Lecture 5.

Note Title 3/1/2008

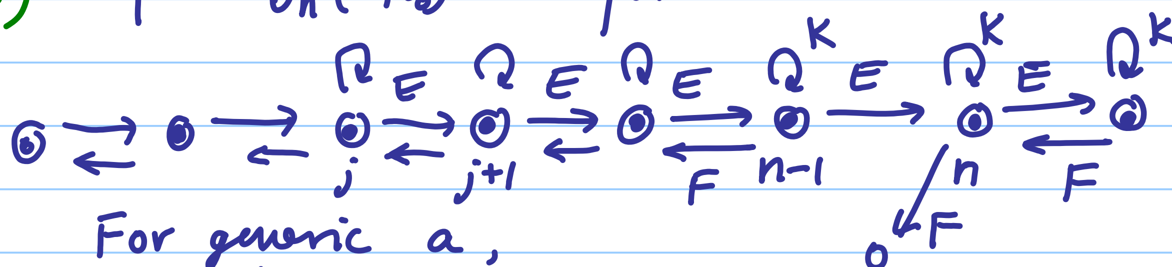
① We constructed infinite-dim, $\mathbb{Z}$-graded representations $V(a,b) = \bigoplus_j \mathbb{C} v_j$

$$K v_j = a q^{2j} v_j ,$$

$$E v_j = v_{j+1} , \quad F v_j = b_j(a, b_0)\, v_{j-1} ,$$

(a) $V(a,b)$ is irreducible for generic a,b

(b) if $b_n(a,b_0) = 0$ for some $n \in \mathbb{Z}$:



For generic a,
the subspace $V_n(a) \subset V(a,b)$ is an irreducible lowest weight module

$$V_n(a) = \bigoplus_{j \geq n} \mathbb{C} v_j , \qquad F v_n = 0$$

$$K v_n = a q^{2n} v_n$$

$$b_n = 0 \Rightarrow \frac{(a q^{2n} - a^{-1} q^{2})(q^{2n} - 1)}{(q - q^{-1})(q^2 - 1)} + b_0 = 0$$

From here

$$b_j = \frac{(a q^{2n} - a^{-1} q^{2-2j})(1 - q^{2j-2n})}{(q - q^{-1})(q^2 - 1)}$$

Thus, for generic a and b_0 as above:

$$0 \to V_n(a) \to V(a, b_0) \longrightarrow \underbrace{V(a,b_0)/V_n(a)}_{} \to 0$$

The quotient representation $U_n(a)$ and the subrepresentation $V_n(a)$ are irreducible.

(c) The coefficient can vanish for $j = m \in \mathbb{Z}$

if $\quad a q^{2n} - a^{-1} q^{2-2m} = 0$

i.e. $\quad a = \varepsilon\, q^{1-n-m},$

- If $m > n$ this means $V_n(a)$ becomes reducible with the irreducible subrepresentation

$$W^{+}_{m-n}(\varepsilon) = \bigoplus_{j \geq m} \mathbb{C} v_j \,, \quad v_j = v^{+}_{j-n}$$

with $\quad K v_j^+ = \varepsilon q^{1-m+n+2j} v_j^+$

$$E v_j^+ = v_{j+1}^+$$

$$F v_j^+ = \varepsilon q^{1+m-n} \frac{(q^{2n-2m} - q^{-2j})(1-q^{2j})}{(q-q^{-1})(q^2-1)} v_j^+$$

$j \geq m-n \geq 0$

The quotient representation is finite-dimensional

$$V_{m-n}^{(\varepsilon)} = \bigoplus_{j=n}^{m-n-1} \mathbb{C} v_j$$

2. For generic q the category of finite-dimensional modules $\underline{U_q(sl_2)\text{-mod}}$ is semisimple.

- The subcategory $\underline{U_q(sl_2)-\text{mod}}'$ of direct sums of modules $V_n(+)$

is braided with the braiding induced by the braiding for $U_h sl_2$-modules

- One can show that the whole category $\underline{U_q(sl_2)\text{-mod}}$ (including representations $V_n(-)$ and their $\oplus$) is also braided.

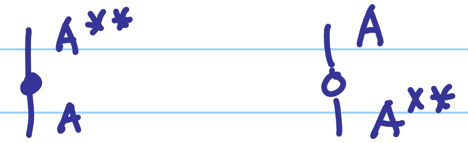
③ The category of balanced decorated diagrams $\widehat{\mathcal{D}}(\mathcal{C})$

Here $\mathcal{C}$ is a balanced category.

