface of the constrains and does not depend on the particular choice of the constraint (as long as it a local cross-section in the neighborhood of each critical point), we can choose them at our convenience at each neighborhood.

In particular, if X is linear, we can deform ϕ to a linear cross-sections at each critical point. Now let us find the asymptotic expansion of the integral

$$\int_{X_a} \exp\left(i\frac{f(x)}{h}\right)\det(L(x))\delta(\phi(x))dx \tag{37}$$

where X_a is an infinitesimal neighborhood of $a \in X$.

Since both X and ϕ are linear, we have a linear subspace $X_\phi = \ker(\phi) \subset X$. Because we assume that $\phi(x) = 0$ is a cross-section through G-orbits $\ker(\phi)$ has a natural orthogonal complement which is the image of $\mathfrak{g}$ in T_aX with respect to the action of G. Denote it by $\mathfrak{g}(a)$. Now we can write $X = X_\phi \oplus \mathfrak{g}(a)$. We will write s_α for

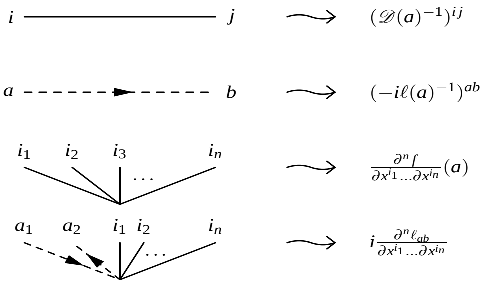


Fig. 13 Weights of Feynman diagrams in (38).

coordinates on X_ϕ and l_a for coordinates on $\mathfrak{g}(a)$. With respect to this decomposition $\phi = 0 \oplus \hat{\phi}$ where $\hat{\phi} : \mathfrak{g}(a) \to \mathfrak{g}(a)$ is invertible.

For the integral (37) we obtain

$$\int_{X_a} \exp\left(i\frac{f(x)}{h}\right) \det(L(x))\delta(\phi(x))dx = \int_{X_\phi(a)} \exp\left(i\frac{f(s)}{h}\right) \det(L(s)) \det(\hat{\phi})^{-1} ds,$$

where $X_\phi(a)$ is an infinitesimal neighborhood of a in X_ϕ. Since the constraint is linear, $L_a^b(s) = \sum_i l_a^i(x)\frac{\partial \phi^a(x)}{\partial x^i}|_{l=0} = \sum_c l_a^c(s)\hat{\phi}_c^b$. Here $l_a^c(s)$ is the matrix describing the action of $\mathfrak{g}$ on $\mathfrak{g}(a)$.

Thus, the integral in question can be written as

$$\int_{X_\phi(a)} \exp\left(i\frac{f(s)}{h}\right) \det(l(s))ds.$$

Its contribution to the asymptotic expansion is given by the formal power series where coefficients are determined by Feynman diagrams with rules described on Fig. 13. This power series does not depend on the choice of X_ϕ. Indeed different choices of ϕ are related by linear transformations in X_ϕ. The contribution from each Feynman diagram is invariant with respect to linear transformations of s-coordinates and therefore does not depend on the choice of ϕ.

Finally, we can write the asymptotic expansion of (30) as

$$\int_X \exp\left(i\frac{f(x)}{h}\right) dx \simeq |G| h^{\frac{d-n}{2}} (2\pi)^{\frac{d+n}{2}}$$
$$\sum_a \frac{1}{\sqrt{|\det(D(a))|}} \det(-il(a)) e^{\frac{i}{h}f(a) + \frac{i\pi}{4} sign(D(a))}$$
$$\left(1 + \sum_{\Gamma \neq \emptyset} \frac{(ih)^{-\chi(\Gamma)}(-1)^{c(D(\Gamma))} F_a(D(\Gamma))}{|\mathrm{Aut}(\Gamma)|}\right). \quad (38)$$

Here $D(a)_{ij} = \frac{\partial^2 f}{\partial s^i \partial s^j}$ where s^i are coordinates on X_ϕ. The coefficients are given by Feynman diagrams with weights of edges and vertices described in Fig. 13, and all other ingredients are as before.

The factor $\exp(\frac{i\pi}{4} sign(B))$ can also be written as $i^N \exp(-\frac{i\pi}{2} n_-(B))$ where $n_-(B)$ is the number of negative eigenvalues of B. This is more or less how the Morse index appear in the semiclassical asymptotic of the propagator in quantum mechanics.

6.5 The BRST differential

The appearance of fermionic variable (Faddeev-Popov ghost fields) in the asymptotic expansion of (30) looks as a bit of a mystery and as a technical trick. In the BRST approach these non-commutative variable attain a natural meaning.

The key observation of Becchi, Rouet, Stora [12] and of Tuytin [55][7] is that the odd operator Q

$$Q = \sum_{a,i=1}^{n,d} c^a L_a^i \frac{\partial}{\partial x^i} - \frac{1}{2} \sum_{a,b,c} f_{bc}^a c^b c^c \frac{\partial}{\partial c^a} + \sum_a \lambda_a \frac{\partial}{\bar{c}_a}$$

acting on the space $C^\infty(L) = Fun(X \times \mathfrak{g}^*) \otimes \mathbb{C}[c^a, \bar{c}_a] = C^\infty(X \times \mathfrak{g}^*) \otimes \wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ of functions on the super-manifold $L = X \times \mathfrak{g}_{odd} \times \mathfrak{g}_{odd} \times \mathfrak{g}^*$ has the properties

$$Q^2 = 0,$$

$$Qf_{FP} = 0.$$

The first property means that the pair $(C^\infty(L), Q)$ is a co-chain complex. Because we assumed that the action of G on X is faithful, its zero cohomology can be naturally identified with $C^\infty(X/G)$, and other cohomology vanish. Note that $Q = Q_{Ch} + Q_{dR}$, where the first term is the differential in the Chevalley complex for $\mathfrak{g}$ with the coefficients in $C^\infty(X)$. The second term $Q_{dR} = \sum_a \lambda_a \frac{\partial}{\bar{c}_a}$ is the de Rham differential for functions on $\mathfrak{g}^*$.

The second property means that the Faddeev-Popov action is a cocycle in the BRST complex. The function f_{FP} is not a co-boundary and therefore defines a non-trivial zero-cohomology class in $H^0(L) \simeq C^\infty(X/G)$. This class is simply the initial function f considered as a function on G-orbits. Indeed, the function f_{FP} can be written as $f_{FP} = f + Q(\sum_a \phi^a \bar{c}_a)$.

To see how the integral over the super-space L appear in this setting consider first a simple fact in linear algebra.

Let C be a super-vector space and $d : C \to C$ be an odd linear operator with $d^2 = 0$. Assume D is another super-vector space with an odd differential $d^* : D \to D, d^{*2} = 0$ and a non-degenerate pairing $< .,. >: D \otimes C \to \mathbb{C}$ such that $< d^* l, a >= (-1)^{\bar{l}} < l, da >$.

[7] The original formulation uses the supersymmetry concept and have slightly different appearance.

We will think of (D,d^*) and (C,d) as co-chain complexes and say that $l \in D$ and $a \in C$ are cocycles if $d^*l = 0$ and $da = 0$. Denote by $[l] \in H(D) = \ker(d^*)/\mathrm{Im}(d^*)$ and $[a] \in H(C) = \ker(d)/\mathrm{Im}(d)$ cohomology classes of cocycles l and a.

Lemma 3. *If l and a are cocycles, then*

$$< l,a >=< [l],[a] >,$$

where $< [l],a] >$ is the induced pairing on the cohomology spaces.

Indeed, the cocycle properties imply that

$$< l + d^*m, a + dc >=< l,a >,$$

which defines the pairing on the cohomology spaces and proves the lemma.

Now we should identify ingredients of this lemma in the FP-BRST setting. The G-invariance of the measure of integration $dxd\bar{c}dcd\lambda$ in the FP integral which we will denote dl implies[8]

$$\int_L Qgdl = 0$$

Considering the integral as a linear functional on $C = C^\infty(L)$ with the differential we can think of it as an element of D which is annihilated by d^*.

Applying the lemma to a cocycle $g \in C^\infty(L)$, i.e. to a function, such that $Qg = 0$ we arrive to the identity

$$\int_L gdl = \int_Y [g]dy \tag{39}$$

Here Y is the super-manifold such that $H^0(C^\infty(L)) = C^\infty(Y))$, i.e. the appropriate topological version of X/G. If we would be in algebro-geometric setting, the variety Y would be spectrum of the commutative algebra H^0. We also made an assumption that all cohomologies except H^0 are vanishing, which is in our setting equivalent to the faithfulness of the G-action on X.

The equation (39) implies, in particular, that if $Qg = 0$ (i.e. if g is G-invariant) and if the measure is G-invariant, then

$$\int_L \exp\left(\frac{if_{FP}}{h}\right) gdl = \int_Y \exp\left(\frac{if}{h}\right) [g]dy$$

This puts Faddeev-Popov method into a natural algebraic setting and ”explains” the algebraic meaning of fermionic ghost fields. It also shows that the method can

[8] The operator Q can be regarded as an odd vector field on L. The invariance of the measure dl is equivalent to the zero-divergence condition of the vector-field (with respect to measure dl). Recall that for any vector field Q we have:

$$\int_L Qgdl = \int_L g\,div_{dl}(Q)dl$$

where $div_{dl}(Q)$ is the divergence of the vector field Q with respect to the volume measure dl.

be extended to any complex which has $C^\infty(X/G)$ as its cohomology space. Because of the formula (39) it does not matter with which complex $(C^\infty(L),Q)$ we started, as long as the cohomology space is $C^\infty(X/G)$. This observation leads to an important notion of cohomological field theories [58] and to a natural notion of quasi-isomorphic field theories.

