1. (3 points) Find the Legendre transform of the function

$$f(x) = |x|^\alpha \text{ if } |x| < 1$$
$$f(x) = |x|^\beta \text{ if } |x| \geq 1$$

Here $\alpha, \beta > 1$.

Recall that the Legendre transform of f is

$$f^\vee(y) = \max_x (xy - f(x))$$

2(3 points) Let H_n be the space of Hermitian $n \times n$ matrices.

Compute the integral

$$Z_n = \int_{H_n} \exp\left(-N\frac{1}{2}tr(M^2)\right) dM$$

where dM is the Euclidean measure on H_n.

Define

$$< f >= \frac{1}{Z_n} \int_{H_n} \exp\left(-N\frac{1}{2}tr(M^2)\right) f(M) dM$$

Prove the formula

$$< M_{ij} M_{st} >= \frac{\delta_{it}\delta_{js}}{N}$$

4 (3 points) prove the formula

$$< M_{i_1 j_1} \dots M_{i_k, j_k} >= \sum_m \prod_{(a,b)\in m} < M_{i_a, j_a} M_{i_b, j_b} >$$

Here the sum is taken over all perfect matchings m on the set $1, 2, \dots, k$ and the product is taken over the set of pairs (a, b) in m. The formula holds for k even. Otherwise the left side is zero.

For example

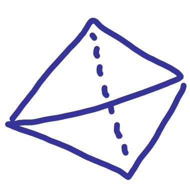
$$< M_{i_1,j_1} M_{i_2,j_2} M_{i_3,j_3} M_{i_4,j_4} >=< M_{i_1,j_1} M_{i_2,j_2} >< M_{i_3,j_3} M_{i_4,j_4} > +$$
$$< M_{i_1,j_1} M_{i_3,j_3} >< M_{i_2,j_2} M_{i_4,j_4} > + < M_{i_1,j_1} M_{i_4,j_4} >< M_{i_2,j_2} M_{i_3,j_3} >$$

6. (6 points) Use the previous problem to prove the formula

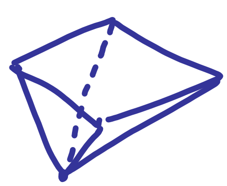
$$\frac{\left\langle \prod_{a \geq 3} \operatorname{tr}(M^a)^{n_a} \right\rangle}{\prod_{a\geq 3} n_a!\, a^{n_a}} = \sum_{\Sigma_{n_3 n_4 \cdots}} \frac{N^{\chi(\Sigma_{n_3 n_4 \cdots})}}{\operatorname{Aut}(\Sigma_{n_3 n_4 \cdots})}$$

Here $\Sigma_{n_3 n_4 \cdots}$ is a surface which is glued out n_3 triangles, n_4 rectangles, n_5 5-gons,...

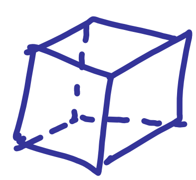
Ex.



$n_3 = 3$
$n_i = 0$
$i > 3$



$n_3 = 4$
$n_4 = 1$
$n_i = 0$
$i > 5$



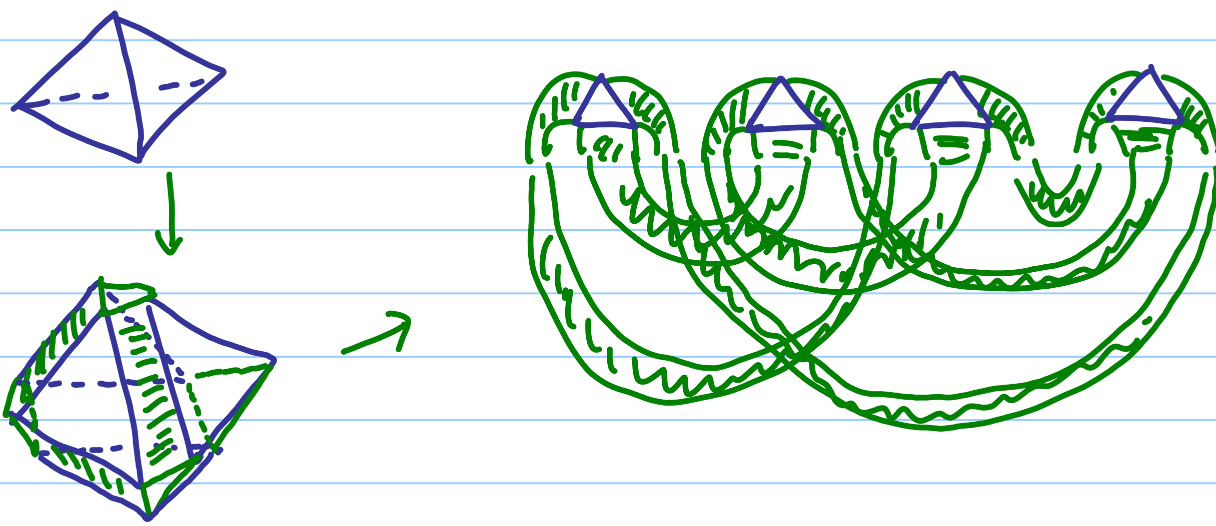
$n_4 = 6$
$n_i = 0$, $i = 3, 5, 6 \cdots$

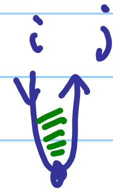
$\chi(\Sigma) = |\text{faces}| - |\text{edges}| + |\text{vertices}|$ (Euler character. of Σ)

$\text{Aut}(\Sigma)$ = the number of automorphisms of the cell approximation.

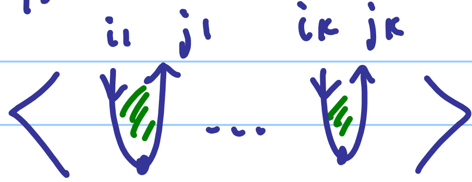
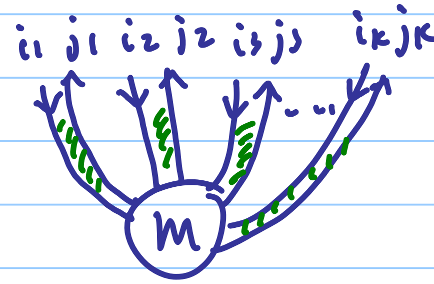
Hint

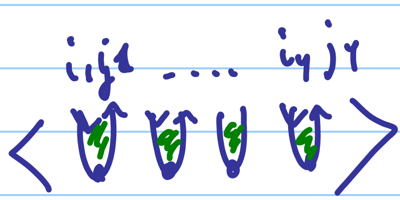
1. Represent the cell complex $\Sigma_{n_3 n_4 \dots}$ as a fat-graph:

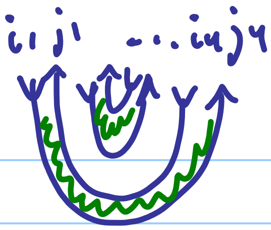


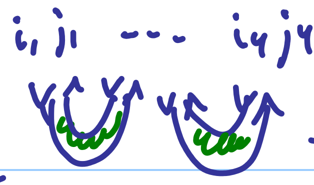
2. Use graphical notation: $M_{ij} \rightsquigarrow$ 

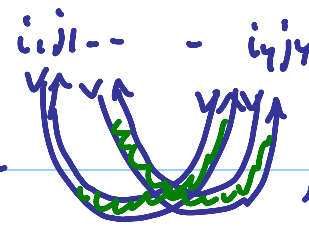
Then

 $= \sum_{\text{perfect matings}} \frac{1}{N^K}$ 

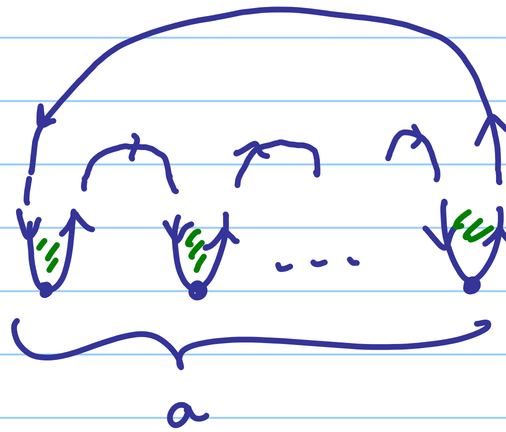
Ex:  =

$\frac{1}{N^2}$ 

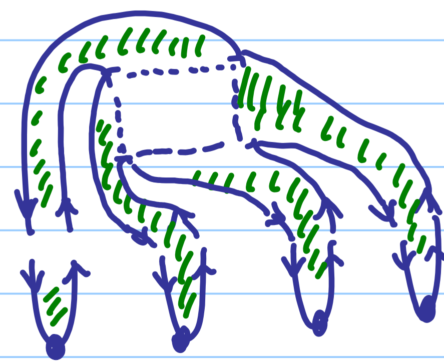
$+ \frac{1}{N^2}$ 

$+ \frac{1}{N^2}$ 

,

3. $\mathrm{tr}(M^a) \rightsquigarrow$ 

Ex.

$\mathrm{tr}(M^4) \rightsquigarrow$ 

7. (to complete the exercise on matrix models in quantum field theory, and just for fun) Prove the following identity between formal power series in t_i:

$$\left\langle e^{N \sum_{a \geq 3} \frac{t_a}{a!} \operatorname{tr}(M^a)} \right\rangle =$$

$$= \sum_{n_3 n_4 \cdots \geq 0} t_3^{n_3} t_4^{n_4} \cdots \sum_{\Sigma_{n_3 n_4 \cdots}} N^{\chi(\Sigma_{n_3 n_4 \cdots})} \frac{1}{\operatorname{Aut}(\Sigma_{n_3 n_4 \cdots})}$$