0.1 Connections in principal fiber bundles

0.1.1 Connections in fiber bundles

Let $\pi : P \to M$ be a fiber bundle. It induces $d\pi : TP \to TM$. The kernel of this map is the sub-bundle $ker(d\pi) \subset TP$ with the fiber $ker(d\pi_p)$ over p. We have the exact sequence

$$0 \to ker(d\pi_p) \hookrightarrow T_pP \to T_{\pi(p)}M \to 0$$

Definition 0.1.1. A connection in P is a subbundle in TP which projects isomorphically to TM. ⋄

In other words a connection in P is a choice of the subspace $(T_pP)_A \subset T_pP$ for each $p \in P$ which depends smoothly on p and for each p:

$$T_pP = ker(d\pi) \oplus (T_pP)_A$$

and $(T_pP)_A \simeq T_{\pi(p)}M$.

Equivalently, we can say a connection is a mapping $\hat{A} : TP \to ker(d\pi)$, such that $i \circ \hat{A} = id$. It is clear that such mapping defines the decomposition $T_p(P) = ker(d\pi_p) \oplus ker(\hat{A}_p)$ and that $ker(\hat{A}_p) \simeq T_{\pi(p)}M$

Note that $ker(d\pi_p)$ is isomorphic to the tangent space to the fiber :

$$ker(d\pi_p) \simeq T_p(P_{\pi(p)})$$

0.1.2 The parallel transport

A connection $\hat{A} : TM \to TP$ lifts a vector field on M to a vector field on P:

$$M \to TM \to (TP)_A \subset TP$$

Let $\gamma_{x,y}$ be a smooth path connecting x and y. We can lift tangent vectors to this path to P. Fixing $p \in \pi^{-1}(x)$ defines a lift $\tilde{\gamma}$ of the path γ to the total space P. This lifting can be described as follows.

Parameterize $\gamma = \{\gamma(t)\}_{t\in[0,1]}$. Then the lifting is the solution to the differential equation

$$\frac{d\tilde{\gamma}(t)}{dt} = \alpha_{\tilde{\gamma}(t)}(\frac{d\gamma(t)}{dt})$$

with the initial condition $\tilde{\gamma}(0) = p$. Here $\alpha_p : T_{\pi(p)}M \to T_pP$ is the lift of the tangent vector by the connection $\hat{A}$.

This path ends at a point $p' \in \pi^{-1}(y)$. Thus, a connection on P gives an identification of fibers $\pi^{-1}(x)$ and $\pi^{-1}(y)$ which is known as the *parallel transport*.

0.1.3 Connections on principal G-bundles

Let G be a Lie group and $P \to M$ a principal G-bundle i.e. G acts simply transitively on the fibers of the bundle. Note that since the action is simply transitive, $M = P/G$.

Definition 0.1.2. A *connection* on a principal G-bundle $P \to M$ is a G-invariant distribution on P that projects isomorphically to TM. ⋄

Again we can think of a connection as the mapping $\hat{A} : TP \to ker(d\pi)$ such that $i \circ \hat{A} = \text{id}$. So, just as in a fiber bundle we have the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(d\pi) \arrow[r, shift left, "i"] & TP \arrow[l, shift left, "\hat{A}"] \arrow[r, "d\pi"] & TM \arrow[r] & 0 \end{tikzcd}$$

but in addition, the mapping $\hat{A}$ should be G-invariant.

The left translation on G by $g \in G$ is the mapping $l_g : G \to G$, $h \mapsto gh$. It induces the mapping of tangent spaces $dl_g : T_hG \to T_{gh}G$. This gives the trivialization of $TG \simeq G \times T_eG$ with $dl_{g^{-1}}$:

$T_gG \to T_eG$. The tangent space T_eG is the Lie algebra of G, so we have a trivialization $TG \simeq G \times \mathfrak{g}$. From now on assume that the tangent bundle to a Lie group is trivialized by left translations.

As we will see, a connection can be viewed as an element $A \in \Omega^1(P, \mathfrak{g})^G$, where G acts on $\mathfrak{g}$ by the adjoint action. For this we need the following lemma.