Perhaps one of the most important developments along these lines is the extension of the BRST observations to a more general class of degenerate Lagrangian (i.e. degenerate critical points of f). This generalization known as Batalin-Vilkovisky quantization (BV) works even in the case when the Lagrangian is invariant with respect to the action of vector fields, which do not necessary form a Lie algebra. One of the most striking applications of this technique was the quantization of the Poisson sigma model and the construction of the *-product for an arbitrary Poisson manifold. But this subject goes beyond the goal of present lectures.

7 Semiclassical quantization of a scalar Bose field

The classical theory of scalar Bose field is described in section 2.5 Let us define the amplitude $Z(M)$ as a semiclassical expansion of a (non-existing) path integral given by Feynman diagrams similar to how the asymptotic expansion looks for finite dimensional integrals.

This definition can be motivated by finite dimensional approximations to the path integral, which provide an acceptable definition of infinite-dimensional integrals such as Wiener integral and path integrals in low-dimensional Euclidean quantum field theories [29].

In the semiclassical quantum field theory path integrals are defined as formal power series which have the same structure if they were asymptotical expansions of existing integrals. The coefficients in these expansions are given by Feynman integrals. We will show how it works in quantum mechanics, and how it compares with the semiclassical analysis of the Schrödinger equation for $d=1$, and will have a brief discussion of $d>1$ case.

7.1 Formal semiclassical quantum mechanics

7.1.1 Semiclassical asymptotics from the Schrödinger equation

To be specific, we will consider here quantum mechanics of a point particle on a Riemannian manifold N in a potential $V(q)$(see sections 2.3) .

Let $\{\gamma_c(t)\}_{t_1}^{t_2}$ be a solution to the Euler-Lagrange equations for a classical Lagrangian $\mathscr{L}(\xi,q)$ with Dirichlet boundary conditions $\gamma(t_1)=q_1, \gamma(t_2)=q_2$. Denote by $S^{(c)}_{t_2-t_1}(q_2,q_1)$ the value classical action functional on γ_c:

$$S^{(c)}_{t_2-t_1}(q_2,q_1)=\int_{t_1}^{t_2}\mathscr{L}(\dot\gamma_c(t),\gamma_c(t))dt$$

Let $U_t(q_2,q_1)$ be the kernel of the integral operator representing the evolution operator (11). Solving Schrödinger equation (12) in the limit $h\to 0$ we obtain the following asymptotics of the evolution kernel as $h\to 0$

$$U_t(q_2,q_1)\sim\sum_{\gamma_c}(2\pi i)^{-\frac{n}{2}}\exp\left(\frac{i}{h}S^{(c)}_t(q_2,q_1)+\frac{i\pi\mu(\gamma_c)}{2}\right)$$
$$\left|\wedge^n\left(\frac{\partial^2 S^{(c)}_t(q_2,q_1)}{\partial q_2\partial q_1}dq_2\wedge dq_1\right)\right|^{\frac{1}{2}}\left(1+\sum_{n\geq 1}h^nU^{(n)}_c(q_2,q_1)\right). \quad (40)$$

Here

$$\wedge^n\left(\frac{\partial^2 S(a,b)}{\partial a\partial b}da\wedge db\right)$$
$$=\wedge^n d_ad_bS(a,b)=\det\left(\frac{\partial^2 S(a,b)}{\partial a^i\partial b^j}\right)da^1\wedge\dots da^n\wedge db^1\wedge\dots db^n \quad (41)$$

$\mu(\gamma_c)$ is the Morse index of γ_c (i.e., the number of focal points of the trajectory in $T^*\mathbb{R}^n$ induced by γ_c relative to fibers of the cotangent bundle). The coefficients $a^{(c)}_k=(2\pi i)^{-\frac{n}{2}}(\det(\frac{\partial^2 S(a,b)}{\partial a^i\partial b^j}))^{\frac{1}{2}}U^{(n)}_c$ satisfy the transport equation

$$\frac{\partial a^{(c)}_k}{\partial t}+\frac{1}{2m}\Delta S^{(c)}a^{(c)}_k+\frac{1}{m}\sum_{j=1}^n\frac{\partial S}{\partial q_j}\frac{\partial a^{(c)}_k}{\partial q_j}+\frac{i}{2m}\Delta a^{(c)}_{k-1}=0$$

But the initial condition $\lim_{t\to+0}U_t(q,q')=\delta(q,q')$ can no longer be imposed since we consider the asymptotical expansion when $h<<t$. Instead, to determine coefficients $a^{(c)}_k$ one should use the semi-group property of the propagator:

$$U_tU_s=U_{s+t}$$

The kernel of the integral operators representing the evolution operator satisfies the identity

$$\int_N U_t(q_3,q_2)U_s(q_2,q_1)=U_{s+t}(q_3,q_1). \quad (42)$$

Here the first factor is a half-density in q_3,q_2, the second is a half-density in q_2,q_1. The product is a density in q_2 and it is integrated over N.

As $h\to 0$ the semigroup property implies that the asymptotical expansion should satisfy the identity

$$\sum_{k,l\geq 0}\int_N \exp\left(\frac{i(S_t^{(c')}(q_3,q_2)+S_s^{(c")}(q_2,q_1))}{h}\right)a_k^{(c')}a_l^{(c")}=\sum_k \exp\left(\frac{iS_t^{(c)}(q_3,q_1)}{h}\right)a_k^{(c)}.$$

Here by the integral of the product of two half-densities on N we mean the formal asymptotic expansion (17), and $\gamma_{c'},\gamma_{c"}$ are parts of the path $\{\gamma_c\}_{t=0}^{t+s}$ when $0<\tau<s$ and $s<\tau<s+t$ respectively.

It is not difficult to derive first coefficients of the asymptotical expansion (40) from this equation. Moreover, this equation alone defines all higher order terms in the semiclassical expansion.

For more details on the semiclassical analysis see for example [51].

7.1.2 Semiclassical expansion from the path integral

Looking at the expression (40), it is natural to imagine that it may be interpreted as a semiclassical asymptotics of an oscillating integral over the space of paths connecting point q_1 and q_2. Critical points in this integral are classical trajectories.

This point of view was put forward in quantum mechanics by R. Feynman and it can be supported by many very convincing arguments [22]. Eventually a mathematically meaningful definition of a path integral for the Euclidian version (when the integral is rapidly decaying instead of oscillating) emerged, and was developed further in the framework of constructive field theory. The Wiener integral, which was introduced in probability theory, even earlier, is an example of such an object.

Here we will not try to make the definition of the integral rigorous. Instead of this we will define its semiclassical expansion in such a way that it has an appearance of the semiclassical expansion of an infinite-dimensional integral. After this we will check that it satisfies the semigroup property. This is an illustration of a semiclassical quantum field theory, where the partition function Z_M depends on the boundary condition, and integrating over possible boundary conditions has the replicating gluing property (9). The difference is that in quantum mechanics we have the Schrd̈inger equation as a reference point to compare any definition of the path integral. In more complicated models of quantum field theory, the gluing axioms seems to be the only major structural requirement (beyond unitarity and causality, which we do not discuss here).

So, we are looking for a formal power series which would look like the asymptotic expansion of the integral

$$Z_t(q_2,q_1)=\int_{\gamma(0)=q_1,\gamma(t)=q_2} e^{\frac{i}{h}S[\gamma]}D\gamma.$$

We will focus in this section on the point particle of mass m in $\mathbb{R}^n$ in the potential $V(q)$ (3). By analogy with the finite dimensional case we define the asymptotical expansion as

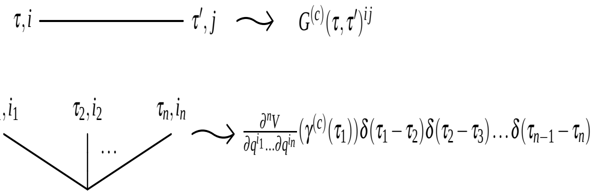


Fig. 14 Weights of Feynman diagrams in (43).

$$Z_t(q_2,q_1) = C\sum_{\gamma_c} \exp\left(\frac{i}{h}S_t^{(c)}(q_2,q_1) + \frac{i\pi\eta(K^{(c)})}{4}\right)$$
$$|\det{}'(K^{(c)})|^{\frac{1}{2}}|dq_1|^{\frac{1}{2}}|dq_2|^{\frac{1}{2}}\left(1+\sum_{\Gamma\neq\emptyset}(ih)^{-\chi(\Gamma)}\frac{F_c(\Gamma)}{|Aut(\Gamma)|}\right). \quad (43)$$

Here
$$(K^{(c)})_{ij} = -mh^2\frac{d^2}{d\tau^2}\delta_{ij} + \frac{\partial^2 V}{\partial x^i\partial x^j}(\gamma_c(\tau))$$
is the matrix differential operator which acts on the space of functions on $[0,t]$ with values in $\mathbb{R}^n$ (trivialized tangent bundle to N restricted to $\gamma^{(c)}$ in local coordinates) with the Dirichlet boundary conditions $f(0) = f(t) = 0$. The half-density $|dq|^{\frac{1}{2}}$ is the "square root" of the Riemannian volume density on N. The sum is taken over classical trajectories connecting q_1 and q_2, and $\eta(K^{(c)})$ is the index of the operator $K^{(c)}$, C is some constant. The weights for Feynman diagrams in (43) are given on Fig. 14 where $G^{ij}(x,y)$ is the kernel of the integral operator which is the inverse to $K^{(c)}$.