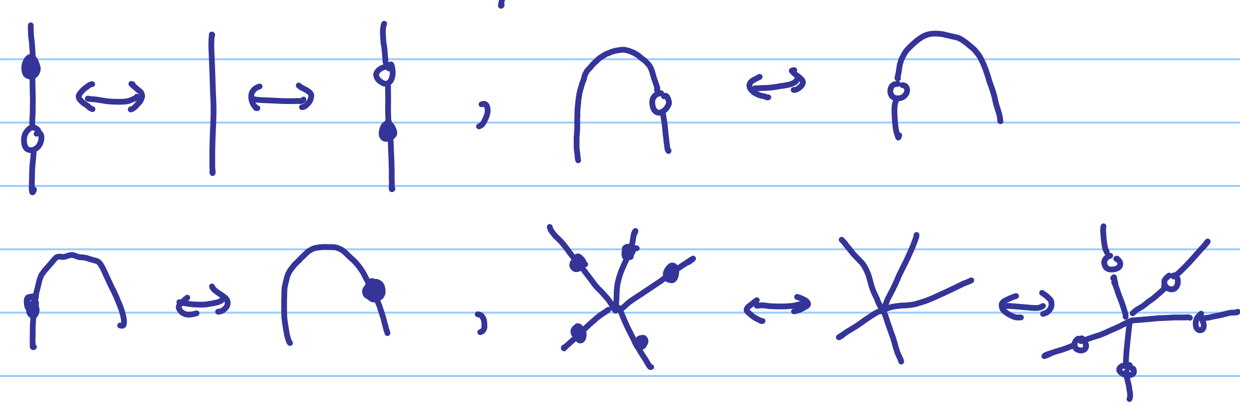
The category $\widehat{\mathcal{D}}(\mathcal{C})$:

Objects: $(\varepsilon_1, A_1) \dots (\varepsilon_n, A_n)$, $\varepsilon_i = \pm$, $A_i \in Ob(\mathcal{C})$

Morphisms: oriented diagrams decorated by objects of $\mathcal{C}$ (as before) with additional vertices



taken modulo equivalences



Just as the category $\tilde{\mathcal{D}}(\mathcal{C})$ it is a braided monoidal category and the following is clear:

<u>Thm</u>. (i) The functor $\tilde F : \tilde{\mathcal{D}}(\mathcal{C}) \to \mathcal{C}$ extends to the functor $\hat F : \hat{\mathcal{D}}(\mathcal{C}) \to \mathcal{C}$ with

$$\hat F(\mathcal{D}) = \tilde F(\mathcal{D}) \quad \text{if } \mathcal{D} \text{ has no new vertices}$$

$$\tilde F\left(\bullet\, {}^{A^{**}}_{A} \right) = \mathscr{b}_A : A \to A^{**}, \quad \mathscr{b}_A = u_A\, \theta_A^{-1}$$

$$\tilde F\left(\circ\, {}^{A}_{A^{**}} \right) = \mathscr{b}_A^{-1} : A^{**} \to A,$$

(ii) The diagram

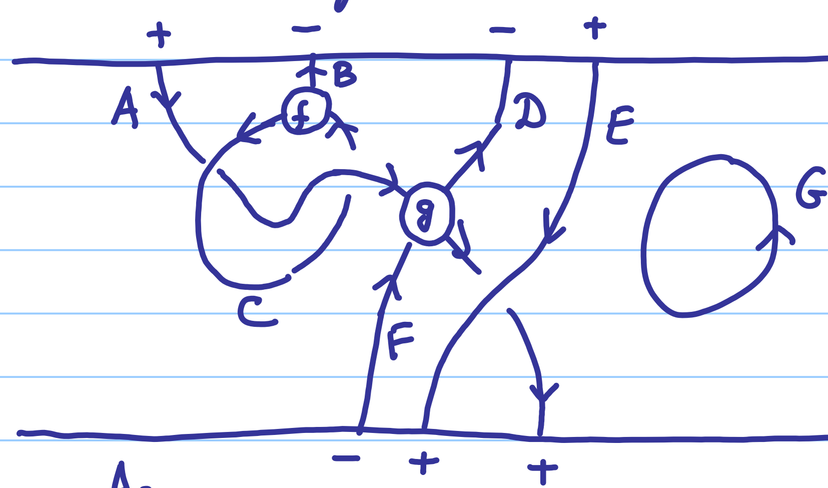
$$\begin{tikzcd} \tilde{\mathcal{D}}(\mathcal{C}) \arrow[rr, hook] \arrow[dr, "\tilde F"'] & & \hat{\mathcal{D}}(\mathcal{C}) \arrow[dl, "\hat F"] \\ & \mathcal{C} & \end{tikzcd}$$