Lemma 0.1.3. $\ker(d\pi) \simeq \mathfrak{g}$ *and this isomorphism is* G*-invariant with respect to the natural action of* G *on* $ker\hat{A}$ *and the adjoint action on* $\mathfrak{g}$ *.*

Proof. Locally, the bundle P is trivial: $TP|_U \simeq TG \times TU$. Trvializing TG by left translations we have $TP|_U \simeq \mathfrak{g} \times G \times TU$. Therefore $T_pP \simeq \mathfrak{g} \times T_{\pi(p)}M$. This implies that $ker(d\pi) \simeq \mathfrak{g}$. ☐

This lemma implies that $\hat{A}$ is a G-invariant mapping $TP \to \mathfrak{g}$. But this is exactly what 1-forms do. Indeed, an element $\omega \in \Omega^1(P, \mathfrak{g})^G$ can be regarded as a G-equivariant mapping $TP \to \mathfrak{g}$, $(p, t \in T_pP) \mapsto < \omega(p), t >$ where $< ., . >$ is a natural pairing between tangent and co-tangent spaces at $p \in P$.

Thus, the space of connections on P is a subspace in $\Omega^1(G, \mathfrak{g})^G$.

0.1.4 The Maurer-Cartan from on a Lie group

Definition 0.1.4. The Maurer-Cartan form on G is $\theta \in \Omega^1(G, \mathfrak{g})^G$, i.s a G-invariant mapping $TG \to \mathfrak{g}$ defined as

$$\theta_g(t) = dr_{g^{-1}}(t) \in T_eG = \mathfrak{g}$$

Here $t \in T_gG$ and r_g is the right translation by g in G. ◇

This form is left-invariant, $dl_g(\theta) = \theta$ and

$$dr_g(\theta) = Ad_{g^{-1}}(\theta)$$

It satisfies the Maurer-Cartan equation

$$d\theta + [\theta \wedge \theta] = 0.$$

Here we use the notation which will be used a lot for $\mathfrak{g}$ valued forms. In local coordinates

$$[\theta \wedge \eta] = \sum_{\{i\},\{j\}} [\theta_{\{i\}}, \eta_{\{j\}}] dx^{\{i\}} \wedge dx^{\{j\}}$$

Let $\{x^i\}$ be local coordinates in the vicinity of the unit element in G and e_i be the corresponding basis in the tangent space T_eG, in the Lie algebra $\mathfrak{g}$. The Maurer-Cartan form in these coordinates is

$$\theta = \sum_{ijk} C^k_{ij} e_k x^i dx^j$$

where C^K_{ij} are structural constants of $\mathfrak{g}$ in the basis e_k.

For $G = GL_n$ and for other matrix groups the Maurer-Cartan form can be written as

$$\theta = \sum_{ij} e_{ij} (g^{-1})_{jk} (dg)_{ki} = g^{-1} dg$$

Each point $p \in P$ gives the identification of the fiber $P_{\pi(p)}$ with G. Indeed, since G acts simply-transitively, for each $q \in P_{\pi(p)}$ there exists unique $g \in G$ such that $p = qg$. Let

$$\tau_p : P_{\pi(p)} \simeq G$$

be such isomorphism. It induces the isomorphism of vector spaces between the tangent space to the fiber at $p \in P$ with the Lie algebra $\mathfrak{g}$: $T_p(P_{\pi(p)}) \simeq \mathfrak{g}$. The image of the Maurer-Cartan form with respect to this isomorphism gives the 1-form $\theta_p \in \Omega^1(P_p, \mathfrak{g})$ with

$$dr^*_g(\theta_p) = Ad_{g^{-1}}(\theta_p)$$

$$d\theta_p + [\theta_p \wedge \theta_p] = 0$$

0.1.5

Let $i_p : P_p \hookrightarrow P$ be the natural inclusion of the fiber containing p into the bundle.

Theorem 0.1.5. *A connection on P can be identified with 1-form $A \in \Omega^1(P, \mathfrak{g})^G$ such that*

$$i_p^*(\hat{A}) = \theta_p,$$

Notice that the G-invariance of the connection means $dr_g^*(\hat{A}) = Ad_{g^{-1}}(\hat{A})$.

Proof. We already proved that a connection on P can be identified with a G-invariant form $\hat{A} \in \Omega^1(P, \mathfrak{g})$. When the bundle is trivial, $P = M \times G$, $p = (x, g)$, the embedding i_p depends only on x and $\hat{A}(x, g) = A(x) + \theta(g)$ where $\theta(g)$ is the Maurer-Cartan form at $g \in G$. In this case $i_x(\hat{A}(x, g)) = \theta(g)$, and the theorem follows. The rest of the proof follows by choosing a local trivialization of P. □

0.1.6 The space of connections

The space of connections is not a vector space. Indeed, if $\hat{A}_1$ and $\hat{A}_2$ are two connections, their sum is not a connection:

$$i_p^*(\hat{A}_1 + \hat{A}_2) = 2\theta_p,$$

and for a connection the r.h.s should have been θ_p.

But for the difference we have

$$i_p^*(\hat{A}_1 - \hat{A}_2) = 0 \tag{0.1.6}$$

which means that the difference is the pull-back of a $\mathfrak{g}$-valued form on M:

$$\hat{A}_1 - \hat{A}_2 = \pi^*(a)$$

where $a \in \Omega^1(M, \mathfrak{g})$. This means that the space of connection is a subspace in $\Omega^1(P, \mathfrak{g})$ which is affine over $\Omega^1(M, \mathfrak{g})$. Recall the formal definition of an affine space:

Definition 0.1.7. A set S is an affine space over a vector space V if it is given together with the mapping $\alpha : S \times S \to V$, which we denote as $\alpha(a, b) = a - b$ such that

- $(a - b) + (b - c) = a - c$
- $\alpha_b : S \to V$, $\alpha_b(a) = b - a$ is a bijection.

◇

In other words, S is a set with a simply transitive action of the Abelian group V.

Proposition 0.1.8. *The space of connections $\mathcal{A}(P)$ is the subspace in $\Omega^1(P, \mathfrak{g})^G$ which satisfy (0.1.6). It is an affine space with respect to the subspace $\pi^*(\Omega^1(P, \mathfrak{g}))$.*

0.1.7 The curvature

Let $\hat{d} : \Omega^{\cdot}(P, \mathfrak{g})^P \to \Omega^{\cdot}(P, \mathfrak{g})^P$ be the differential for G-invariant, $\mathfrak{g}$-valued forms on P.

The differential twisted by the connection $\hat{A}$ acts on forms as

$$\hat{d}_{\hat{A}} f = \hat{d} f + \frac{1}{2}[\hat{A} \wedge f]$$

It is known mostly as the *covariant derivative* for the connection $\hat{A}$. In general it is not a differential (in a sence that its square is not zero):

$$\hat{d}_{\hat{A}}^2 = F(\hat{A}) = \hat{d}\hat{A} + \frac{1}{2}[\hat{A} \wedge \hat{A}]$$

This expression is known as the *curvature* of the connection.

Proposition 0.1.9. *The curvature is a pull-back of a $\mathfrak{g}$-valued 2-form on M, $F(\hat{A}) \in \pi^*(\Omega^1(M, \mathfrak{g}))$.*

Connections with zero curvature are called *flat connections*. The covariant derivative squares to zero and defines a cohomology theory theory on the space $\Omega^1(P, \mathfrak{g})^G$.

0.1.8 Trivial G-bundle

When P is the trivial principal G-bundle, $P = M \times G$,

$$\Omega^1_{(x,g)}(M \times G, \mathfrak{g})^G = \Omega^1_x(M, \mathfrak{g}) \oplus \Omega^1_g(G, \mathfrak{g})^G$$

where $x \in M$ and $g \in G$. The connection $\hat{A}$ splits as

$$\hat{A} = A + \theta$$

where θ is the Maurer-Cartan form on G and A is a $\mathfrak{g}$-valued 1-form on M.

The covariant derivative $\hat{d}_{\hat{A}}$ in this case splits into the sum of two:

$$\hat{d}_{\hat{A}} = (d_G + \frac{1}{2}[\theta\wedge\ ]) + (d + \frac{1}{2}[A\wedge\ ])$$

The first term is the standard differential in the Lie group cohomology with coefficients in $\mathfrak{g}$. The second term $d_A = d + A$ is the de Rahm differential twisted by A.

$$\hat{d}^2_{\hat{A}} = d^2_A = F(A) = dA + \frac{1}{2}[A \wedge A]$$

0.1.9 Coordinate charts

0.1.10 Gauge transformations

Let $\phi : P \to P$ be a mapping which commutes with the G-action and such that the diagram

$$\begin{tikzcd}
P \arrow[rr, "g"] \arrow[dr, "\pi"] & & P \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

is commutative. Such mapping is called a bundle automorphism, or a *gauge transformation.* These transformations form a group, known as the *gauge group.*

Proposition 0.1.10. *Gauge transformations can be identified with mappings*

$$\hat{g} : P \to G$$

which commute with the right G-action.

Indeed, to the bundle automorphism ϕ we assign the mapping $\hat{g}_\phi : P \to G$ defined by the equation $\phi(p) = p\hat{g}_\phi(p)$. To the mapping $\hat{g} : P \to \mathfrak{g}$ we assign the bundle automorphism $\phi_{\hat{g}}(p) = p\hat{g}(p)$. It is clear that this gives a bijection.

When the bundle is trivial (i.e. when there exists a global section $s : M \to G$). The mappings $P \to G$ commuting with the right G-action and $M \to G$ can be identified by $g(x) = \hat{g}(s(x))$.

Gauge transformations act on connections as

$$\phi^*(\hat{A}) = Ad_{g_\phi^{-1}}(\hat{A}) + \mathfrak{g}^*_\phi(\theta)$$

where θ is the Maurer-Cartan form on G and $\mathfrak{g}_{\phi}^{*}(\theta)$ is its pull-back to P. The curvature $F(\hat{A})$ transforms as a tensor:

$$\phi^{*}(F(\hat{A})) = Ad_{g_{\phi}^{-1}}(F(\hat{A}))$$

When the bundle P is trivial, gauge transformations can be identified with mappings $g : M \to G$. When, in addition G is a matrix group, the connection form $A \in \Omega^{1}(M, \mathfrak{g})$ transforms as

$$g : A \mapsto A^{g} = g^{-1}Ag + g^{-1}dg$$

The curvature of A transforms as a $\mathfrak{g}$-valued two form on M:

$$F(A^{g}) = g^{-1}F(A)g$$

0.1.11 Graph connections

Consider M_T a cell decomposition of M.

Definition 0.1.11. A fiber bundle over M_T is an assignment to each vertex (0-cell) $x \in V(M_T)$ a fiber P_x such that $P_x \cong P_y$ (non-canonically) for all $x, y \in V(M_T)$. $\diamond$

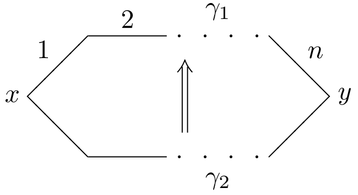
Clearly such fiber bundle is always trivial. A fiber bundle over M induces a fiber bundle over M_T by restriction but not vice versa.

Definition 0.1.12. A connection on P over M_T, is the collection of G-invariant isomorphisms $\alpha(e) : P_{e_+} \to P_{e_-}$. After a choice of trivialization $P \simeq V(M_T) \times G$ a connection becomes a mapping $\alpha : E(M_T) \to G$, $e \mapsto \alpha(e)$. $\diamond$

A gauge transformation is a bundle automorphism. It can be regarded as a mapping $V(M_T) \to G$. It acts on fibers by left multiplications. It acts on connections as $\alpha(e) \mapsto g(e_+)\alpha(e)g(e_-)^{-1}$.

Flatness

. Suppose we have two graph paths which are related by a homotopy in the cell complex.



Given α, we can define parallel transport along γ as $h_{\gamma}(\alpha) = \alpha(e_n) \cdots \alpha(e_2)\alpha(e_1)$. We say that α is *flat* if $h_{\gamma_1}(\alpha) = h_{\gamma_2}(\alpha)$ for two paths which are homotopy equivalent in M_T. In particular, a connection on M which is flat in the sense of differential geometry, induces a flat connection on the cell decomposition.

The moduli space of flat connections $\mathcal{M}_{M_T}^{G}$ is the space of gauge classes of flat connections on P. It is a subspace is the space $\mathcal{A}_{M_T}^{G} = \{\text{connections on } P\}/G_{M_T}$ of gauge classes of connections.

There is an isomorphism $\mathcal{M}_{M_T}^{G} \cong (\pi_1(M) \to G)/G$ which between the moduli space of graph connections on a cell complex and the representation variety of $\pi_1(M)$ in G.