The expansion is not the result of computation, it is a definition, which based on the idea that the path integral exists in some sense and its asymptotical expansion as $h\to 0$ is given by the formula similar to the finite dimensional case. It turns out that despite very different appearance the semiclassical expansion of U_t coincide with this series.

One can show easily (see for example [51]) that
$$|\det{}'(K^{(c)})| = \left|\det\left(\frac{\partial^2 S(a,b)}{\partial a_i\partial b_j}\right)\right|,$$
as well as that $\mu(\gamma^{(c)})$ can be identified with properly defined $\eta(K^{(c)})$. This shows that the leading terms of (43) and (40) are the same. Now the question is whether the two power series are the same.

We will state without the proof the following theorem.

Theorem 4. *The expansion $Z_t(q_2,q_1)$ is equal to the asymptotic expansion of the kernel of the propagator and it satisfies the gluing property.*

The details will appear in a paper by T. Johnson-Freyd [37] when $N=\mathbb{R}^d$ with flat metric.

As an immediate corollary of this theorem we have

Corollary 1. *Functions*

$$U_c^{(n)}(q_2,q_1)) = \sum_{-\chi(\Gamma)=n} \frac{F_c(\Gamma)}{|Aut(\Gamma)|}.$$

are coefficients of the asymptotical expansion of the propagator, and, being properly normalized satisfy the transport equation. Here $\chi(\Gamma)=|V|-|E|$ *is the Euler characteristic.*

Let us write the semiclassical propagator as

$$Z_t(q_2,q_1) = \sum_c \exp\left(\frac{i}{h} S_t^{(c)}(q_2,q_1)\right) J_t^{(c)}(q_2,q_1)$$

The semigroup property of the propagator implies that this power series satisfies the following gluing/cutting identity:

$$\exp\left(\frac{i}{h} S_t^{(c)}(q_3,q_1)\right) J_t^{(c)}(q_3,q_1) = \int_{q_2\in N} \exp\left(\frac{i}{h}\left(S_s^{(c)}(q_3,q_2)+S_{t-s}^{(c)}(q_2,q_1)\right)\right) J_s^{(c)}(q_3,q_2) J_{t-s}^{(c)}(q_2,q_1). \quad (44)$$

Here the integral is taken in a sense of the semiclassical expansion as the sum of corresponding Feynman diagrams. It is easy to check that the identity (44) determines uniquely not only the higher order coefficients but also the leading order factor.

7.2 $d>1$ and ultraviolet divergencies

In the semiclassical theory of scalar Bose field on a compact Riemannian manifold the partition function for the theory is given by the formal power series

$$Z_M(b) = C\sum_{\phi_c} \exp\left(\frac{iS_M(\phi_c)}{h} + \frac{i\pi}{4}\eta(K_{\phi_c})\right) |\det{}'(K_{\phi_c})|^{-\frac{1}{2}} \left(1+\sum_{\Gamma\neq\emptyset} (ih)^{-\chi(\Gamma)} \frac{F_{\phi_c}(\Gamma)}{|Aut(\Gamma)|}\right) \quad (45)$$

Here we assume that there are finitely many solutions ϕ_c to the Euler-Lagrange equation (4) with the Dirichlet boundary conditions $\phi_c|_{\partial M}=b$. The number $\eta(K)$ is the index of the differential operator

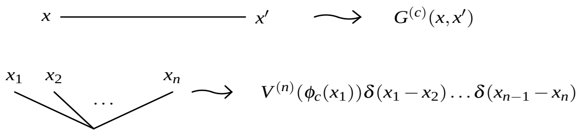


Fig. 15 Weights of Feynman diagrams in 45.

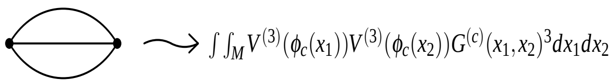


Fig. 16 An example of the Feynman diagram of order one.

$$K_{\phi_c} = \Delta + V''(\phi_c(x))$$

acting on the space of functions on M with the boundary condition $f(x) = 0, x \in \partial M$, and $det'(K_\phi)$ is its regularized determinant ($\eta(K)$ can be thought as the argument of the square root of the determinant). The ζ-function regularization is one of the standard ways to define det$'$ (see for example [6]). The weights of Feynman diagrams are given on Fig. 15 where $G^{(c)}(x,y)$ is the kernel of the integral operator which is inverse to K_{ϕ_c}.

An example of order 1 Feynman diagrams is given on Fig. 16.

The kernel $G(x,y)$ behaves at the diagonal as

$$G(x,y) \sim |x-y|^{2-d}$$

which means that the Feynman integrals converge for $d = 1$ (quantum mechanics), diverge logarithmically for $d = 2$, and diverges as a power of the distance for $d > 2$.

This is a well known problem of ultraviolet divergencies in the perturbation theory. The usual way to deal with divergencies is a two step procedure.

Step 1. The theory is replaced by a family of theories where the Feynman integrals converge (regularized theories). There are several standard way to do this:

- Pauli-Willars regularization replaces the theory with the one where the quadratic part of the action has terms with higher derivatives. In the regularized theory the propagator $G(x,y)$ is not singular at the diagonal.
- Lattice regularization replaces the theory on a smooth Riemannian manifold M by a metrized cell approximation of M. The path integral becomes finite-dimensional and Feynman diagrams describing the semiclassical expansion become finite sums.
- Dimensional regularization is more exotic. It replaces Feynman d-dimensional integrals where d is integer, by formal expressions where d is not integer. It is very convenient computationally for certain tasks (see for example [19] and references therein).

Step 2. After the theory is replaced by the family of theories where Feynman integrals converge, one should compute them and pass to the limit corresponding to the original theory. Of course the limit will not exist since some terms will have singularities. In some cases it is possible to make the parameters in the regularized theory (for example, coefficients in $V(\phi)$) depend on the parameters of the regularization in such a way that the sum of Feynman diagrams of order up to n remain finite when the regularization is removed. Such theories are called renormalizable in orders up to n.

The compatibility of the gluing/cutting axiom, i.e. an analog of the identity (44) and the renormalization is, basically, an open problem for $d > 1$, which requires further investigation. Notice that for $d = 2$ the integration over the boundary fields does not introduce Feynamn diagrams with ultraviolet divergencies but these diagrams will diverge for $d > 2$. This problem was addressed in case of Minkowski flat space-time by K. Symanzik in [49].

One should expect that the constant C in (45) is determined by the gluing property and that the gluing property should hold if it is compatible with the renormalization.

8 The Yang-Mills theory

The classical Yang-Mills theory with Dirichlet boundary conditions was described in section 2.6.

In this section we will define Feynman diagrams for the Yang-Mills theory following the analogy with the finite-dimensional case. In these notes will do it "halfway" leaving the most important part concerned with the ultraviolet divergencies aside.

Naively, the path integral quantization of the classical d-dimensional Yang-Mills theory can be constructed as follows. Let G be a compact Lie group.

- To a closed oriented $(d-1)$-dimensional Riemannian manifold with a principal G-bundle P we assign the space of functionals on the space of connections on P.
- To a d-dimensional Riemannian manifold M with a principal G-bundle on it, define the functional Z on the space of connections on $P|_{\partial M}$ as

$$Z_M(b) = \int_{i^*(A)=b} \exp\left(\frac{i}{h} S_{YM}(A)\right) DA.$$

where $i : \partial M \hookrightarrow M$ is the tautological inclusion of the boundary and $i^8(A)$ is the pullback of the connection A to the boundary.

Now we can use the Faddeev-Popov Feynman diagrams to *define* the semiclassical expansion of this integral. In the finite dimensional case Feynman diagrams were derived as an asymptotic expansion of the existing integral. To define such expansion we should do the gauge fixing and then define the Feynman rules. The Feynman diagrams for the Yang-Mills are divergent because the propagator is singular at the diagonal (ultraviolet divergence). Nevertheless, the theory is renormalizable, as in the previous example, even better it is asymptotically free [30]. We will not go into the details of the discussion of renormalization but will make few remarks at the end of this section.

8.1 The gauge fixing

As we have seen in the finite dimensional case the constraint (gauge fixing) needed to construct the asymptotic expansion of the integral (30) has to be a cross-section through the orbits only in the vicinity of critical points (critical orbits) of the action function. To define Feynman diagrams for the Yang-Mills theory we can follow the same logic. In particular, we can choose a linear Lorentz gauge condition for connections in the vicinity of the classical solution A.

