Lecture 2

Note Title 2/11/2008

1 The double of a Hopf algebra (Drinfeld double):

Let H, H^*, $\langle \cdot , \cdot \rangle$ be a dual pair of Hopf algebras.

Denote by H° the Hopf algebra structure on H^* with the same multiplication as on H^*, but with the opposite comultiplication:

$$\Delta_{H^\circ}(x) = \sigma \circ \Delta_{H^*}(x)$$

where $\sigma(a \otimes b) = b \otimes a$.

Define the following bilinear operation on $D(H, H^*) = H \otimes H^*$

$$(a \otimes \ell)(b \otimes \ell) = \sum_{b,\ell} a b_{(2)} \otimes \ell_{(2)} m \,.$$

$$\cdot \langle b_{(1)}, S^{-1}(\ell_{(1)}) \rangle \langle b_{(3)}, \ell_{(3)} \rangle$$

He we used notations

$$\Delta_H^{(3)}(b) = \sum_b b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$$

$$\Delta_{H^\circ}^{(3)}(\ell) = \sum_\ell \ell_{(1)} \otimes \ell_{(2)} \otimes \ell_{(3)}$$

<u>Thm</u>. (1) This operation defines an associative algebra structure on $\mathcal{D}(H, H^*)$.

(2) This algebra is a Hopf algebra with

$$\Delta(a \otimes 1) = \sum_a (a_{(1)} \otimes 1) \otimes (a_{(2)} \otimes 1)$$

$$\Delta(1 \otimes \ell) = \sum_\ell (1 \otimes \ell_{(1)}) \otimes (1 \otimes \ell_{(2)})$$

(3) This Hopf algebra is quasitriangular with $R = \sum_i (e_i \otimes 1) \otimes (1 \otimes e^i)$

Example ① $U_h b_+ = \langle H, E \mid [H,E] = 2E \rangle / \mathbb{C}[[h]]$

$\Delta H = H \otimes 1 + 1 \otimes H$,

$\Delta E = E \otimes \exp\left(\frac{hH}{2}\right) + 1 \otimes E$

Consider the algebra

$U_h b_+^{\circ} = \langle H^{\vee}, F \mid [H^{\vee}, F] = -\frac{h}{2} F \rangle / \mathbb{C}[[h]]$

It is a topological Hopf algebra with the comultiplication

$\Delta H^{\vee} = H^{\vee} \otimes 1 + 1 \otimes H^{\vee}$

$\Delta F = F \otimes 1 + \exp(-2H^{\vee}) \otimes F$

The algebra $U_h b_+^{\circ}$ is filtered by the degree of elements

$\deg(H^{\vee}) \leq 1$, $\deg(F) \leq 1$,

$\deg(ab) \leq \deg(a)\deg(b)$

The exponent in the coproduct is

a formal power series convergent in the -adic topology corresponding to this filtration.

<u>Proposition</u> (1) The exists unique Hopf pairing

$$\langle \cdot, \cdot \rangle : U_h b_+ \otimes U_h b_+^{\circ} \longrightarrow \mathbb{C}[[h]]$$

with

$$\langle H, H^\vee \rangle = 1, \quad \langle H, F \rangle = 0$$

$$\langle E, H^\vee \rangle = 0, \quad \langle E, F \rangle = 1$$

(2) Monomials

$$\frac{1}{n!\,(m)!} H^{\vee n} F^{m}, \quad (m)! = \frac{sh(\frac{hm}{2})\, sh(\frac{h(m-1)}{2}) \dots sh(\frac{h}{2})}{sh(\frac{h}{2})\, sh(\frac{h}{2}) \dots sh(\frac{h}{2})}$$

form a basis in $U_h b_+^*$ dual to $H^n E^m$ in $U_h b_+$

The following proposition describes the double $\mathcal{D}(U_h b_+, U_h b_+^*)$ in terms of generators and relations.

Proposition. The double $\mathcal{D}(U_h b_+, U_h b_+^*)$ is generated by $H, E, H^\vee, F$ with defining relations

$$[H, H^\vee] = 0, \quad [H, E] = 2E, \quad [H, F] = -2F$$

$$[H^\vee, E] = \frac{h}{2}E, \quad [H^\vee, F] = -\frac{h}{2}F$$

$$[E, F] = e^{\frac{hH}{2}} - e^{-2H^\vee},$$

According to general properties of $\mathcal{D}(H, H^*)$, it is a Hopf algebra with

$$\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta H^\vee = H^\vee \otimes 1 + 1 \otimes H^\vee$$

$$\Delta E = E \otimes e^{\frac{hH}{2}} + 1 \otimes E, \quad \Delta F = F \otimes 1 + e^{-2H^\vee} \otimes F$$