commutes. Here $\tilde{\mathcal{D}}(\mathcal{C}) \hookrightarrow \hat{\mathcal{D}}(\mathcal{C})$ is the natural embedding.

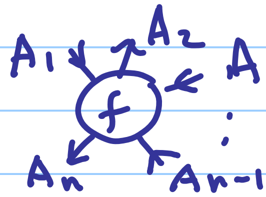
1 The category of colored diagrams $\mathcal{D}(\mathcal{C})$

Objects: $(A_1, \varepsilon_1) \cdots (A_n, \varepsilon_n)$, $A_i \in Ob(\mathcal{C})$, $\varepsilon_i = \pm$

Morphisms: $Mor((A,\sigma),(B,\varepsilon))$ are oriented diagrams with edges colored by objects of $\mathcal{C}$, inner vertices colored by morphisms of $\mathcal{C}$ such that coloring of edges and their orientation agree with objects at the boundary.



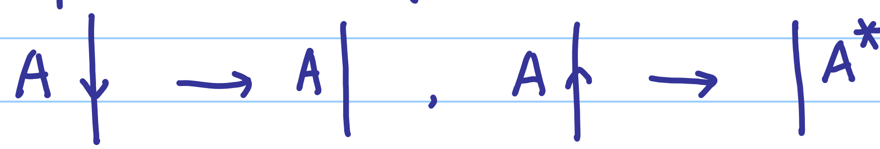
Recall that for each vertex a total order on adjacent edges is fixed



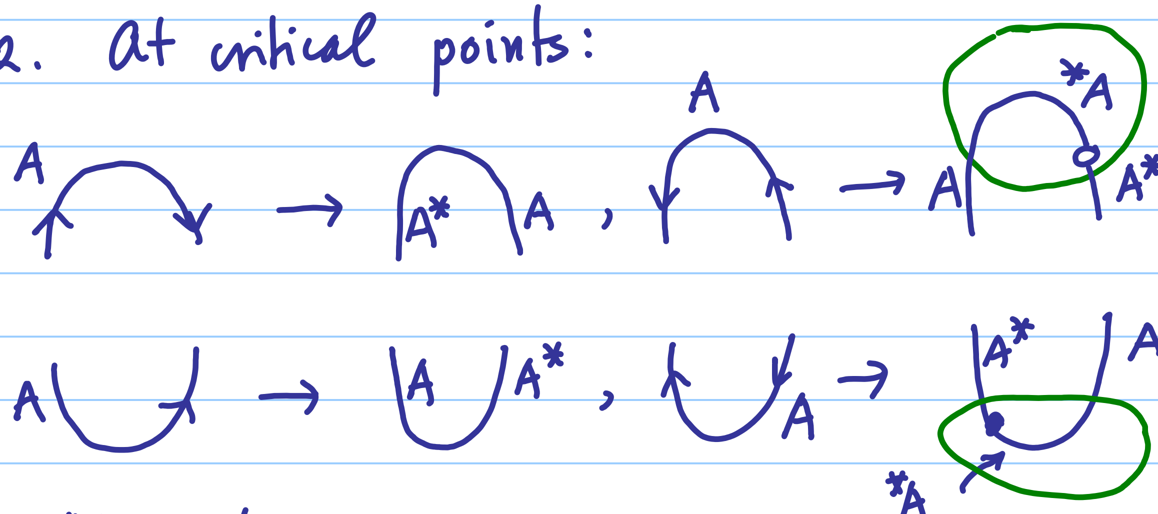
, $f : A_1 \otimes A_2^* \otimes A_3 \otimes \cdots \otimes A_{n-1} \otimes A_n^* \to \mathbb{1}$

Now let us describe the embedding of colored diagrams into the category of balanced decorated diagrams.

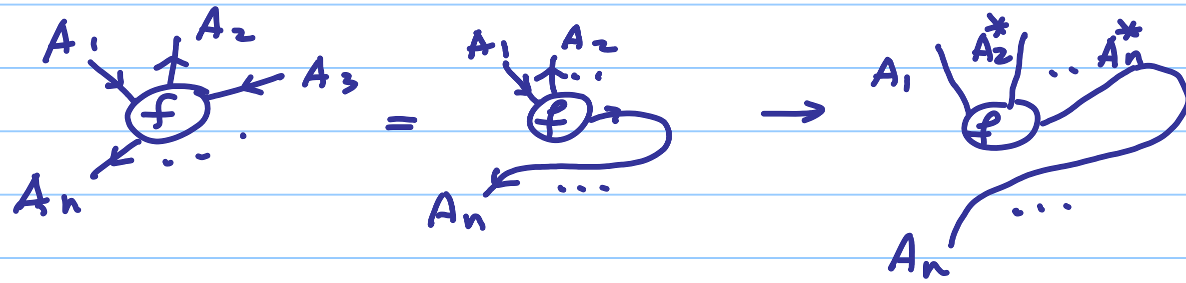
1. Replace a coloring by a decoration as:



2. At critical points:



3. At vertices:

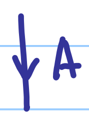


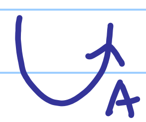
Denote the restriction of $\hat{F} : \hat{\mathcal{D}}(\mathcal{C}) \to \mathcal{C}$ to $\mathcal{D}(\mathcal{C}) \hookrightarrow \hat{\mathcal{D}}(\mathcal{C})$ by $F : \mathcal{D}(\mathcal{C}) \to \mathcal{C}$.

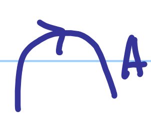
It acts as:

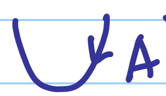
$$F((A_1,\varepsilon_1)\dots(A_n,\varepsilon_n)) = A_1^{\varepsilon_1} \otimes \cdots \otimes A_n^{\varepsilon_n},$$

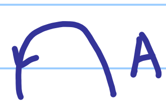
$$A^+ = A, \quad A^- = A^*$$

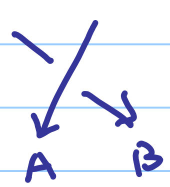
F() $= id_A$

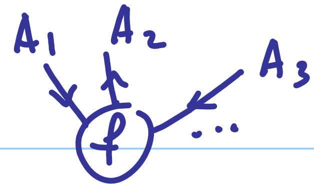
F() $= i_A : \mathbb{1} \to A \otimes A^*$,

F() $= e_A : A^* \otimes A \to \mathbb{1}$,

F() $= (b_{*A} \otimes id_A)\, i_{*A} : \mathbb{1} \to A^* \otimes A$

F() $= e_{*A} \circ (id_A \otimes b_{*A}^{-1}) : A \otimes A^* \to \mathbb{1}$

F() $= c_{AB} : A \otimes B \to B \otimes A$

F() $= f : \mathbb{1} \to A_1 \otimes A_2^* \otimes A_3 \otimes \cdots$

5 Invariants of $\mathcal{C}$-colored tangles

Def. A $\mathcal{C}$-coloring of a framed tangle is the assignment:

- objects of $\mathcal{C}$ to edges
- morphisms of $\mathcal{C}$ to vertices

If  and $e_i \mapsto A_i$, then

$$v \mapsto f_v : \mathbb{1} \to A_1 \otimes A_2^* \otimes \cdots \otimes A_n$$

Proposition. Any $\mathcal{C}$-colored tangle with a black-board framing can be projected to a $\mathcal{C}$-colored diagram, and any $\mathcal{C}$-colored diagram lifts to a $\mathcal{C}$-colored framed tangle with black-board framing.

Theorem. $\mathcal{C}$-colored $\underbrace{\text{tangles}}^{\text{(framed)}}$ are in bijection with framed Reidemeister classes of $\mathcal{C}$-colored diagrams.

Theorem. The functor $F : \mathcal{D}(\mathcal{C}) \to \mathcal{C}$ is constant on framed R. equivalence classes

Corollary. F depends only on the tangle which is represented by a diagram and $\Rightarrow$ is an invariant of $\mathcal{C}$-colored tangles.