For a connection $A+\alpha$ where α is a 1-form (quantum fluctuation around A) Lorentz gauge condition is

$$d_A * \alpha = 0, \tag{46}$$

where $*$ is the Hodge operation. This condition defines a subspace in the linear space of $\mathfrak{g}$-valued 1-forms, so we can use the formula (38) which uses no Lagrange multipliers. The contribution to the path integral from a vicinity of A is, then an "integral" over the space of 1-forms α from $Ker(d_A^*)$. In other words, to define the semiclassical asymptotic of the partition function for the Yang-Mills theory we can try the Faddev-Popov expansion with the Lorenz gauge condition.

8.2 The Faddeev-Popov action and Feynman diagrams

Following the analogy with the finite dimensional case define the Faddeev-Popov action for pure Yang-Mills theory as the following action with fields $\alpha(x), \bar{c}(x), c(x)$:

$$S_A(\alpha) = S_{YM}(A) + \int_M \frac{1}{2} tr < F_A(\alpha), F_A(\alpha) > dx \\ - \frac{ih}{2}\int_M *d_A\bar{c}\wedge d_A c - \frac{ih}{2}\int_M *d_A\bar{c}\wedge[\alpha, c] \tag{47}$$

Here A is a background connection which is the solution to the classical Yang-Mills equations and α is a $\mathfrak{g}$-valued 1-form on M. The bosonic part of this action is simply $S_{YM}(A+\alpha)$.

The quadratic part in α and the quadratic part in $\bar{c}, c$ of the action (47) are given by the differential operator $d_A^* d_A$ which is invertible on the space $Ker(d_A^*)$ with Dirichlet boundary conditions. Other terms define the weights of Feynmann diagrams. The weights are shown on Fig. 17. Functions G_1^A and G_0^A are Green's functions of the Laplace-Beltrami operator $\Delta = d^*d + dd^*$ on 1- and 0-forms respectively.

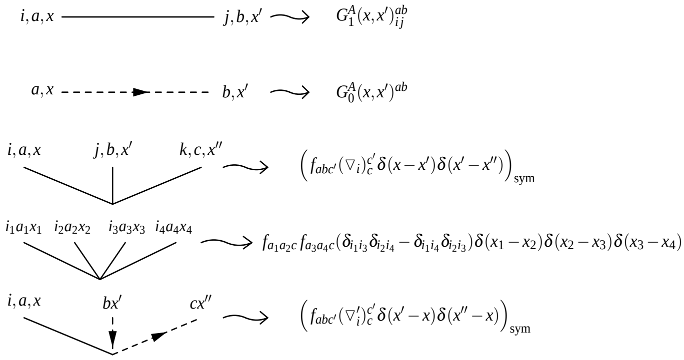


Fig. 17 Weights of Feynman diagrams in the semiclassical expansion for the Yang-Mills theory .

8.3 The renormalization

The propagator in the Yang-Mills theory is singular at the diagonal for $d > 1$ and just as in the scalar Bose field contributions from Feynman diagrams to the partition function diverge. However, just as in the scalar Bose field when $d \leq 4$, after the renormalization procedure Feynman diagrams become finite and there is a well defined semiclassical formal power series for the Yang-Mills given by renormalized diagrams. This fact was discovered by t'Hooft [32] who invented the dimensional regularization of Feynmann diagrams and showed that taking into account Fadeev-Popov ghost fields makes Yang-Mills into a renormalizable theory.

Moreover, the renormalization in the Yang-Mills theory is remarkable because it gives an asymptotically free theory. This was discovered in [30] and it means that particles in such theory has to behave as non-interacting, free, particles at high energies. This prediction perfectly agrees with experimental data and this is why the Yang-Mills theory is part of the Standard Model unifying the theory of strong, weak, and electromagnetic interactions.

The super-symmetric $N = 4$ Yang-Mills theory is expected to have a particularly remarkable renormalization. It turns out the divergent contributions from Feynman diagrams cancel each other in each order of the expansion in h . This was proven in the light-cone gauge and is believed to be true for other guages. This Yang-Mills theory is particularly important for Topological Quantum Field Theories [38][26] and in particular to the quantum field theoretical interpretation of the geometric Langlands program.

Finally, few words on correlation functions. Since the Yang-Mills theory is gauge invariant, natural observables should also be gauge invariant. Such observables are known as Wilson loops or, more generally, as Wilson graphs.

Recall the definition of Wilson loops. Let V be a finite dimensional representation of the Lie group G. The Yang-Mills potential A (the field in the Yang-Mills theory) is a connection in a principal G-bundle P. It induces a connection in the vector bundle

$V_P = P \times_G V$. Let

$$h_A(C_x) = P\exp\left(\int_{C_x} A\right) \tag{48}$$

be the parallel transport in V_P along a path C_x which starts and ends at $x \in M$ defined by the connection A. here P stands for the iterated path ordered integral.

The Wilson loop observable is

$$W_A^V(C) = Tr_{V_x}(h_A(C_x)) \tag{49}$$

Here the trace is taken over the fiber V_x of V_P over $x \in M$. The definition of more general gauge invariant observables, Wilson graphs, will be given later, when we will discuss observables in the Chern-Simons theory.

An important conjecture about the Yang-Mills theory, and another fundamental fact expected from this theory, is the dynamical mass generation. In terms of expectation values of Wilson loops this conjecture means that

$$< W_A(C) > \propto \exp(-ml(C)), \tag{50}$$

as $l(C) \to \infty$. Here on the left side we have the expectation value of the Wilson loop and on the right side $l(C)$ is the length of C in the Riemannian metric on M. This conjecture is based on the conjecture that the Yang-Mills theory can be defined non-perturbatively.

The parameter m in (50) characterizing the radius of correlation. In massless theory such as Yang-Mills theory there are no reasons to expect that $m \neq 0$. The appearance of such parameter with the scaling dimension of the mass, is known as dynamical mass generation. For more details about this conjecture see [35].

9 The Chern-Simons theory

In this section M is a compact oriented manifold. The classical Chern-Simons theory with compact simple Lie group G was described in section 2.8. As in the pure Yang-Mills theory, fields in the Chern-Simons theory are connections in a principal G-bundle over the space-time M. In contrast with the Yang-Mills theory, the Chern-Simons action is the first order action. One of the implications of this is the difference in Hamiltonian formulations. The other is that the path integral quantization for the Chern-Simons theory for manifolds with the boundary is more involved. Some of the aspects of this theory on manifolds with boundaries can be found in references [23] [7].

From now on we assume that the space-time M is a compact, oriented, and closed 3-manifolds. The Chern-Simons action is topological, i.e. its definition does not require a choice of metric on M . This is why it is natural to expect that the result of quantization, the partition function $Z(M)$, also depends only on the homeomorphism class of M. This gives a powerful criterium for consistency of the definition of Feynman diagrams: the sum of Feynman diagrams for any given order should depend only on the topology of the manifold.

The path integral formulation of the Chern-Simons theory on manifolds with a boundary is a bit more involved then the one for the Yang-Mills. This is because the Chern-Simons is the first order theory. The space of states assigned to the boundary is the space of holomorphic sections of the geometric quantization line bundle over the moduli space of flat connections in a trivial principal G-bundle over the boundary (provided we made a choice of complex structure). This space is a quantum counterpart to the boundary conditions for the Chern-Simons theory when the pull-back to the boundary is required to be holomorphic. For more details on the quantization of the Chern-Simons on manifolds with boundary see for example [7].

So, the goal of this section is to make sense of the expression

$$Z_M = \int e^{ikCS(A)} DA \tag{51}$$

or, more generally, of

$$Z_{M,\Gamma} = \int e^{ikCS(A)} W_\Gamma(A) DA, \tag{52}$$

where $W_\Gamma(A)$ is a gauge invariant functional (Wilson graph, or any other gauge invariant functional) which will be defined later, and k is an integer, which guarantees that the exponent is gauge invariant. The integral is supposed to be taken over the space of all connections on a principal G-bundle on M.

The integrals (51)(52) are not defined as mathematical objects. However, one can try (as in previous examples of the scalar Bose field and of the Yang-Mills theory) to define a combination of formal power series in k^{-1} resembling the expansion of finite dimensional integrals studied in the previous section. In the case of the Chern-Simons theory there is a natural requirement for such expansion: every term should be an invariant of 3-manifolds. Remarkably, such power series exists and is more or less unique. This program was originated by Witten in [57] who outlined the basic structure of the expansion. It was developed in a number of subsequent works, in particular, in [41][8][9][13][14][16] for the partition function for closed 3-manifolds and in [11][31][3] for (52), and others, when Γ is a link.

9.1 The gauge fixing

Let us use the same gauge fixing as in the Yang-Mills theory. For this we need to choose a metric on M.

Since classical solutions in the Chern-Simons theory are flat connections, the covariant derivative $d_A = d + A$ is a differential i.e. $d_A^2 = 0$ (twisted de Rham differential) acting on $\mathfrak{g}$-valued forms on M [9]. Denote the cohomology spaces by H^i_A. Because of the Poincare duality we have natural isomorphisms $H^0_A \simeq H^3_A$ and $H^1_A \simeq H^2_A$.

In a neighborhood of a classical solution A connections can be written as $A+\alpha$ where α is a $\mathfrak{g}$-valued 1-form on M. As in the Yang-Mills theory the Lorenz gauge condition for such connections is:

$$d^*_A \alpha = 0$$

We will use this gauge condition in the rest of the paper.

