Lecture 11.

Note Title 4/27/2008

The description of $\mathcal{C}_\varepsilon(sl_2)$ in terms of q-6j symbols

Let V_i , $i = 0, 1 \dots, \ell-2$ be simple $\mathring{U}_\varepsilon(sl_2)$-modules with non-zero quantum dimensions. Their tensor product decomposes as

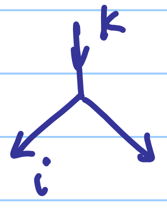
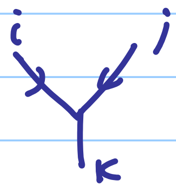
$$V_i \otimes V_j \cong V_{|i-j|} \oplus \cdots \oplus V_{\min(i+j, 2\ell-4-j-i)} \oplus Z_{ij}$$

where Z_{ij} is a vanishing module (see lecture 10).

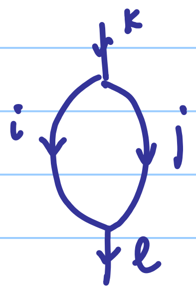
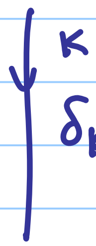
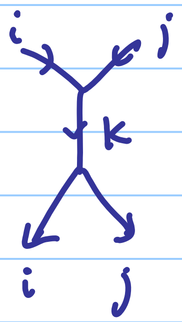
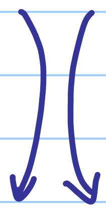
Fix $\mathring{U}_\varepsilon(sl_2)$ - linear maps

$$\alpha^{ij}_K : V_i \otimes V_j \to V_K \, , \quad \beta^{ij}_K : V_K \to V_i \otimes V_j$$

These morphsims will correspond to 3-valent vertices:

$\alpha^{ij}_k \longleftrightarrow$  , $\beta^{ij}_k \longleftrightarrow$ 

Let us choose them such that

 $=$  $\delta_{k,l}$ and $\sum_k$  $=$ 

The las identity means that for any $f : V_i \otimes V_j \circlearrowleft$

$$\sum_k \mathrm{tr}_{V_i\otimes V_j}\left(f\, \beta^{ij}_k\, \alpha^{ij}_k\right) = \mathrm{tr}_{V_i\otimes V_j}(f)$$

We can use α's and β' to define

$$V_i \overline{\otimes} V_j = V_{i-j} \oplus \cdots \oplus V_{\min(i+j,\, 2l-4-i-j)}$$

Here

For each object U in $\mathcal{C}_\varepsilon(sl_2)$ we have:

$$\mathrm{Hom}\,(U, V_i \bar{\otimes} V_j) = \bigoplus_{k=|i-j|}^{\min(i+j,\,2l-4-i-j)} \mathrm{Hom}\,(U, V_k),$$

$$\mathrm{Hom}(V_i \bar{\otimes} V_j, U) = \bigoplus_{k=|i-j|}^{\min(i+j,\,2l-4-i-j)} \mathrm{Hom}(V_k, U),$$

For triple tensor products we have

$$\mathrm{Hom}(V_s, V_i \bar{\otimes} (V_j \bar{\otimes} V_k)) = \bigoplus_t \mathrm{Hom}(V_s, V_i \bar{\otimes} V_t) \otimes$$
$$\otimes \mathrm{Hom}(V_t, V_j \bar{\otimes} V_k)$$

and

$$\mathrm{Hom}(V_s, (V_i \bar{\otimes} V_j) \bar{\otimes} V_k) = \bigoplus_t \mathrm{Hom}(V_s, V_t \bar{\otimes} V_k) \otimes$$
$$\mathrm{Hom}(V_t, V_i \bar{\otimes} V_j)$$

Since we fixed $\beta_k^{ij} \in \mathrm{Hom}(V_k, V_i \bar{\otimes} V_j)$

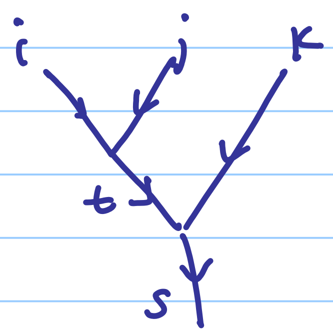
$$\mathrm{Hom}(V_s, (V_i \bar{\otimes} V_j) \bar{\otimes} V_k)$$

and $\mathrm{Hom}(V_s, V_i \bar{\otimes} (V_j \bar{\otimes} V_k))$

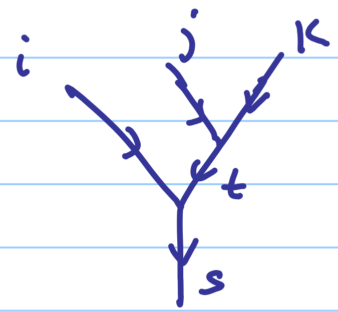
can be identified. We will write

$$\mathrm{Hom}(V_s, V_i \bar{\otimes} V_j \bar{\otimes} V_k) \text{ for them.}$$

But since we fixed α's and β's we have two natural bases:



and



Ex. Check that

$$\#\{t \mid |i-j| \le t \le \min(i+j, 2\ell-4-i-j),\ |t-k| \le s \le \min(k+t, 2\ell-4-k-t)\}$$
$$= \#\{t \mid |i-k| \le t \le \min(i+k, 2\ell-4-i-k),\ |i-t| \le s \le \min(i+t, 2\ell-4-i-t)\}$$

The associativity constraint

$$a_{ijk} : (V_i \bar{\otimes} V_j) \bar{\otimes} V_k \to V_i \bar{\otimes} (V_j \bar{\otimes} V_k)$$

acts on the spaces

$$\mathrm{Hom}(V_s, V_i \bar{\otimes} V_j \bar{\otimes} V_k)$$

by mapping one basis to the other: