Lecture 8.

Note Title 3/7/2008

Solutions to some h.w. problems

① Braiding for $U_q(sl_2)$ for generic q

(i) Consider the algebra $\mathring{U}_q(sl_2)$ generated by x, P_n, $n \in \mathbb{Z}$, E, F with defining relations:

$$P_n E = E P_{n-2}, \quad P_n F = F P_{n+2}$$

$$1 = \sum_{n \in \mathbb{Z}} P_n, \quad x \in \text{center}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad K = x \sum_{n \in \mathbb{Z}} q^n P_n$$

(ii) It is a Hopf algebra with

$$\Delta P_n = \sum_{n_1 + n_2 = n} P_{n_1} \otimes P_{n_2}$$

$$\Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F$$

(iii) The Hopf algebra embedding

$$U_q(sl_2) \hookrightarrow \mathring{U}_q(sl_2)$$

$K \mapsto K$, $E \mapsto E$, $F \mapsto F$

<u>Thm</u>. The Hopf algebra $\mathring{U}_q(sl_2)$ is quasitriangular with

$$R = \sum_{n,m \in \mathbb{Z}} q^{\frac{nm}{2}} x^n P_m \otimes x^m P_n \sum_{\ell \geq 0} \frac{(q-q^{-1})^\ell}{(\ell)_q!} E^\ell \otimes F^\ell$$

<u>Proof.</u> Denote

$$R_0 = \sum_{n,m \in \mathbb{Z}} q^{\frac{nm}{2}} x^n P_m \otimes x^m P_n$$

$$R_1 = \sum_{n \geq 0} \frac{(q-q^{-1})^n}{(n)_q!} E^n \otimes F^n$$

It is easy to verify that

$$(\Delta\otimes\mathrm{id})(R_0) = (R_0)_{13}(R_0)_{12}$$
$$(\mathrm{id}\otimes\Delta)(R_0) = (R_0)_{13}(R_0)_{23}$$

and that

$$R_0(E\otimes 1)R_0^{-1} = E\otimes K, \quad R_0(1\otimes E)R_0^{-1} = K\otimes E,$$
$$R_0(F\otimes 1)R_0^{-1} = F\otimes K^{-1}, \quad R_0(1\otimes F)R_0^{-1} = K^{-1}\otimes F,$$
$$R_0(K\otimes 1)R_0^{-1} = K\otimes 1, \quad R_0(1\otimes K)R_0^{-1} = 1\otimes K,$$

Using this and the identities for the R-matrix in $U_h sl_2$ we conclude:

(i) $\quad R\,\Delta(a)\,R^{-1} = \Delta^{op}(a)$

(ii) $\quad (\Delta\otimes\mathrm{id})(R_1) = (R_0)_{12}^{-1}(R_1)_{13}(R_0)_{12}(R_1)_{12}$

$\quad\quad (\mathrm{id}\otimes\Delta)(R_1) = (R_0)_{23}^{-1}(R_1)_{13}(R_0)_{23}(R_1)_{23}$

This proves the quasitriangularity $\blacksquare$

(iv) <u>Thm</u> The category of finite-dim. modules over $\mathring{U}_q(sl_2)/\langle x^2-1\rangle$ is naturally equivalent to $\underline{U_q(sl_2)\text{-mod}}$

<u>Proof</u>. Both categories are semisimple. There is a bijection between irreducible modules.

<u>Corollary</u>. $\underline{U_q(sl_2)\text{-mod}}$ is a braided category.

② The center of $U_\varepsilon(sl_2)$

Let $\varepsilon^\ell = 1$, $\ell = $ odd, and $U_\varepsilon(sl_2)$ be the specialization of $U_q(sl_2)$ to $q = \varepsilon$.

He we will use the identify

$$E^n F^m = F^m E^n + \sum_{t \geq 1} \frac{[m]!}{[m-t]!} F^{m-t} \cdot \prod_{s=1}^{t} \left(\frac{q^{2t-n-m-s+1} K - q^{-2t+n+m+s-1} K^{-1}}{q - q^{-1}} \right) E^{n-t} \qquad (*)$$

(a) $m = \ell$, $n = 1$,

$$E F^\ell - F^\ell E = \frac{q^\ell - q^{-\ell}}{q - q^{-1}} F^{\ell-1} \frac{q^{-\ell+1} K - q^{\ell-1} K^{-1}}{q - q^{-1}}$$

Thus in $U_\varepsilon(sl_2)$:

$$E F^\ell - F^\ell E = 0$$

(b) similarly $\quad K^\ell E = E K^\ell$, $K^\ell F = F K^\ell$

and $E^{\ell} F = F E^{\ell}$

(c) The identity (*) also implies that

$$E^{\ell} F^{\ell} = F^{\ell} E^{\ell}$$

Thus, we proved.

Theorem. The subalgebra Z_0 generated by $K^{\pm \ell}, E^{\ell}, F^{\ell}$ is central.

But we know also that

$$c = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2},$$

is central in $U_q(sl_2)$ and therefore its image in $U_\varepsilon(sl_2)$ is also central.

Thm. $Z(U_\varepsilon(sl_2))$ is generated by Z_0

and c with the polynomial relation

$$P_\ell(c) = 0$$

where coefficients of $P_\ell(c)$ are in Z_0. The exact form of the polynomial is given below.

<u>Proof.</u> PBW theorem implies that monomials

$$E^n K^m F^k$$

form a linear basis in $U_q(sl_2)$ ....