9.2 The Faddeev-Popov action in the Chern-Simons theory

According to our finite dimensional example we should add fermionic ghost fields $c(x)$ and $\bar{c}(x)$ and the Lagrange multipliers $\lambda(x)$ to the action if we want to define Feynman diagrams in this gauge theory. However, as we argued in section 6.4, the gauge condition can be chosen linearly near each critical point of the action, and therefore we can use the version without Lagrange multipliers. In this case we just have to add fermionic ghost fields to the action.

According to the rules of section 6.4 the Faddeev-Popov action for the Chern-Simons theory is the sum of the classical Chern-Simons action and the ghost terms which are identical to those for the Yang-Mills theory:

$$CS_A(\alpha) = CS(A) + \int_M \frac{1}{2} tr\left(\alpha\wedge d_A\alpha - \frac{2}{3}\alpha\wedge\alpha\wedge\alpha\right) - \frac{ih}{2}\int_M *d_A\bar{c}\wedge d_A c - \frac{ih}{2}\int_M *d_A\bar{c}\wedge[\alpha, c] \quad (53)$$

where h stands for $\frac{1}{k}$. We will focus in the discussion below mostly on the special case of isolated flat connections, when $H^1_A = \{0\}$. Quite remarkably [8], the field α and ghost fields in the Chern-Simons theory can be combined into one odd "superfield":

$$\Psi = c + \alpha + ih * d_A\bar{c}$$

Here c, α, and $*d_A\bar{c}$ are 0, 1, and 2 forms respectively. The action (53) can be written entirely in terms of Ψ [10]:

$$CS_A(\alpha) = CS(A) + \frac{1}{2}\int_M tr\left(\Psi\wedge d_A\Psi - \frac{2}{3}\Psi\wedge\Psi\wedge\Psi\right)$$

---

[9] Because a principal G-bundle over any compact oriented 3d-manifolds is trivializable, we choose a trivialization and identify $\Omega(M, ad(E))$ with $\Omega(M,\mathfrak{g})$

[10] This form of the Faddeev-Popov action for the Chern-Simons theory has a simple explanation in the framework of the Batalin-Vilkovisky formalism, see for example [17]. However, we will not discuss it in these notes.

The quadratic part of the action is the de Rham differential twisted by the flat connection A.

If $H^2_A(M,\mathfrak{g}) = \{0\}$ (equivalently, $H^1_A = \{0\}$) the gauge condition $d_A^*\alpha = 0$ together with the special form of the last term in Ψ, is equivalent to $d_A^*\Psi = 0$. The inverse is also true: $d_A^*\Psi = 0$ implies $d_A^*\alpha = 0$ together with $\Psi^{(2)}$ being Hodge dual to an exact form.

The quadratic part of this action is $(\Psi, *d_A\Psi)$ where

$$(\Phi,\Psi) = \int_M tr(\Phi \wedge *\Psi).$$

The surface of the constraint $d_A\Psi = 0$ is the super-space $\Omega^0(M,\mathfrak{g}) \oplus Ker((d_A^*)_0^*) \oplus Im((*d_A)_0) \subset \Omega(M,\mathfrak{g})[1]$ where the first and the third summands are odd and the second is even. The operator $D_A = *d_A + d_A*$ restricted to this subspace describes the quadratic part of the action. Indeed, we have

$$\int_M tr(\Psi \wedge d_A\Psi) = \frac{1}{2}(\Psi, (*d_A + d_A*)\Psi)$$

The operator D_A maps even forms to even and odd form to odd, $D_A : \Omega^i \to \Omega^{2-i} \oplus \Omega^{4-i}$. It plays a prominent place in the index theory [6]. It can be considered as a Dirac operator in a sense that

$$D_A^2 = \Delta_A = d_A^* d_A + d_A d_A^*,$$

where Δ_A is Hodge Laplace operator. The operator D_A effectively appears in the quadratic part only being restricted to odd forms. This operator will be denoted $D_A^- : \Omega^1 \to \Omega^1 \oplus \Omega^3; \Omega^3 \to \Omega^1$.

Now the question is whether the operator D_A^- is invertible on the surface of the constraint. In other words, if the Lorenz gauge is really a cross-section through gauge orbits.

9.2.1 The propagator

First, assume that the complex $(\Omega^i(M,\mathfrak{g}), d_A)$ is acyclic, i.e. $H^i(M,\mathfrak{g}) = \{0\}$ for all $i = 0,1,2,3$ (by Poincare duality $H^i \simeq H^{3-i}$, so it is enough to assume the vanishing of H^0 and H^1). In this case the representation of $\pi_1(M)$ in G defined by holonomies of flat connection A is irreducible (implied by $H^0 = \{0\}$) and isolated (implied by $H^1 = \{0\}$).

Because the spaces H^i can be naturally identified with harmonic forms and therefore with zero eigenspaces of Laplace operators, in this case all Laplace operators are invertible and so is D_A. Denote by G_A the inverse to Δ_A, i.e. the Green's function, then

$$P_A = (D_A^-)^{-1} = D_A^- G_A = G_A D_A^-$$

Thus, in this case the quadratic part is non-degenerate and we can write contributions from Feynman diagrams as multiple integrals of the kernel of the integral operator $(D_A^-)^{-1}$. The analysis of the contributions of Feynman diagrams to the the partition function was studied in this case by Axelrod and Singer in [8][9], and by Kontsevich [41].

Another important special case when the flat connection is reducible but still isolated. For example a trivial connection for rational homology spheres [13][14] [16] has such property. In this case we still have $H^1 = H^2 = \{0\}$ and the Lorenz gauge for α together with the exactness of $*\Psi^{(2)}$ is still equivalent to the Lorenz gauge for Ψ, i.e. $d^*\Psi = 0$. However now there are harmonic forms in $\Omega^0(M)$ and $\Omega^3(M)$ corresponding to the fundamental class of M and because of this, D_A^- is not invertible on the space of all forms.

Nevertheless, in this case (and in a more general case when $H^1 \neq \{0\}$) one can construct an operators which is "almost inverse" to D_A^-. Such operator is determined by the chain homotopy $K : \Omega^i \to \Omega^{i-1}$ and the Hodge decomposition of Ω. For details about such operator P see [8][9][13][14] [16] and section 3.2 of [17].

An important case of rational homology spheres is S^3 itself. In this case the inverse to D^- for trivial connection can be constructed explicitly by puncturing of S^3 at one point (the infinity). The punctured S^3 is homeomorphic to $\mathbb{R}^3$ where the fundamental class is vanishing and D^- is invertible. The restriction of $(D^-)^{-1}$ to 1-forms is the integral operator with the kernel

$$\omega(x,y) = \frac{1}{8\pi}\sum_{ijk=1}^{3} \varepsilon^{ijk}\frac{(x-y)^i dx^i \wedge dy^k}{|x-y|^3} I \tag{54}$$

where ε^{ijk} is the totally antisymmetric tensor with $\varepsilon^{123} = 1$, and I is the identity in $End(V)$. It acts on the form $\sum_i \alpha_i(x)dx^i$ as

$$\omega \circ \alpha(x) = \frac{1}{8\pi}\sum_{ijk=1}^{3} \varepsilon^{ijk}\int_{\mathbb{R}^3} \frac{(x-y)^i}{|x-y|^3}\alpha_j(x) d^3y dx^k. \tag{55}$$

In all cases the propagator P_A is defined as the restriction of the restriction of the "inverse" to D_A^- (a chain homotopy, when D^- is not invertible) to one-forms. It is an integral operator with the kernel being an element of the skew-symmetric part of $\Omega^2(M \times M, \mathfrak{g} \times \mathfrak{g})$. If e_a is an orthonormal basis in $\mathfrak{g}$, and x^i are local coordinates

$$P_A(x,y) = P_A^{ab}(x,y)_{ij} e_a \otimes e_b dx^i \wedge dy^j, \quad P_A^{ab}(x,y)_{ij} = P_A^{ba}(y,x)_{ji}$$

9.3 *Vacuum Feynman diagrams and invariants of* 3*-manifolds*

9.3.1 Feynman diagrams

As in other examples of quantum field theories such as scalar Bose field, and the Yang-Mills field we want to define the semiclassical expansion of the partition function and of correlation function imitating the semiclassical expansion of finite dimensional integrals.