It is also quasitriangular with

$$R = \exp(H \otimes H^\vee) \sum_{m \geq 0} \frac{1}{(m)!} E^m \otimes F^m$$

(2) Notice that in $\mathcal{D}(U_h b_+, U_h b_+^*)$ elements $H - \frac{h}{4} H^\vee$ generate a Hopf ideal (ideal in the algebra and $\Delta I \subset I \otimes H + H \otimes I$)

$$U_h sl_2 = \mathcal{D}(U_h b_+, U_h b_+^*)/I$$

Since the quotient over I preserves quasitriangularity, $U_h sl_2$ is quasitriangular with

$$R = \exp\left(\frac{h}{4} H \otimes H\right) \sum_{m \geq 0} \frac{1}{(m)!} E^m \otimes F^m$$

The algebra $U_h sl_2$ is generated by H, E, F with defining relations

$$[H, E] = 2E, \quad [H, F] = -2F$$

$$[E, F] = \frac{sh\left(\frac{h}{2} H\right)}{sh\left(\frac{h}{2}\right)}$$

with the co-multiplication

$$\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta E = E \otimes e^{\frac{hH}{2}} + 1 \otimes E$$

$$\Delta F = F \otimes 1 + e^{-\frac{hH}{2}} \otimes F.$$

(3) We can take a different topological version of $\mathcal{D}(U_h b_+, U_h b_+^*)$.

(a) Define $C_h(B_-)$ as the Hopf algebra generated by invertible L and by F with defining relations

$$L F = e^{-h} F L$$

and with the co-product

$$\Delta L = L \otimes L, \quad \Delta F = F \otimes 1 + L^{-1} \otimes F$$

It is not difficult to see that

- $C_h(B_+) \to U_h b_+^{\circ}$

$$L \mapsto \exp(2H^{\vee}), \quad F \mapsto F$$

is a homomorphism of algebras

- the pairing defined on generators by

$$\langle H, F \rangle = 0 \,, \quad \langle H, L \rangle = 2$$

$$\langle E, F \rangle = 1 \,, \quad \langle E, L \rangle = 0$$

extends uniquely to a Hopf pairing $U_h b_+ \otimes C_h(B_-) \to \mathbb{C}[[h]]$

(b) - The algebra $U_h b_+$ is a formal Hopf algebra deformation of $U b_+$

- The algebra $C_h(B_-)$ is a formal deformation of the commutative Hopf algebra with generators $L^{\pm 1}, F$ and with the comultiplication

$$\Delta L = L \otimes L \,, \quad \Delta F = F \otimes 1 + L^{-1} \otimes F$$

This Hopf algebra can be naturally identified with polynomial functions on the algebraic group of lower triangular matrices

(*) $\begin{pmatrix} 1 & 0 \\ F & L^{-1} \end{pmatrix}$ with coordinate functions L, F

- The algebra $U_h\mathfrak{b}_+^{\circ}$ at $h=0$ is the algebra of functons on the formal neighborhood of the identity on the group (*) with coordinate functions $H^\vee = \frac{1}{2}\log L$ and F

We will denote this algebra $F(B_-)$

(c) Thus, at $h=0$ we have $U\mathfrak{b}_+$ and to different topological vertions of the dual Hopf algebras, both of them are functions on lower triangular matrices.

(d) The double $\mathcal{D}(U_h b_+, U_h b_+^*)$ at $h=0$ becomes $\mathcal{D}(Ub_+, F(B_-))$. It is generated by $H, H^\vee, E, F$ with defining relations

$[H, H^\vee]=0$, $[H,E]=2E$, $[H,F]=-2F$

$[H^\vee, E]=0$, $[H^\vee, F]=0$, $[E,F]=1-e^{-2H^\vee}$

It is a topological Hopf algebra, complete in the topology defined by $\deg H^\vee \le 1$ $\deg F \le 1$ with

$$R = \exp(H \otimes H^\vee)\exp(E \otimes F)$$

4 Ribbon Hopf algebras

Def. A quasitriangular Hopf algebra (H, R) is a ribbon Hopf algebra if $\exists\, \tau \in H$, s.t.

- $\varepsilon(\tau) = 1$
- $S(\tau) = \tau$

$$\cdot \quad \Delta(\tau) = (\tau \otimes \tau)\sigma(R)R$$

where $\sigma(a \otimes b) = b \otimes a$.

<u>Example</u>. $U_h sl_2$ is a ribbon Hopf algebra with

$$\tau = e^{-\frac{hH}{2}} \sum_i S(\beta_i)\alpha_i$$

We will see how to prove this statement later, when we will discuss invariants of tangles.

<u>Claim</u> The category of f. dim modules over a ribbon Hopf algebra is a ribbon category with $\theta_V : (\pi, V) \circlearrowleft$ given by

$$\theta_V = \pi_V(\tau)$$