Following this strategy and the computations of the Faddeev-Popov action for the Chern-Simons theory in Lorenz gauge presented above it is natural to define the partition function $Z(M)$ (the "integral" (51)) for the Chern-Simons theory as the following combination of formal power series

$$\sum_{[A]} \exp\left(i\frac{CS_M(A)}{h} + \frac{i\pi}{4}\eta(A)\right) |\det{}'(D_A^-)|^{-1/2} \det{}'(\Delta_A^0) \left(1 + \sum_{n\geq 1} (ih)^n I_A^{(n)}(M,g)\right), \tag{56}$$

where h stands for k^{-1}, det' are regularized determinants of corresponding differential operators, Δ_A^0 is the Laplace-Beltrami operator acting on $C^\infty(M,\mathfrak{g})$, D_A^- is the operator D_A acting on odd forms, and $\eta(A)$ is the index of the operator D_A^-. The sum is taken over gauge classes of flat connections on M (we assume that there is a finite number of such isolated flat connections). The n-th order contribution is given by the sum of Feynman diagrams

$$I_A^{(n)}(M,g) = \sum_{\Gamma, -\chi(\Gamma)=n} \frac{I_A(D(\Gamma),M,g)(-1)^{c(D(\Gamma))}}{|Aut(\Gamma)|} \tag{57}$$

In the Chern-Simons case these are graphs with $2n$ vertices (each of them being 3-valent). The contribution $I_A(D(\Gamma),M,g)$ is an appropriate trace of the integral over M^m of the product of propagators. In other words this is the contribution from the Feynman diagram $D(\Gamma)$ with weights from Fig. 18[11]

Because in this case we have only 3-valent vertices only two first order diagrams Fig. 5 will survive. Among these two only the "theta graph" will give non-zero contribution due to the skew-symmetry of the propagator. The contribution from the theta graph is

---

[11] The weights in Feynman diagrams for the Chern-Simons theory are the same as we would have without the ghost fields (without Faddeev-Popov determinant). For the Chern-Simons theory the ghost fields change the bosonic Feynman diagrams (which we would have in the naive perturbation theory) to the fermionic one (with the sign $(-1)^{c(D(\Gamma))}$). It happens because Ψ is an odd field and therefore the Feynman diagrams have fermionic nature. Orientation of graphs used in [41] is another way to encode the fermionic nature of Feynman diagrams for the Chern-Simons theory.

With the fermionic sign the sum of Feynman diagrams of is finite in each order [8]. Without this sign the sum would diverge because of the singularity of the propagator at the diagonal. It is similar to the effect of ghost fields in the Yang-Mills theory. Without ghost fields the Yang-Mills theory is not renormalizable. With ghost fields, as it was shown by t'Hooft it becomes renormalizable.

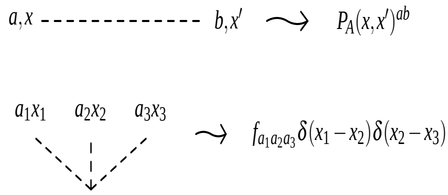


Fig. 18 Weights in Feynman diagrams for the Chern-Simons theory, i.e propagators and vertices for the Ψ-field.

$$\int_M \int_M \sum_{\{a\},\{b\}} f_{a_1a_2a_3} f_{b_1b_2b_3} P^{a_1b_1}(x,y) P^{a_2b_2}(x,y) P^{a_3b_3}(x,y) dxdy. \tag{58}$$

According to what we expect from the heuristic formula (51) the expression (56) should depend only on the homeomorphism class of M and should not depend on the choice of metric (gauge condition). But first of all we should make sure that every term in this series is defined. The problem is that individual integrals in the definition of $I^{(n)}$ diverge.

The remarkable property of Feynman diagrams in the Chern-Simons case is that the sum of Feynman diagrams of any given order is finite. It is relatively easy to see that (58) is finite because of the skew-symmetry of the propagator and because it is asymptotically equivalent to (55) near the diagonal (i.e. when $x \to y$). The finiteness of the sum of Feynmann diagrams in each order was proven in all orders by Axelrod and Singer in [8][9] for acyclic connections, and by Kontsevich in[41] for trivial connections in rational homology spheres. This illustrates that the Chern-Simons theory is very different from the Yang-Mills theory where the renormalization procedure is necessary.

9.3.2 Metric independence

Now let us focus on the metric dependence of (56). Because we expect the quantum field theory to be topological, the leading terms and each coefficient in the expansion in powers of h should not depend on metric. First, assume that A is an isolated irreducible flat connection.

The most singular term in the exponent is $CS_M(A)$ which is clearly metric independent. Taking into account that $\Delta = D^2$ and the natural isomorphism $\Omega^3 \simeq \Omega^0$ the absolute value of the determinant of D_A^- can be written as

$$|\det{}'(D_A^-)| = \det{}'(\Delta_A^1)^{\frac{1}{2}} \det{}'(\Delta_A^3)^{\frac{1}{2}} = \det{}'(\Delta_A^1)^{\frac{1}{2}} \det{}'(\Delta_A^0)^{\frac{1}{2}},$$

where Δ_A^i is the action of the Laplacian twisted by A on i-forms. Using this identity we can rearrange the determinants as

$$|\det{}'(D_A)|^{-1/2}\det{}'(\Delta_A^0) = \frac{\det'(\Delta_A^0)^{\frac{3}{4}}}{\det'(\Delta_A^1)^{\frac{1}{4}}}.$$

This expression is exactly the square root of the Ray-Singer analytical torsion [44], which is also the Reidemeister torsion, and is known to be independent of the metric [12].

The exponent $\frac{i\pi}{4}\eta(A)$, which involves the index of D_A^- also known as the η-invariant can be written as

$$\frac{2\pi\eta(A)}{8} = d\frac{2\pi\eta(g,M)}{4} + c_2(G)CS(A) - \frac{2\pi}{4}I_A - \frac{d\pi(1+b^1(M))}{4} + \frac{2\pi(\dim(H^0)+\dim(H^1))}{8} \quad (mod 2), \quad (59)$$

Here $\eta(g,M)$ is the index of the operator $D = *d + d*$ acting on odd forms on M, $d = \dim(G)$, $c_2(G)$ is the value of the Casimir element for $\mathfrak{g} = Lie(G)$ on the adjoint representation (also known as the dual Coxeter number $h^\vee$ for the appropriate normalization of the Killing form on $\mathfrak{g}$), and $b^1(M)$ is the first Betti number for M. The quantity $I_A \in \mathbb{Z}/8\mathbb{Z}$ is the spectral flow of the operator

$$\begin{pmatrix} *d_{A_t} & -d_{A_t}* \\ d_{A_t}* & 0 \end{pmatrix}$$

acting on $\Omega^1(M,\mathfrak{g}) \oplus \Omega^3(M,\mathfrak{g})$. Here A_t, $t \in [0,1]$, is a path in the space of connections joining A with the trivial connection. The spectral flow I_A depends neither on the metric on M nor on the path.

The index $\eta(g,M)$ depends on the metric g on M. Recall that a framing of M is the homotopy class of a trivialization of the tangent bundle TM. Given a framing $f : M \to TM$ of M we can define the gravitational Chern-Simons action

$$I_M(g,f) = \frac{1}{4\pi}\int_M f^* Tr(\omega \wedge d\omega - \frac{2}{3}\omega\wedge\omega\wedge\omega), \quad (60)$$

---

[12] When A is an isolated irreducible flat connection the Ray-Singer torsion is defined as a positive number $\tau(M,A)$ such that

$$\tau(M,A) = \prod_{i\geq 1}\det{}'(\Delta_A^i)^{i(-1)^{i+1}/2}$$

Here and in the main text $\det'(\Delta)$ is the zeta function regularization of the determinant: $\det'(\Delta) = \exp(-\zeta'_\Delta(0))$, where

$$\zeta(s) = Tr(\Delta^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}Tr(e^{t\Delta})dt,$$

Taking into account that for Riemannian manifolds we have natural isomorphisms $\Omega^0 \simeq \Omega^3$ and $\Omega^1 \simeq \Omega^2$ we obtain

$$\tau(M,A) = \det{}'(\Delta_A^0)^{\frac{3}{2}}\det{}'(\Delta_A^1)^{-\frac{1}{2}}.$$

When the H^i are not all zeroes, $\tau(M,A)^{\frac{1}{2}}$ can be regarded as a volume element on ”zero” modes, i.e. on the space $H^0 \oplus H^1$.

where g is the metric on M, ω is the Levi-Civita connection on M, and the integrand is the pullback via f^* of the Chern-Simons form on TM.

According to the theorem of Atiyah-Patodi-Singer the expression

$$\frac{1}{4}\eta(g,M)+\frac{1}{12}\frac{I_M(g,f)}{2\pi}$$

depends only on the homeomorphism class of the manifold M with the framing f but not on the metric, and this is true for any framing f.

These arguments suggest [57] [24] that for manifolds with only irreducible and isolated flat connections, the leading term in the expression (56) should be proportional to

$$\exp\left(d\frac{i\pi}{4}\eta(g,M)+i\frac{d}{24}I_M(g,f)-\frac{di\pi}{4}\right)$$
$$\sum_{[A]}\exp\left(-\frac{2\pi i I_A}{4}+i(\frac{1}{h}+c_2(G))CS_M(A)\right)\tau(M,A)^{1/2}(1+O(1/k)) \quad (61)$$

where $\tau(M,A)$ is the Ray-Singer torsion. This expression differs from the original guess (56) by the extra factor $\exp(i\frac{d}{24}I_M(g,f))$.

Let us emphasize that this formula is *not a computation of the path integral*, as there is nothing to compute. It is the rearrangement and adjustment of the natural guess for the leading terms of the semi-classical expansion of the quantity to be defined. The adjustment was made on the base of the concept that the expression should not depend on the metric. Remarkably, at the end it does not depend on the metric, though it still depends on the framing.

Now let us look into higher order terms.

In the finite dimensional case, when Feynman diagrams represent an asymptotic expansion of an existing (convergent) integral the sum of Feynman diagrams in each order does not depend on the choice of gauge condition simply by the nature of these coefficients.

In the infinite dimensional case we are defining the integral as the sum of Feynman diagrams. Therefore the independence of the sum of Feynman diagrams on the choice of the gauge condition (a metric on M in the case of Lorenz gauge condition for Chern-Simons theory) should be checked independently in each order. This was done by Axelrod and Singer in [8][9] for acyclic connections and by Bott and Cattaneo [13][14] for trivial connections and rational homology spheres. One of the important tools for the proof of such fact is the graph complex by Konstevich [41].

More precisely the following has been proven. First write the sum of higher order contributions as

$$1+\sum_{n\geq 1}(ih)^n I_A^{(n)}(M,g)=\exp\left(\sum_{n\geq 1}(ih)^n J_A^{(n)}(M,g)\right),$$

where

$$J_A^{(n)}(M,g) = \sum_{-\chi(\Gamma)=n}^{(c)} \frac{I_A(D(\Gamma),M,g)(-1)^{c(D(\Gamma))}}{|Aut(\Gamma)|}.$$

Here the sum is taken over connected graphs only. As it follows from [8][9][13][14] this expression can be written as

$$J_A^{(n)}(M,g) = F_A^{(n)}(M,f) + \beta_n I_M(g,f),$$

for some $F_A^{(n)}(M,f)$ and constants β_n. Here $I(g,f)$ is the gravitational Chern-Simons action (60).

Thus, the sum of contributions of connected Feynman diagrams of fixed order gives, after the substraction of the gravitational Chern-Simons action with an appropriate numerical coefficient, does not depend on the metric, and, therefore, is an invariant of framed rational homology spheres (in the works of Bott and Cattaneo) or of a 3-manifold with an acyclic flat connection in a trivial principal G-bundle over it (in the works of Axelrod and Singer, and Kontsevich).

Finally, all these results can be summarized as the following proposal for the partition function of the semiclassical Chern-Simons theory. It depends on the framing and is proportional to

$$\exp\left(c(h)\left(\frac{i\pi}{4}\eta(g,M) + i\frac{1}{24}I_M(g,f)\right)\right) e^{-\frac{id\pi(1+b^1(M))}{4}} \sum_{[A]} e^{i(\frac{1}{h}+c_2(G))CS_M(A)}$$
$$e^{-\frac{2\pi i I_A}{4}} \tau(M,A)^{1/2} \exp\left(\sum_{n\geq 1} (ih)^n F_A^{(n)}(M,f)\right), \quad (62)$$

where $c(h) = d + O(h)$. Witten suggested [57] the exact form of $c(h)$:

$$c(h) = \frac{d}{1+hc_2(G)} = \frac{kd}{k+h^\vee}$$

where $k = \frac{1}{h}$. This is the central charge of the corresponding Wess-Zumino-Witten theory.

In order to define the full TQFT from this proposition one should define the partition function in case when flat connections are not necessary irreducible and when they are not isolated. For most recent progress in this direction see [17]. Also, in order to have a TQFT we should define partition functions for manifolds with boundaries. In the semiclassical framework this is an open problem.

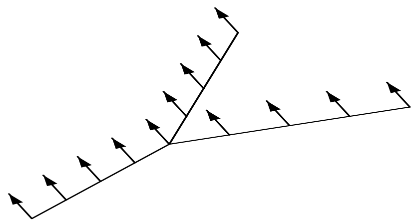


Fig. 19 Parallel framing at the trivalent vertex.

9.4 Wilson loops and invariants of knots

Arguing ”phenomenologically” one should expect that expectation values of topological[13] gauge invariant observables in Chern-Simons theory, which do not require metric in their definition, should depend only on topological data and, therefore, give some topological invariants.

9.4.1 Wilson graphs

An example of topological observables are Wilson loops (49), or, more generally, Wilson graphs. Let us clarify the notion of the Wilson loop observable in the perturbative Chern-Simons theory. Wilson loops are defined for a collection of circles embedded into M otherwise known as a link. Our goal is to define the power series which would be similar to the perturbative expansion (34), as (64) is similar to the perturbative expansion (38). Most importantly, such power series should not depend on the choice of a metric on M (the choice of the gauge condition). As we have seen above, this is possible but one should choose a framing $f : M \to TM$ of the 3-manifold.

Framed Wilson graph observable (or simply Wilson graph) is a gauge invariant functional on connections defined as follows. Let Γ be a framed graph[14] embedded in M. Here by the framing we mean a section of the co-normal bundle $x \in \Gamma \to TM/T_x\Gamma$ for a generic point $x \in \Gamma$ which agree on vertices.

Framing, together with the orientation of M defines a cyclic ordering of edges adjacent to each vertex. It is illustrated on Fig. 19

To define a Wilson graph we should make the following choices:

1. Choose a total ordering of edges adjacent to each vertex which agrees with the cyclic ordering defined by the framing.

---

[13] Topological observables do not require a metric in their definition

[14] The embedding $C \subset M$ induces the embedding $TC \subset TM$ and therefore a framing on M induces a framing on C, i.e the mapping $C \to (T_C M/TC)^{perp}$. A metric on M defines the splitting $T_C M = NC \oplus TC$ where NC is a normal bundle to C. A framing $f : M \to TM$ defines the framing $f_C : C \to NC$ of C by attaching a normal vector $f_C(x) \in N_x C$ for every $x \in C$.

2. Choose an orientation of each edge.
3. Choose a total ordering of vertices of Γ.
4. Choose a finite dimensional representation V for each edge of Γ.
5. Choose a G-invariant linear mapping $\nu : \mathbb{C} \to V_1^{\varepsilon_1} \otimes \cdots \otimes V_k^{\varepsilon_k}$ for each vertex. Here numbers $1, \dots, k$ enumerate edges adjacent to the vertex, $\varepsilon_i = +$ if the edge i is incoming to the vertex, $\varepsilon_i = -$ if the edge i is outgoing from the vertex, $V_i^+ = V_i$, $V_i^- = V_i^*$, V_i is the representation assigned to the edge i, and V_i^* is its dual.

As in the case of Feynman diagrams, the ordering of vertices, and on the edges adjacent to each vertex, defines a perfect matching on endpoints of edges. Choose such total ordering.

Use the coloring of edges by finite dimensional G-modules and the orientation of edges to define the space $V_{a_1}^{\alpha_1} \otimes V_{a_2}^{\alpha_2} \otimes V_{a_3}^{\alpha_3} \otimes V_{b_1}^{\beta_1} \otimes V_{b_1}^{\beta_2} \otimes \cdots$. Here indices $1, 2, \cdots$ enumerate vertices, letters $a_i, b_i, c_i, \dots$ enumerate edges adjacent to the vertex i, and $\alpha, \beta, \cdots = \pm$ indicate the orientations of edges $a, b, c, \cdots$ ($+$ if the edges is incoming, and $-$ if the edge is outgoing). The number of factors in the tensor product is equal to the number of endpoints of edges.

The coloring of vertices defines the vector

$$\nu_1 \otimes \nu_2 \otimes \cdots \in V_{a_1}^{\alpha_1} \otimes V_{a_2}^{\alpha_2} \otimes V_{a_3}^{\alpha_3} \otimes V_{b_1}^{\beta_1} \otimes V_{b_1}^{\beta_2} \otimes \cdots .$$

The holonomy $h_e(A)$ along the edge e is an element of $End(V_e)$, and therefore, it is a vector in $V_e \otimes V_e^*$, where V_e is the finite dimensional G-module assigned to the edge. The tensor product of holonomies defines a vector $\otimes_e h_e(A)$ in the space dual to $V_{a_1}^{\alpha_1} \otimes V_{a_2}^{\alpha_2} \otimes V_{a_3}^{\alpha_3} \otimes V_{b_1}^{\beta_1} \otimes V_{b_1}^{\beta_2} \otimes \cdots$.

The Wilson graph observable is the functional on the space of connections defined as

$$W_\Gamma(A) = < \otimes_e h_e(A), \nu_1 \otimes \nu_2 \otimes \cdots >$$

Here is an example of the Wilson graph observable for the "theta graph" :

$$\sum_{i_1, i_2, i_3} (h_{e_1}(A))^{i_1}_{j_1} (h_{e_1}(A))^{i_1}_{j_1} (h_{e_1}(A))^{i_1}_{j_1} \nu_{i_1, i_2, i_3} \nu_{j_1, j_2, j_3}$$

The indices i_k, j_k enumerate a basis in the representation V_k assigned to the edge $k = 1, 2, 3$ and ν, μ are G-invariant vectors in the corresponding tensor products. Here we used an orthonormal basis in $\mathfrak{g}$ which explains upper and lower indices.

9.4.2 Feynman diagrams for Wilson graphs

As in case of the partition function define the expectation value (52) of the Wilson graph Γ as a combination of formal power series, similar to the formula (34) for the asymptotic expansion of corresponding finite dimensional integrals.

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Fig. 20 Weights of trivalent vertices were two solid edges meet dashed edge.

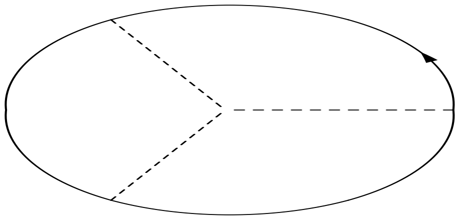

Fig. 21 An example of order one graph.

Taking into account all we know for the partition functions of the Chern-Simons theory we arrive to the following proposal. The semiclassical ansatz for the expectation value of the Wilson graph W_Γ is

$$\sum_{[A]} \exp\left(i\frac{CS(A)}{h} + \frac{id\pi}{4}\eta(A)\right) |\det{}'(D_A)|^{-1/2} \det{}'((\Delta_A)_0)\left(W_\Gamma(A) + \sum_{n\geq 1}(ih)^n I_A^{(n)}(M,\Gamma)\right) \quad (63)$$

Here we assume that all flat connections are irreducible and isolated. All quantities are the same as in (56) except

$$I_A^{(n)}(M,\Gamma) = \sum_{\Gamma', \chi(\Gamma)-\chi(\Gamma')=n} \frac{I_A(\Gamma',\Gamma)}{|Aut(\Gamma)|}.$$

The Feynman diagram rules in the presence of Wilson graphs are essentially the same as for the partition function with weights given on Fig. 18. The difference is that now there are two types of edges, and two types of propagators (linear operators assigned to edges). As for the partition function we have dashed edges with 3-valent vertices. But now we also have solid edges, see an example on Fig. 21, vertices where only solid edges meet and vertices where where two solid edges (with opposite orientations) meet a dashed edge. The subgraph formed by solid edges is always Γ. The weights of vertices where only solid edges meet is given by the coloring of this edge in Γ. The weights of vertices where two solid edges meet a dashed edges and weights of solid edges are described on Fig. 20.

One can show [11][31][3][15][52], [13][14] that the sum of integrals corresponding to Feynman diagrams of order n is finite for each n. Similarly to the vacuum partition function from the previous section the semiclassical ansatz for the expectation value of the Wilson graph depends on the framing, but remarkably not on the metric. When flat connections are irreducible and isolated we arrive to the following expression

$$\exp\left(c(h)(\frac{i\pi}{4}\eta(g,M)+\frac{i}{24}I_M(g,f))-\frac{id\pi(1+b^1(M))}{4}\right)$$
$$\sum_{[A]}\exp\left(i(\frac{1}{h}+c_2(G))CS_M(A)-\frac{2\pi i I_A}{4}\right)\tau(M,A)^{1/2}$$
$$\left(W_\Gamma(A)+\sum_{n\geq 1}(ih)^n J_A^{(n)}(M,\Gamma,f)\right) \quad (64)$$

Here the coefficients $J_A^{(n)}(M,\Gamma,f)$ do not depend on the metric but depends on the framing f of M. This formula defines the path integral semiclassically. Let us emphasize again, that it is not a result of computation of an integral. It is a definition, modeled after the semiclassical expansion of integrals in terms of Feynman graphs. A remarkable *mathematical* fact is that every terms is defined (integrals do not diverge), and that it does not depend on the metric.

More careful analysis includes powers of h. A conjecture for counting powers of h when $H_A^0, H_A^1 \neq \{0\}$ was proposed in [24][36][46]. It agrees with the finite dimensional analysis from previous sections and states that, in general, we should expect that the partition function is proportional to

$$\exp\left(d\frac{i\pi}{4}\eta(g,M)+i\frac{c(h)}{24}I_M(g,f)-\frac{d\pi i(1+b^1(M))}{4}\right)$$
$$\sum_{A}(2\pi(k+h^\vee)^{\frac{\dim(H_A^0)-\dim(H_A^1)}{2}}\frac{1}{Vol(G_A)}$$
$$\exp\left(i(k+h^\vee)CS_M(A)-\frac{2\pi i I_A}{4}-i\pi\frac{\dim(H_A^0)+\dim(H_A^1)}{2}\right)$$
$$\int_{M_A}\tau^{1/2}\left(W_\Gamma(A)+\sum_{n\geq 1}(ih)^n J_A^{(n)}(M,\Gamma,f)\right) \quad (65)$$

Here the sum is taken over representatives A of connected components M_A of the moduli space of flat connections in a principal G-bundle over M. The torsion τ is an element of $\otimes_i det(H_A^i)^{\otimes(-1)^i} \simeq (det(H_A^0)\otimes det(H_A^1)^*)^{\otimes 2}$. The Lie algebra $\mathfrak{g}$ has an invariant scalar product and therefore $H_A^0 \subset \mathfrak{g}$ has an induced volume form. Pairing this volume form with the square root of the torsion gives a volume form on H_A^1. Assuming the connected component is smooth we can integrate functions with respect to this volume form. The factor $Vol(G_A)$ is the volume of the stabilizer of the flat connection.

9.5 Comparison with combinatorial invariants

Invariants of 3-manifolds with framed graphs also can be constructed combinatorially (as a combinatorial topological quantum field theory). In [45] such invariants were constructed using modular categories and the representation of 3-manifolds as a surgery on S^3 or on a handlebody along a framed link. Another combinatorial construction, based on the triangulation, was developed in [53]. This construction uses certain class of monoidal categories which are not necessarily braided.

These two constructions are related:

$$Z_M^{RT}(\mathscr{C})Z_{\overline{M}}^{RT}(\mathscr{C}) = Z_M^{TV}(\mathscr{C}) = Z_M^{RT}(D(\mathscr{C}))$$

Here $Z_M^{RT}(\mathscr{C})$ is the invariant obtained by the surgery [45], $Z_M^{TV}(\mathscr{C})$ is the invariant obtained by the triangulation, and the category $D(\mathscr{C})$ is the center (the double) of the category $\mathscr{C}$, see for example [39], and $\overline{M}$ is the manifold M with the reversed orientation.

Most interesting known examples of modular categories are quotient categories of finite dimensional modules over quantized universal enveloping algebras at roots of unity, see [45][2][28]. Such categories are parametrized by pairs $(\varepsilon, \mathfrak{g})$, where $\varepsilon = \exp(\frac{2\pi i m}{r})$ with mutually prime m and r and $\mathfrak{g}$ is a simple Lie algebra. Denote the truncated category of modules over $U_\varepsilon(\mathfrak{g})$ by $\mathscr{C}_\varepsilon(\mathfrak{g})$ (see [45][2][28] for details). When $m = 1$ and $r = k + c_2(\mathfrak{g})$ this category is naturally equivalent to the braiding-fusion category of the WZW conformal field theory at level k, i.e. to the category of integrable modules over the affine Lie algebra $\hat{\mathfrak{g}}$ at level k with the fusion tensor product [40]. This conformal field theory is directly related to the Chern-Simons theory at level k. The arguments in favor of this are not perturbative [7]. They are based on ideas of geometric quantization.

For other values of m it is also equivalent to the braiding-fusion category of a conformal field theory, but this conformal field theory is not directly related to the Chern-Simons theory.

The main conjecture relating the combinatorial and geometric approaches is that the *following power series are identical*:

- The asymptotic expansion of the combinatorial TQFT based on the category $\mathscr{C}_\varepsilon(\mathfrak{g})$ when $\varepsilon = \exp(\frac{2\pi i}{k+c_2(\mathfrak{g})})$ and $k \to \infty$.
- The semiclassical expansion for the Chern-Simons path integral in terms of Feynman diagrams.

Of course this is rather an outline of a number of conjectures rather than a conjecture. The main reason is that the semi-classical partition functions for the Chern-Simons theory in terms of Feynman integrals are not worked out yet.

The precise statement about the correspondence between these formal power series was first worked out in [24] followed by [36][27][46][47][1].

To compare these invariants one should first choose a canonical 2-framing on M [4]. The 2-framing on M is a section of $TM \times TM$. The Levi-Civita connection on TM defined by the Riemannian structure on M induces a connection on $TM \times TM$.

the canonical 2-framing defines the branch of the gravitational Chern-Simons action with the property

$$d\frac{\pi}{4}\eta(g,M)+\frac{c(h)}{24}I_M(g,f)=0.$$

One should expect that the choice of such 2-framing presumably fixes the framing in higher order corrections, though this part is still conjectural.

When the moduli space of flat connection on a principal G-bundle over M is a collection of isolated points each corresponding to an irreducible flat connection, one should expect the following:

$$Z_M^{RT}\sim\frac{1}{|Z(G)|}e^{-\frac{d\pi i}{4}}\sum_{[A]}e^{\frac{-2i\pi I_A}{4}}e^{(k+h^\vee)CS_M(A)}(1+O(1/k)),$$

where $|Z(G)|$ is the number of elements in the center of G, and Z^{RT} is the combinatorial invariant corresponding to the category $\mathscr{C}_\varepsilon(\mathfrak{g})$. When connected components of the moduli space have non-zero dimension and are smooth, the expected asymptotic behavior is

$$\begin{aligned}Z_M^{RT}\sim\exp\left(-\frac{d\pi i(1+b^1(M))}{4}\right)&\sum_{[A]}(2\pi(k+h^\vee))^{\frac{\dim(H_A^0)-\dim(H_A^1)}{2}}\frac{1}{Vol(G_A)}\\ \exp\left(i(k+h^\vee)CS_M(A)-\frac{2\pi iI_A}{4}-i\pi\frac{\dim(H_A^0)+\dim(H_A^1)}{2}\right)&\\ \int_{M_A}\tau^{1/2}W_\Gamma(A)\,(1+O(1/k))\,.&\qquad(66)\end{aligned}$$

Many examples confirming this prediction were analyzed in [24][36][27][46][1].

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